Bringing Parametric Mortality Indexes to Practice: A Generalized CBD Model with Stochastic Socioeconomic Differentials in Mortality Improvements

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Outline

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Parametric mortality indexes

Parametric mortality indexes are created from the time-varying parameters of a suitable stochastic mortality model.

Uses of parametric mortality indexes:

- Summarize mortality information at different time
- Quantify the level of mortality and longevity risks
- Construct standardized mortality-linked securities
- Develop effective index-based longevity hedges

The CBD mortality indexes

The CBD mortality indexes (Chan et al., 2014) are developed based on the Cairns-Blake-Dowd model (Cairns et al., 2006):

$$\text{logit}(q_{x,t}) = K_t^{(1)} + K_t^{(2)}(x - \bar{x}),$$

where

- $q_{x,t}$ is the probability of death for age x in year t,
- $\bar{x}$ is the average age over the sample age range, and
- $K_t^{(1)}$ and $K_t^{(2)}$ are two time-varying parameters.

The time-t values of the CBD mortality indexes are defined by $K_t^{(1)}$ and $K_t^{(2)}$.

Previous studies on CBD mortality indexes

- Chan et al. (2014) introduced K-forwards, which are forward contracts based on the CBD mortality indexes.
- Tan et al. (2014) examined how a static K-forward hedge can be calibrated using the duration- matching approach.
- Hao et al. (2017) studied the counterparty credit risk associated with K-forwards.
- Li et al. (2021) proposed K-options with closed-form risk-neutral valuations for developing static and dynamic hedges.

The research question

The previous studies have shown that the CBD mortality indexes can be used to effectively reduce longevity risk exposures in idealized settings.

However, a number of important factors in a more realistic environment has not been investigated:

- Population basis risk
- Socioeconomic differentials
- Coherence assumption

Our objectives

The objectives of this study are

- To examine the relationship between the CBD mortality indexes and the mortality dynamics of populations with socioeconomic mortality differentials.
- To develop a generalized CBD model that can capture socioeconomic mortality differentials based on the CBD mortality indexes.
- To investigate the impact of population basis risk and socioeconomic mortality differentials on CBD index-based longevity hedges.

Outline

Data

Population	Source	Gender	Age range	Sample period
Reference	HMD US Total	Female	35-99	1999-2018
Book	SOA US Quintile 1 SOA US Quintile 3 SOA US Quintile 5	Female	35-99	1999-2018

The CBD model

Assume that there exist two CBD indexes, $K_t^{(1)}$ and $K_t^{(2)}$:

$$\text{logit}(q_{x,t}) = K_t^{(1)} + K_t^{(2)}(x - \bar{x}),$$

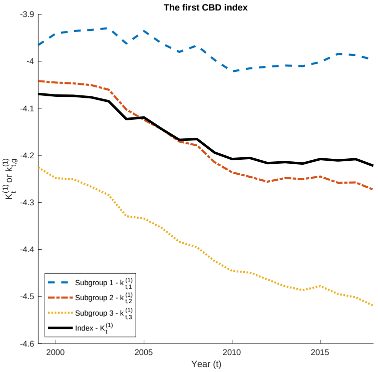
where $q_{x,t}$ is the probability of death for age x in year t of the reference population.

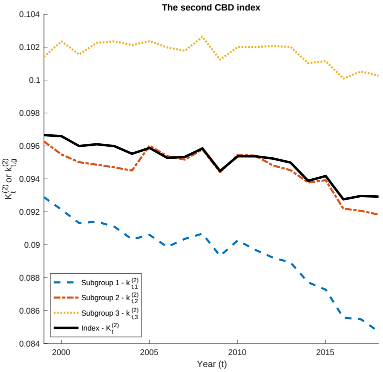
Further assume that the book population's subgroups can each be described by the CBD model:

$$\text{logit}(q_{x,t,g}) = k_{t,g}^{(1)} + k_{t,g}^{(2)}(x - \bar{x}),$$

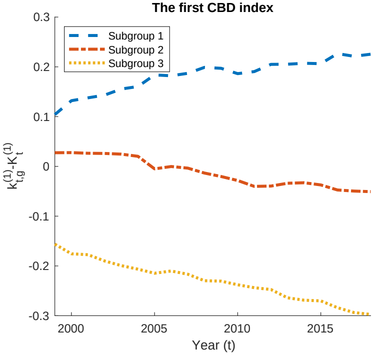
where $q_{x,t,g}$ is the probability of death for age x in year t of the g-th subgroup, $g \in \{1, 2, 3\}$.

