Robust Longevity Risk Management

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Longevity 10

3 - 4, September, Santiago

Motivation: Background

- Longevity risk has gained greater and greater attention from pension plans and annuity providers (insurers).
- Total disclosed pension liabilities of companies in the FTSE100 index increased by 31 billion in 2012; total deficit increased by 8 billion. (Bor and Cowling 2013)
- Insurers can reduce longevity risk exposure with customized/index-based mortality-linked derivatives (Dawson et al. 2006, 2010; Cairns 2013).

Motivation: Managing Longevity risk

- Hedging with mortality-linked derivatives requires accurate forecasts of mortality rates of both the reference population and the portfolio-specific population — a difficult task.
  - Different mortality models calibrated to different sample sizes produce significantly different mortality forecasts (Cairns et al. 2006; Cairns et al. 2011; Li et al. 2013).
  - It's not clear which model, and which calibration window, should be use — ambiguity of forecasted mortalities exists!
- Robust hedging strategies w.r.t. mortality forecasts are needed.

Literature review

- Continuous time
  - dynamic hedging (Dahl et al. 2008; Barbarin 2008, etc)
- Discrete time
  - value hedging (Dowd et al. 2011; Cairns 2013; Cairns et al. 2014)
  - cash-flow hedging (Cairns et al. 2008)
  - key-q duration (Li and Luo 2012)
  - "MV + CVaR" approach (Cox et al. 2013)
- Most studies do not take into account the ambiguity of forecasted mortalities (except for Cairns (2013) and Cox et al. (2013))

Contribution

- We treat the ambiguity of the forecasted mortalities in a systematic way by solving the robust mean-variance and conditional-value-at-risk (CVaR) hedging problems of the insurer which
  - are applicable to most existing mortality models and mortality-linked derivatives;
  - can be solved in a tractable way.
- The robust optimal hedges significantly out-perform the nominal (non-robust) ones in a very realistic numerical study.

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Liabilities

- $k = 1$ reference population; $k = 2$ portfolio-specific population
- $p(t, x_j, k)$: the probability that an individual aged x_j in year 0 in population k is alive in year t.
- X: the set of cohorts in the portfolio.
- n_j: number of normalized annuities sold to cohort x_j.
- Original discounted (random) liabilities

$$\sum_{x_j \in X} n_j \sum_{t=1}^{T} \frac{p(t, x_j, 2)}{(1 + r)^t}$$

An example of mortality-linked derivatives: survivor swaps

- $S(x_j, k)$: a survivor swap contingent on a single cohort x_j in population k.
- $k = 1$: index-based swap. $k = 2$: customized swap.
- At year t
  - Fixed rate payment: time 0 best estimated survival rate of cohort x_j at year t + risk premium τ_j;
  - Floating rate payment: realized survival rate.
- Discounted cash flows generated by one unit of $S(x_j, k)$

$$\sum_{t=1}^{T} \frac{p(t, x_j, k) - (1 + \tau_j) E_P[p(t, x_j, k)]}{(1 + r)^t}$$

Hedges

- Discounted (random) hedged liabilities

$$L(a,k) = \sum_{x_j \in X} n_j \sum_{t=1}^{T} \frac{p(t, x_j, 2)}{(1+r)^t} + \sum_{x_j \in X_S} a_j \sum_{t=1}^{T} \frac{(1+\tau_j) E_P[p(t, x_j, k)] - p(t, x_j, k)}{(1+r)^t}$$

  - a_j: units of $S(x_j, k)$ held by the insurer.
  - $X_S \subseteq X$: set of cohorts with survivor swaps.
- Source of longevity risks:
  - cohort basis risk ($X_S \subseteq X$);
  - population basis risk (if $k = 1$).

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Nominal vs. Robust

- $p(t, x_j, k)$-s are assumed to be captured by a parametric model, P_θ.
- Denote by
  - P_{θ^0} the true (but unknown) model;
  - $P_{\hat{\theta}}$ the best estimated model of the insurer given data up to time 0 (the reference model).

Nominal vs. Robust (cont.)

- Nominal optimization: $\hat{\theta} = \theta^0 \implies P_{\hat{\theta}} = P_{\theta^0}$.
  - The true model is assumed to be known to the insurer.
  - Solve

$$\min_a \ \text{objective}$$

- Robust optimization : $\hat{\theta} \neq \theta^0$, but $\theta^0 \in \Theta(\hat{\theta})$ (Θ a compact "confidence interval").
  - The insurer does not know the true probability law, and optimizes against the worst-case scenario.
  - Solve

$$\min_a \max_{\theta \in \Theta} \ \text{objective}$$

Objective functions

- Mean-variance

$$E_\theta[L(a,k)] + \lambda \mathrm{Var}_\theta[L(a,k)] \tag{1}$$

- Conditional-value-at-risk (Rockafellar and Uryasev 2000)

$$F_\alpha(a,\xi,\theta,k) = \xi + \frac{1}{1-\alpha}\int [L(a,k)-\xi]^+ P_\theta(dy) \tag{2}$$

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Step I: discretization

- The above optimization problems are intractable (even for the nominal ones):
  - high dimensional integrations;
  - nonlinear objective functions.
- Therefore, discretize the optimization problems
  - the (unknown) true model: $P_{\theta^0} \to \pi^0 \in R^I$;
  - the best estimated model: $P_{\hat{\theta}} \to \hat{\pi} \in R^I$;
- Consider $\Pi(\hat{\pi})$ instead of $\Theta(\hat{\theta})$.

