The Impact of Joint Mortality Modelling on Hedging Effectiveness of Mortality Derivatives

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Joint Mortality Models

- modelling the mortality dynamics of multiple populations.
  - males and females of a population
  - gender-specific population of different nations
  - a population and its sub-populations

Joint Mortality Models

- generally stem from single-population mortality models.
  - Lee-Carter model (Lee and Carter,1992)
  - Age-Period-Cohort model (Osmond, 1985)
  - Cairns-Blake-Dowd model (Cairns et al., 2006)
  - Renshaw-Haberman model (Renshaw and Haberman, 2006)
- five broad categories:
  - associated mortality indices
  - common and specific factors
  - ratio of death rates
  - historical simulation
  - continuous-time models

Category I: Associated Mortality Indices

- Carter and Lee (1992):
  - separate fitting of the Lee-Carter (LC) model for each population;
  - a bivariate random walk with drift for the LC mortality indices.

$$\begin{aligned} \ln m_{x,t,i} &= \alpha_{x,i} + \beta_{x,i}\kappa_{t,i} \\ \mathbf{K}_t &= \Theta + \mathbf{K}_{t-1} + \mathbf{\Delta}_t \end{aligned} \tag{1}$$

- Li and Hardy (2011):
  - A co-integrated model for the LC mortality indices.

$$\begin{aligned} \ln m_{x,t,i} &= \alpha_{x,i} + \beta_{x,i}\kappa_{t,i} \\ \kappa_{t,1} &= \theta + \kappa_{t-1,1} + \delta_t \\ \kappa_{t,2} &= a_0 + a_1\kappa_{t,1} + \omega_t \end{aligned} \tag{2}$$

Category I: Associated Mortality Indices

- Yang and Wang (2013):
  - A Vector Error Correction model (VECM) for the LC mortality indices.

$$\begin{aligned} \ln m_{x,t,i} &= \alpha_{x,i} + \beta_{x,i}\kappa_{t,i} \\ \mathbf{K}_t - \mathbf{K}_{t-1} &= \mathbf{\Theta} + \mathbf{\Pi}\mathbf{K}_{t-1} + \mathbf{\Gamma}\left(\mathbf{K}_{t-1} - \mathbf{K}_{t-2}\right) + \mathbf{\Delta}_t \end{aligned} \tag{3}$$

- Zhou et al. (2013):
  - common β_x in the Lee-Carter model;
  - a random walk plus a first order autoregressive process (RWAR) for the LC mortality indices.

$$\begin{aligned} \ln m_{x,t,i} &= \alpha_{x,i} + \beta_x\kappa_{t,i} \\ \kappa_{t,1} &= \theta + \kappa_{t-1,1} + \delta_t \\ \kappa_{t,1} - \kappa_{t,2} &= b_0 + b_1\left(\kappa_{t-1,1} - \kappa_{t-1,2}\right) + \omega_t \end{aligned} \tag{4}$$

Category I: Associated Mortality Indices

- Cairns et al. (2011):
  - separate fitting of the Age-Period-Cohort (APC) model for each population;
  - a RWAR process for the period effect parameters; an AR(2) process for the cohort effect parameters.

$$
\begin{aligned}
\ln m_{x,t,i} &= \alpha_{x,i} + n_a^{-1}\kappa_{t,i} + n_a^{-1}\iota_{t-x,i} \\
\kappa_{t,1} &= \theta + \kappa_{t-1,1} + \delta_t \\
\kappa_{t,1} - \kappa_{t,2} &= b_0 + b_1\left(\kappa_{t-1,1} - \kappa_{t-1,2}\right) + \omega_t \\
\tilde{\iota}_{h,1} &= c_{0,1} + c_{1,1}\tilde{\iota}_{h-1,1} + c_{2,1}\tilde{\iota}_{h-2,1} + \epsilon_{h,1} \\
\iota_{h,1} - \iota_{h,2} &= c_{0,2} + c_{1,2}\left(\iota_{h-1,1} - \iota_{h-1,2}\right) \\
&\quad + c_{2,2}\left(\iota_{h-2,1} - \iota_{h-2,2}\right) + \epsilon_{h,2} \qquad (5)
\end{aligned}
$$

Category I: Associated Mortality Indices

- Dowd et al. (2011):
  - A gravity model for the period and cohort effect parameters of the smaller subpopulation.