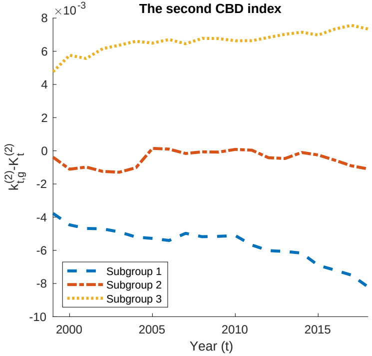
$K_t^{(i)}$ and $k_{t,g}^{(i)}$


The first CBD index
-3.9
-4
-4.1
-4.2
-4.3
-4.4
-4.5
-4.6
2000
2005
2010
2015
Year (t)
K_t^(1) or k_t,g^(1)
Subgroup 1 - k_t,1^(1)
Subgroup 2 - k_t,2^(1)
Subgroup 3 - k_t,3^(1)
Index - K_t^(1)


The second CBD index
0.104
0.102
0.1
0.098
0.096
0.094
0.092
0.09
0.088
0.086
0.084
2000
2005
2010
2015
Year (t)
K_t^(2) or k_t,g^(2)
Subgroup 1 - k_t,1^(2)
Subgroup 2 - k_t,2^(2)
Subgroup 3 - k_t,3^(2)
Index - K_t^(2)

$k_{t,g}^{(i)} - K_t^{(i)}$


The first CBD index
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
Subgroup 1
Subgroup 2
Subgroup 3
$k_{t,g}^{(1)}$-$K_t^{(1)}$
2000
2005
2010
2015
Year (t)


The second CBD index
×10^-3
8
6
4
2
0
-2
-4
-6
-8
-10
Subgroup 1
Subgroup 2
Subgroup 3
$k_{t,g}^{(2)}$-$K_t^{(2)}$
2000
2005
2010
2015
Year (t)

The M5M5 model

To develop an index-based hedge, the hedger needs to establish a relationship between $q_{x,t}$ and $q_{x,t,g}$.

The M5M5 model is sepecified as

$$\text{logit}(q_{x,t,g}) - \text{logit}(q_{x,t}) = \epsilon^{(1)}_{t,g} + \epsilon^{(2)}_{t,g}(x - \bar{x}),$$

which implies that

$$k^{(i)}_{t,g} - K^{(i)}_{t} = \epsilon^{(i)}_{t,g},$$

where $\epsilon^{(i)}_{t,g}$ follows a non-stationary stochastic process.

A cointegrated relationship

Instead of using the M5M5 model's implied relationship:

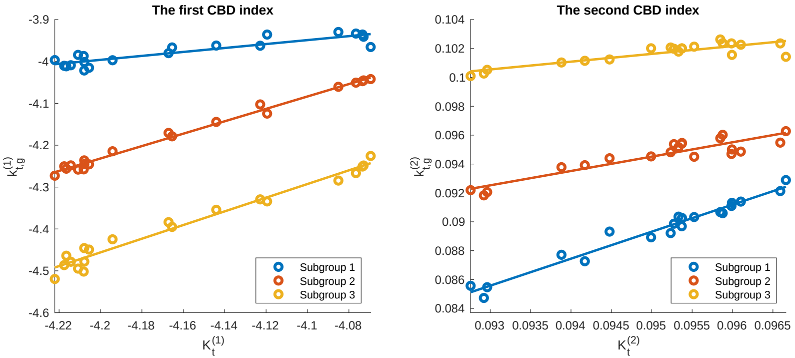
$$k_{t,g}^{(i)} = K_t^{(i)} + \epsilon_{t,g}^{(i)},$$