Step II: construct the uncertainty set

- Kullback-Leibler divergence (Hansen and Sargent 2001, 2008; Ben-Tal et al. 2013; ...)

$$\Pi(\hat{\pi}) = \{\pi \in R^I | \pi_i \geq 0 \ \forall i, \ \sum_{i=1}^{I} \pi_i = 1, \ \sum_{i=1}^{I} \pi_i \log(\frac{\pi_i}{\hat{\pi}_i}) \leq \rho\}, \tag{3}$$

  where

$$\rho = \frac{\chi^2_{d,1-\alpha}}{2N}$$

  - d: number of uncertain parameters;
  - N: sample size used to estimate π^0;
  - α: confidence level.
- $\Pi(\hat{\pi})$ is the $(1-\alpha)\%$ "confidence interval" around $\hat{\pi}$.

Tractable reformulation of the robust problems

- Mean-Variance

$$\min_{a,\eta,\mathcal{K},\xi} \quad \eta\rho + \xi + \eta \sum_{i=1}^{I} \hat{\pi}_i \exp\left(\frac{\lambda \bar{L}^2(z_i, a, k) + (\mathcal{K}+1)\bar{L}(z, a, k) - \xi}{\eta} - 1\right) + \frac{\mathcal{K}^2}{4\lambda}$$

$$\begin{aligned} s.t.\ \ & a \in \mathbf{A}(k) \\ & \mathcal{K} \in R; \\ & \xi \in R; \\ & \eta > 0 \end{aligned}$$

- Conditional-value-at-risk

$$\min_{a,\xi,u,\zeta,\eta} \quad \lambda\xi + \rho\zeta + \eta + \zeta \sum_{i=1}^{I} \hat{\pi}_i \exp\left(\frac{\bar{L}(z_i, a, k) + \frac{\lambda}{1-\alpha} u_i - \eta}{\zeta} - 1\right)$$

$$\begin{aligned} & a \in \mathbf{A}(k) \\ & \xi \in R \\ & \eta \in R \\ & u_i \geq \bar{L}(z_i, a, k) - \xi, \quad \forall i \in \{1, 2, ..., I\} \\ & u_i \geq 0, \quad \forall i \in \{1, 2, ..., I\} \\ & \zeta \geq 0 \end{aligned}$$

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Insurer's portfolio

- Data
  - Reference population: Dutch males population.
  - Portfolio population: Dutch pension portfolios containing about 100,000 male policy holders.
  - Data used to estimate $P_{\hat{\theta}}$ $(\hat{\pi})$: 1980 to 2009.
- Insurer's portfolio
  - $T = 30$
  - $X = \{64, 65, ...68\}$
  - $X_S = \{64\}$
  - Risk aversion parameter $\gamma = 5$

Parametrization of P_θ

- Reference population: Lee-Carter model

$$\log(\mathbf{m}_t) = \boldsymbol{\alpha} + \boldsymbol{\beta}\kappa_t + \boldsymbol{\varepsilon}_t, \quad \boldsymbol{\varepsilon}_t \overset{iid}{\sim} N(0, \Sigma_\varepsilon)$$
$$\kappa_t = d + \kappa_{t-1} + \omega_t, \quad \omega_t \overset{iid}{\sim} N(0, \sigma_\omega^2).$$

- Portfolio population: Plat (2009)

$$\mathbf{q}_t^f = 1 + \mathbf{w}\vartheta_t + \boldsymbol{\varepsilon}_t^f, \quad \boldsymbol{\varepsilon}_t^f \overset{iid}{\sim} N(0, \Sigma_f)$$
$$\vartheta_t = \delta + \omega_t^f, \quad \omega_t^f \overset{iid}{\sim} N(0, \sigma_f^2),$$

where $q^f(t, t-1, x_j) \equiv \frac{q(t,t-1,x_j,2)}{q(t,t-1,x_j,1)}$ is the ratio of death probabilities.