$$
\begin{aligned}
\ln m_{x,t,i} &= \alpha_{x,i} + n_a^{-1}\kappa_{t,i} + n_a^{-1}\iota_{t-x,i} \\
\kappa_{t,1} &= \theta + \kappa_{t-1,1} + \delta_t \\
\kappa_{t,2} - \kappa_{t-1,2} &= b_0 + b_1\left(\kappa_{t-1,1} - \kappa_{t-1,2}\right) + \omega_t \\
\iota_{h,1} - \iota_{h-1,1} &= c_{0,1} + c_{1,1}\left(\iota_{h-1,1} - \iota_{h-2,1}\right) + \epsilon_{h,1} \\
\iota_{h,2} - \iota_{h-1,2} &= c_{0,2} + c_{1,2}\left(\iota_{h-1,2} - \iota_{h-2,2}\right) \\
&\quad + c_{2,2}\left(\iota_{h-1,1} - \iota_{h-1,2}\right) + \epsilon_{h,2} \qquad (6)
\end{aligned}
$$

Category I: Associated Mortality Indices

- Chan et al. (2014):
  - separate fitting of the Cairns-Blake-Dowd (CBD) model with quadratic age and cohort effects (Cairns et al., 2009);
  - a vector autoregressive moving average VARMA(r, s) process for the vector of the d-th difference of period effect parameters; an AR(1) process for the cohort effect parameters.

$$\ln\left(\frac{q_{x,t,i}}{1-q_{x,t,i}}\right) = \kappa_{t,i}^{(1)} + \kappa_{t,i}^{(2)}(x-\bar{x}) + \kappa_{t,i}^{(3)}\left((x-\bar{x})^2 - \hat{\sigma}_x^2\right) + \gamma_{t-x,i}$$

$$\mathbf{Z}_t = \mathbf{\Theta} + \sum_{l=1}^{r} \mathbf{\Phi}_l \mathbf{Z}_{t-l} + \sum_{u=1}^{s} \mathbf{\Lambda}_u \mathbf{\Delta}_{t-u} + \mathbf{\Delta}_t$$

$$\begin{aligned} \gamma_{h,1} &= c_{0,1} + c_{1,1}\gamma_{h-1,1} + \epsilon_{h,1} \\ \gamma_{h,1} - \gamma_{h,2} &= c_{0,2} + c_{1,2}\left(\gamma_{h-1,1} - \gamma_{h-1,2}\right) + \epsilon_{h,2} \end{aligned} \tag{7}$$

Category II: Common and Specific Factors

- Li and Lee (2005):
  - the augmented common factor model;
  - a random walk with drift for common factor; an AR(1) process for specific factor.

$$\begin{aligned} \ln m_{x,t,i} &= \alpha_{x,i} + B_x K_t + \beta_{x,i}\kappa_{t,i} \\ K_t &= \theta + K_{t-1} + \delta_t \\ \kappa_{t,i} &= c_{0,i} + c_{1,i}\kappa_{t-1,i} + \epsilon_{t,i} \end{aligned} \tag{8}$$

Category II: Common and Specific Factors

- Li (2013):
  - the generalized Poisson common factor model;
  - a random walk with drift for common factor; an AR(r) process for specific factors.

$$\begin{aligned} \ln m_{x,t,i} &= \alpha_{x,i} + B_x K_t + \sum_{j=1}^{n} \beta_{x,i,j} \kappa_{t,i,j} \\ K_t &= \theta + K_{t-1} + \delta_t \\ \kappa_{t,i,j} &= c_{0,i} + \sum_{l=1}^{r} c_{l,i,j} \kappa_{t-l,i,j} + \epsilon_{t,i,j} \end{aligned} \tag{9}$$

Category II: Common and Specific Factors

- Li (2012):
  - the two-population logistic model;
  - a bivariate random walk with drift for common factor; a VAR(1) process for specific factors.

$$\ln\left(\frac{m_{x,t,i}}{1-m_{x,t,i}}\right) = K_t^{(1)} + K_t^{(2)}x + \kappa_{t,i}^{(1)} + \kappa_{t,i}^{(2)}x$$
$$\begin{pmatrix} K_t^{(1)} \\ K_t^{(2)} \end{pmatrix} = \begin{pmatrix} \theta_t^{(1)} \\ \theta_t^{(2)} \end{pmatrix} + \begin{pmatrix} K_{t-1}^{(1)} \\ K_{t-1}^{(2)} \end{pmatrix} + \begin{pmatrix} \delta_t^{(1)} \\ \delta_t^{(2)} \end{pmatrix}$$
$$\begin{pmatrix} \kappa_{t,i}^{(1)} \\ \kappa_{t,i}^{(2)} \end{pmatrix} = \begin{pmatrix} c_{0,i}^{(1)} \\ c_{0,i}^{(2)} \end{pmatrix} + \begin{pmatrix} c_{1,i}^{(1)}\kappa_{t-1,i}^{(1)} \\ c_{1,i}^{(2)}\kappa_{t-1,i}^{(2)} \end{pmatrix} + \begin{pmatrix} \epsilon_{t,i}^{(1)} \\ \epsilon_{t,i}^{(2)} \end{pmatrix} \quad (10)$$