where $\epsilon_{t,g}^{(i)}$ follows a non-stationary stochastic process, we consider the following linearly cointegrated relationship between $k_{t,g}^{(i)}$ and $K_t^{(i)}$:

$$k_{t,g}^{(i)} = \alpha_g^{(i)} + \beta_g^{(i)} K_t^{(i)} + \epsilon_{t,g}^{(i)},$$

where $\alpha_g^{(i)}$ amd $\beta_g^{(i)}$ are the coefficients, and $\epsilon_{t,g}^{(i)}$ follows a stationary stochastic process.

$k_{t,g}^{(i)}$ vs. $K_t^{(i)}$


The first CBD index
-3.9
-4
-4.1
-4.2
-4.3
-4.4
-4.5
-4.6
-4.22
-4.2
-4.18
-4.16
-4.14
-4.12
-4.1
-4.08
K(1) t
k(1) t,g
Subgroup 1
Subgroup 2
Subgroup 3
The second CBD index
0.104
0.102
0.1
0.098
0.096
0.094
0.092
0.09
0.088
0.086
0.084
0.093
0.0935
0.094
0.0945
0.095
0.0955
0.096
0.0965
K(2) t
k(2) t,g
Subgroup 1
Subgroup 2
Subgroup 3

Cointegratoin analysis

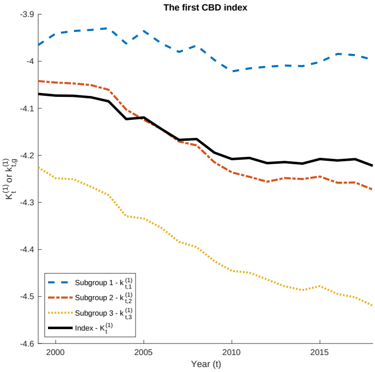
To statistically confirm the proposed cointegrated relationship, we conduct a cointegration analysis on $k_{t,g}^{(i)}$ and $K_t^{(i)}$, following the three steps used in Li and Hardy (2011):

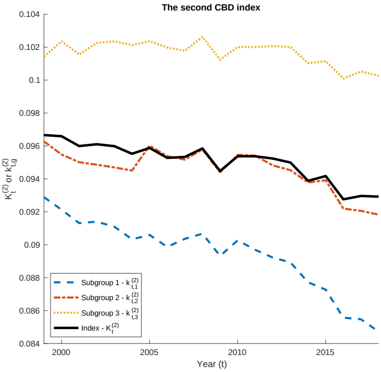
1. Verify that both $k_{t,g}^{(i)}$ and $K_t^{(i)}$ are non-stationary using the augmented Dickey-Fuller test.
2. Determine the values of $\alpha_g^{(i)}$ and $\beta_g^{(i)}$ such that

$$k_{t,g}^{(i)} = \alpha_g^{(i)} + \beta_g^{(i)} K_t^{(i)} + \epsilon_{t,g}^{(i)},$$

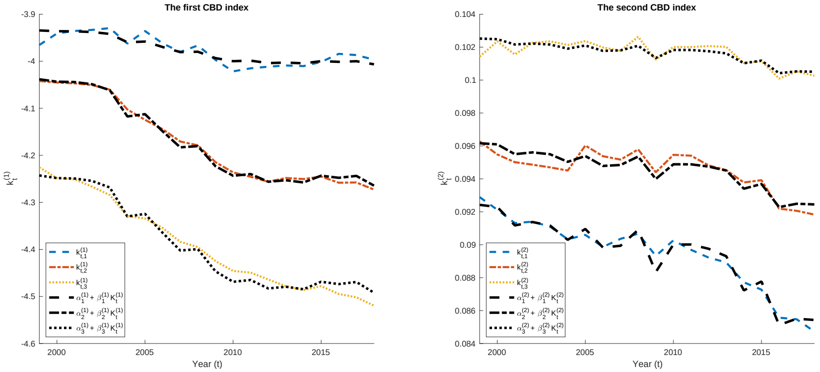
3. Check that $\epsilon_{t,g}^{(i)}$ is stationary using the augmented Dickey-Fuller test.