Parameter Uncertainty

- Consider uncertainty only for crucial parameters (Cairns 2013; Börger et al. 2011; Li et al. 2013)

$$\log(\mathbf{m}_t) = \boldsymbol{\alpha} + \boldsymbol{\beta}\kappa_t + \boldsymbol{\varepsilon}_t, \ \ \boldsymbol{\varepsilon}_t \overset{iid}{\sim} N(0, \Sigma_\varepsilon)$$
$$\kappa_t = d + \kappa_{t-1} + \omega_t, \ \ \omega_t \overset{iid}{\sim} N(0, \sigma_\omega^2).$$

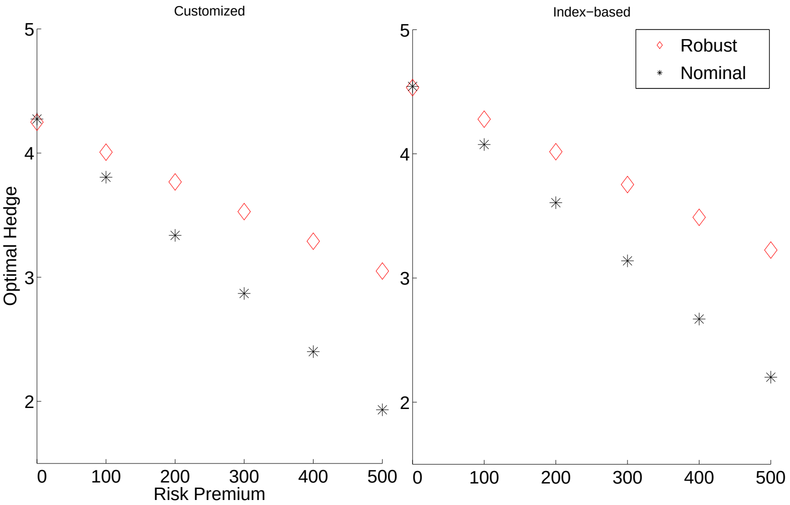
$$\mathbf{q}_t^f = 1 + \mathbf{w}\vartheta_t + \boldsymbol{\varepsilon}_t^f, \ \ \boldsymbol{\varepsilon}_t^f \overset{iid}{\sim} N(0, \Sigma_f)$$
$$\vartheta_t = \delta + \omega_t^f, \ \ \omega_t^f \overset{iid}{\sim} N(0, \sigma_f^2).$$

- So, $\theta = (d, \sigma_\omega^2, \delta)$.

Choice of risk premium

- Mortality-linked derivative market is newly developed — no meaningful risk premiums can be calibrated from existing data.
- Therefore, we look at optimal hedges under different risk premiums of the swaps: $\tau = (0\%, 1\%, 2\%, 3\%, 4\%, 5\%)$.
- Procedure
  1. Obtain optimal hedges under each risk premium.
  2. Evaluate the performance of optimal hedges under a (large) number of perturbed probability laws, $\pi \in \Pi(\hat{\pi})$.

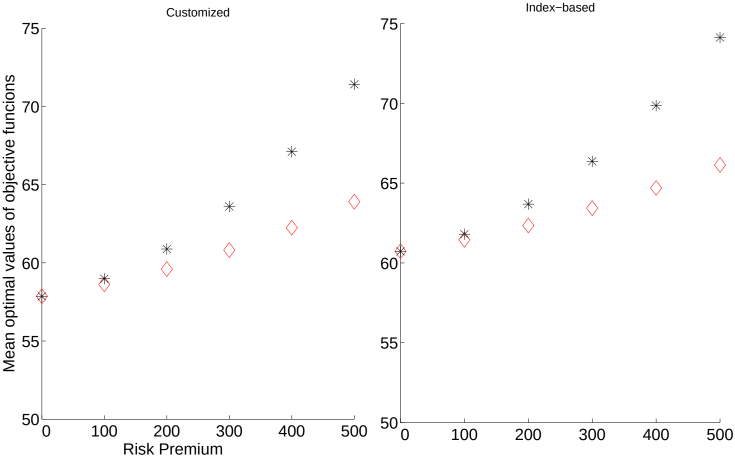
Optimal hedges: Mean-Variance case



Optimal hedges

- Risk premium rises from 0% to 5% $\implies$ optimal hedges decrease by 30%.
  - Higher risk premiums make swaps less attractive.
- Robust optimal hedges are 15% larger than nominal optimal hedges.
  - Robust optimization takes into account model misspecification, thus leads to more conservative results.
- Optimal hedges with customized swaps are 6% larger.
  - Population basis risk lowers hedge efficiency per swap.

Evaluation of optimal hedges: Mean-Variance case



Evaluation of optimal hedges

- Indexed-based survivor swaps $\implies$ 4.5% higher mean optimal value of objective functions.
  - Population basis risk lowers hedge efficiency.
- Robust optimal hedges $\implies$ uniformly lower (12%) mean optimal value of objective functions.

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Conclusion

- Probability distributions of future mortality rates are hard to estimate, which makes the hedge of longevity risk a difficult task.
- We consider the hedging problem of an insurer when she
  - does not know the true probability law governing the future mortality dynamics;
  - considers a "confidence interval" of probability laws;
  - optimizes against the worst-case scenario within the "confidence interval".
- Our model is applicable to most existing mortality models and mortality-linked derivatives.
- The robust optimal hedges uniformly outperforms the nominal optimal hedges in a realistic numerical study.

Conclusion

Thank You!