- Other common period effect K_t models:
  - Carter and Lee (1992), Li and Lee (2005), Delwarde et al. (2006), Debón et al. (2011), Russolillo et al. (2011).

Category III: Ratio of Death Rates

- Plat (2009):
  - the CBD model for the reference population, a function of age and period factors for the ratio of death rates between the portfolio and the reference population;
  - a bivariate random walk with drift for period effect parameters; an AR(1) process for period factors.

$$\ln\left(\frac{q_{x,t,1}}{1-q_{x,t,1}}\right) = \kappa_{t,i}^{(1)} + \kappa_{t,i}^{(2)}(x-\bar{x})$$
$$\frac{q_{x,t,2}}{q_{x,t,1}} = 1 + \psi_x(1)\gamma_t(1) + \cdots + \psi_x(n)\gamma_t(n)$$
$$\begin{pmatrix}\kappa_t^{(1)}\\ \kappa_t^{(2)}\end{pmatrix} = \begin{pmatrix}\theta_t^{(1)}\\ \theta_t^{(2)}\end{pmatrix} + \begin{pmatrix}\kappa_{t-1}^{(1)}\\ \kappa_{t-1}^{(2)}\end{pmatrix} + \begin{pmatrix}\delta_t^{(1)}\\ \delta_t^{(2)}\end{pmatrix}$$
$$\gamma_t(j) = c_0(j) + c_1(j)\gamma_{t-1}(j) + \epsilon_t(j) \qquad (11)$$

Category III: Ratio of Death Rates

- Ngai and Sherris (2011):
  - a logistic model for the reference population, a simple linear model for the ratio of death rates between the portfolio and the reference population.

$$\ln\left(\frac{m_{x,t,1}}{1-m_{x,t,1}}\right)-\ln\left(\frac{m_{x-1,t-1,1}}{1-m_{x-1,t-1,1}}\right)=a_0+a_1x+\omega_{x,t}$$
$$\frac{q_{x,t,2}}{q_{x,t,1}}=b_0+b_1x+\epsilon_{x,t} \tag{12}$$

Category III: Ratio of Death Rates

- Hyndman et al. (2013):
  - the product-ratio model using a function of age and period factors;
  - an ARMA(r, s) process for the period factors.

$$\begin{aligned} \ln \sqrt{m_{x,t,1} m_{x,t,2}} &= \eta_x + \phi_x(1)\lambda_t(1) + \cdots + \phi_x(n_p)\lambda_t(n_p) \\ \ln \sqrt{\frac{m_{x,t,1}}{m_{x,t,2}}} &= \xi_x + \psi_x(1)\gamma_t(1) + \cdots + \psi_x(n_r)\gamma_t(n_r) \\ {}^d\lambda_t(j) &= \tau_0(j) + \sum_{l=1}^{r} \tau_l(j)\,{}^d\lambda_{t-l}(j) + \sum_{u=1}^{s} \xi_u(j)\omega_{t-u}(j) + \omega_t(j) \\ \gamma_t(j) &= c_0(j) + \sum_{l=1}^{r} c_l(j)\gamma_{t-l}(j) + \sum_{u=1}^{s} \zeta_u(j)\epsilon_{t-u}(j) + \epsilon_t(j) \end{aligned} \tag{13}$$

Category III: Ratio of Death Rates

- Villegas and Haberman (2014):
  - the Renshaw-Haberman model for the reference population, a function of age and period factors to the log ratio of death rates between the portfolio and the reference population;
  - a random walk with drift for the period effect parameters.

$$
\begin{aligned}
\ln m_{x,t,1} &= \alpha_x + \beta_x \kappa_t + \iota_{t-x} \\
\ln \frac{m_{x,t,2}}{m_{x,t,1}} &= a_x + b_x k_t \\
\kappa_t &= \theta + \kappa_{t-1} + \delta_t \\
k_t &= c + k_{t-1} + \epsilon_t
\end{aligned} \tag{14}
$$

Category IV: Historical Simulation

- model independent approach
- repeated sampling from historical data
  - e.g., Coughlan et al. (2011), Li and Ng (2011).