$k_{t,g}^{(i)}$ and $K_t^{(i)}$

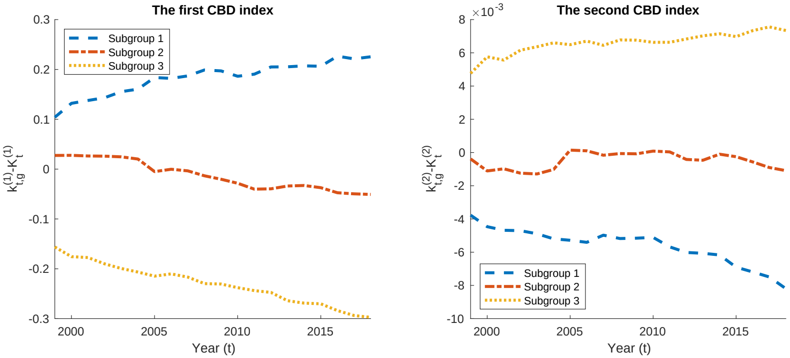

The first CBD index
-3.9
-4
-4.1
-4.2
-4.3
-4.4
-4.5
-4.6
2000
2005
2010
2015
Year (t)
K(1)t or k(1)t,g
Subgroup 1 - k(1)t,1
Subgroup 2 - k(1)t,2
Subgroup 3 - k(1)t,3
Index - K(1)t


The second CBD index
0.104
0.102
0.1
0.098
0.096
0.094
0.092
0.09
0.088
0.086
0.084
2000
2005
2010
2015
Year (t)
K(2)t or k(2)t,g
Subgroup 1 - k(2)t,1
Subgroup 2 - k(2)t,2
Subgroup 3 - k(2)t,3
Index - K(2)t

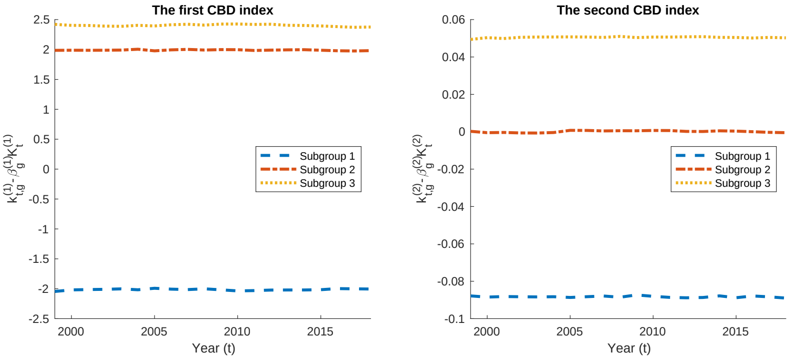
$k_{t,g}^{(i)}$ and $\alpha_g^{(i)} + \beta_g^{(i)} K_t^{(i)}$



$$k_{t,g}^{(i)} - K_t^{(i)}$$


The first CBD index
Subgroup 1
Subgroup 2
Subgroup 3
$k_{t,g}^{(1)}$-$K_t^{(1)}$
Year (t)
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
2000
2005
2010
2015
The second CBD index
$\times 10^{-3}$
8
6
4
2
0
-2
-4
-6
-8
-10
$k_{t,g}^{(2)}$-$K_t^{(2)}$
Year (t)

$$k_{t,g}^{(i)} - \beta_g^{(i)} K_t^{(i)}$$



Outline

The proposed M5M5G model

Given that the linearly cointegrated relationship holds, we substitute the equation ($k_{t,g}^{(i)} = \beta_g^{(i)} K_t^{(i)} + \epsilon_{t,g}^{(i)}$) into the CBD model structure to obtain the proposed M5M5G model as

$$\text{logit}(q_{x,t,g}) = \beta_g^{(1)} K_t^{(1)} + \beta_g^{(2)} K_t^{(2)} (x - \bar{x}) + \epsilon_{t,g}^{(1)} + \epsilon_{t,g}^{(2)} (x - \bar{x}),$$

where

- $K_t^{(1)}$ and $K_t^{(2)}$ are the CBD mortality indexes,
- $\beta_g^{(1)}$ and $\beta_g^{(2)}$ are model parameters, $g \in \{1, 2, 3\}$, and
- $\epsilon_{t,g}^{(1)}$ and $\epsilon_{t,g}^{(2)}$ follow stationary processes.

Model estimation

$$\boxed{\text{logit}(q_{x,t,g}) = \beta_g^{(1)} K_t^{(1)} + \beta_g^{(2)} K_t^{(2)} (x - \bar{x}) + \epsilon_{t,g}^{(1)} + \epsilon_{t,g}^{(2)} (x - \bar{x})}$$

We fit the M5M5G model using the ML method. The likelihood function can be written as

$$\sum_{x,t,g} D_{x,t,g} \ln(q_{x,t,g}) - (E_{x,t,g} - D_{x,t,g}) \ln(1 - q_{x,t,g}).$$

Assume that the CBD indexes $K_t^{(1)}$ and $K_t^{(2)}$ are observable and given. We need to estimate the values of $\beta_g^{(1)}$, $\beta_g^{(2)}$, $\epsilon_{t,g}^{(1)}$ and $\epsilon_{t,g}^{(2)}$ by maximizing the above likelihood function.

Model constraints

$$\text{logit}(q_{x,t,g}) = \beta_g^{(1)} K_t^{(1)} + \beta_g^{(2)} K_t^{(2)} (x - \bar{x}) + \epsilon_{t,g}^{(1)} + \epsilon_{t,g}^{(2)} (x - \bar{x})$$

We consider the following model constraints:

- To ensure that the linearly cointegrated relationship is fully captured,

$$\sum_t \epsilon_{t,g}^{(i)} \left(K_t^{(i)} - \bar{K}_t^{(i)} \right) = 0.$$

- To ensure coherence between the reference and book populations,

$$\sum_g \beta_g^{(i)} = 3.$$

Estimation procedure

With model constraints, we estimate the M5M5G model using the Lagrange multiplier method:

$$\sum_{x,t,g} D_{x,t,g} \ln(q_{x,t,g}) - (E_{x,t,g} - D_{x,t,g}) \ln(1 - q_{x,t,g})$$
$$+ \sum_{g,i} \eta_{1,g}^{(i)} \left(\sum_t \epsilon_{t,g}^{(i)} \left(K_t^{(i)} - \bar{K}_t^{(i)} \right) \right)$$
$$+ \sum_i \eta_2^{(i)} \left(\sum_g \beta_g^{(i)} - 3 \right).$$

We use a sequential quadratic optimization algorithm to maximize the above Lagrangian function.

Outline

Assumptions

- The liability being hedged is a pension plan, in which plan members are aged 65 at time 2018 and can be divided into the **three socioeconomic subgroups**.
- The hedging instruments are **two K-put options** freshly issued at time 2018 with a time-to-maturity of 15 years, and linked to the two CBD indexes.
- A **static delta hedge** is established at 2018 for each subgroup based on the M5M5G model.
- The effectiveness of the static delta hedge is evaluated by the **reduction in variance** between the hedged and unhedged positions of each subgroup.