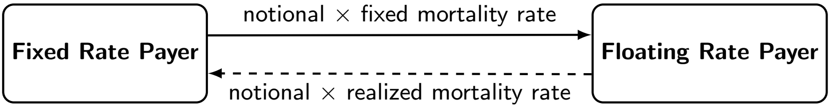
Data

- industry data: the Continuous Mortality Investigation (CMI).
  - UK male assured lives
  - UK male pensioners
  - UK male annuitants
- population data: the Human Mortality Database (HMD).
  - English and Welsh males
- year range: 1983-2006; age range: 65-90.

Model Fitting: Goodness-of-fit

Model	UK male assured lives with EW males	UK male pensioners with EW males	UK male annuitants with EW males
(1)/(2)/(3)	16,113 [9]	16,644 [8]	14,283 [9]
(4)	16,020 [7]	16,485 [6]	14,172 [7]
(5)/(6)	**14,549 [3]**	**15,169 [3]**	**12,754 [3]**
(7)	**14,224 [1]**	**14,676 [1]**	**12,428 [1]**
(8)/(9)	15,906 [6]	16,520 [7]	14,065 [6]
(10)	16,107 [8]	16,755 [9]	14,228 [8]
(11)	15,722 [5]	16,152 [4]	13,877 [5]
(12)	14,835 [4]	16,341 [5]	13,127 [4]
(13)	19,381 [10]	18,676 [10]	19,510 [10]
(14)	**14,377 [2]**	**14,885 [2]**	**12,664 [2]**

q-forwards



- reference age x^* matured at year t^*.
- realized mortality rate q_{x^*,t^*}.
- forward mortality rate $q^f_{x^*,t^*}$.
- at maturity, net settlement will be made.
- payoff at maturity date = notional $\times$ $(q^f_{x^*,t^*} - q_{x^*,t^*})$.

Longevity-Linked Liability

- single person, aged 65 at time 0.
- pension liability: pays $1 at the beginning of each year from age 65 until the person dies or attains age 91.
- mortality experience follows:
  - UK male assured lives
  - UK male pensioners
  - UK male annuitants
  - year range: 1983-2006; age range: 65-90.
- interest rate: 3% flat.

Longevity Hedge

- a portfolio of q-forwards.
- reference ages: 70, 75, 80, 85.
- the pension plan takes position as a fixed rate receiver.
- the hedging objective is to minimize the variability of unexpected cash flows.

Hedging Illustrations

- use joint mortality models to simulate 5000 future scenarios $\mathbf{q}$.
- $L(\mathbf{q})$ is the PV of the pension liability.
- $H(\mathbf{q})$ is the PV of the hedging portfolio.
- $X = L(\mathbf{q}) - E\big(L(\mathbf{q})\big)$ is the PV of unexpected cash flows.
- $X^* = L(\mathbf{q}) - E\big(L(\mathbf{q})\big) - H(\mathbf{q}) + E\big(H(\mathbf{q})\big)$ is the PV of unexpected cash flows when a longevity hedge is in place.
- hedging effectiveness metric:
  longevity risk reduction, $R = 1 - \frac{\sigma^2(X^*)}{\sigma^2(X)}$.

Optimal Hedging Effectiveness under Basis Risk

- basis risk: the payoff of q-forward is linked to EW males.
- maximum value of R (optimal strategy) with basis risk.

Model	UK male assured lives	UK male pensioners	UK male annuitants
(1)	20.1%	5.0%	0.3%
(2)	97.4%	96.9%	81.3%
(3)	95.8%	97.1%	83.6%
(4)	97.4%	95.7%	74.4%
(5)	78.3%	65.5%	2.9%
(6)	78.0%	66.3%	4.6%
(7)	17.8%	12.2%	0.2%
(8)/(9)	97.0%	88.2%	70.8%
(10)	5.8%	10.4%	0.1%
(11)	71.8%	70.0%	43.3%
(12)	1.6%	2.1%	0.1%
(14)	62.7%	28.7%	2.0%

Extensions

- parameter error
- sampling error
- multi-cohort pension liability

Conclusion

- the presence of basis risk reduces the potential hedging effectiveness of q-forwards.
- need to calibrate a longevity hedge that is robust to certain simulated joint mortality models (with higher potential maximum R).
- q-forward may not be a good hedging instrument under certain simulated joint mortality models (with very low potential maximum R).

Q&A