The liability being hedged

The expected value of the pension liabilities for subgroup g, $g \in \{1, 2, 3\}$, at time $t_0 = 2018$ is

$$L_g = \sum_{u=1}^{\infty} (1+r)^{-u} \mathrm{E}\left[\mathcal{S}_{65,t_0,g}(u) | \mathcal{F}_{t_0}\right].$$

The longevity deltas of the pension liabilities are defined as the partial derivatives of L_g with respect to $K_{t_0}^{(1)}$ and $K_{t_0}^{(2)}$:

$$\Delta_g^{(i,L)} = \frac{\partial L_g}{\partial K_{t_0}^{(i)}} = \sum_{u=1}^{\infty} (1+r)^{-u} \frac{\partial}{\partial K_{t_0}^{(i)}} \mathrm{E}\left[\mathcal{S}_{65,t_0,g}(u) | \mathcal{F}_{t_0}\right].$$

The hedging instruments

Consider a (European) K-put option on the i-th CBD mortality index, issued at time $t_0 = 2018$ and matures at time $t_1 = 2033$ with a strike value of K (Li et al., 2021).

The payoff of this K-put option at time t_1 is

$$\mathcal{P}^{(i)}(K) = \max(K - K_{t_1}^{(i)}, 0).$$

The price at time t_0 can be explicitly calculated and is denoted as $P^{(i)}(K)$. The longevity delta is given by

$$\Delta^{(i,P)}(K) = \frac{\partial P^{(i)}(K)}{\partial K_{t_0}^{(i)}}.$$

The static delta hedge

We calibrate a static longevity hedge that focuses on reducing the variability of the cash flows arising from the pension liabilities and the hedging instruments.

The notional amount of the K-put option on the i-th CBD mortality index is calculated as

$$h_g^{(i)}(K) = \frac{\Delta_g^{(i,L)}}{\Delta^{(i,P)}(K)}.$$

As a static hedge, no adjustment will be made after the hedge is established.

The hedge effectiveness

The unhedged position for subgroup $g \in \{1, 2, 3\}$ is denoted as

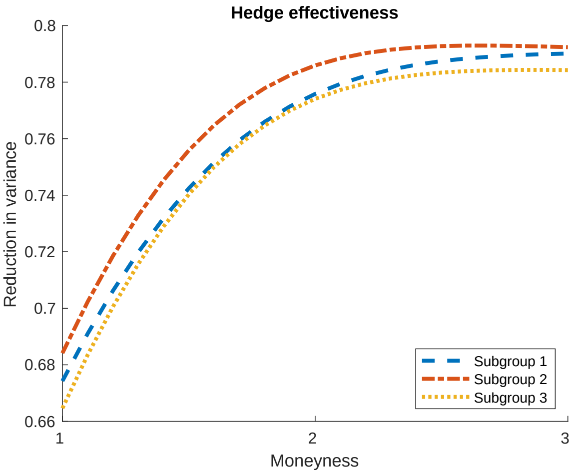
$$\mathcal{L}_g = \sum_{u=1}^{\infty} (1 + r)^{-u} \mathcal{S}_{65,t_0,g}(u),$$

while the hedged position is given by

$$\mathcal{L}_g + \sum_{i=1}^{2} h_g^{(i)}(K) \left(P^{(i)}(K) - (1 + r)^{-(t_1 - t_0)} \mathcal{P}^{(i)}(K) \right).$$

We measure the hedge effectiveness (HE) as the percentage of reduction in variance between the hedged and unhedged positions of each subgroup $g \in \{1, 2, 3\}$.

Baseline results

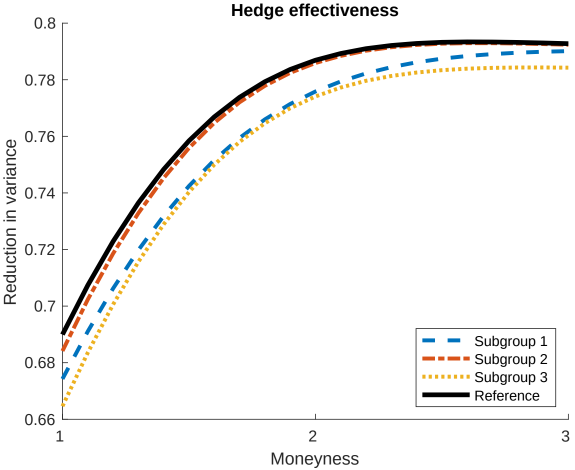


Scenario analysis

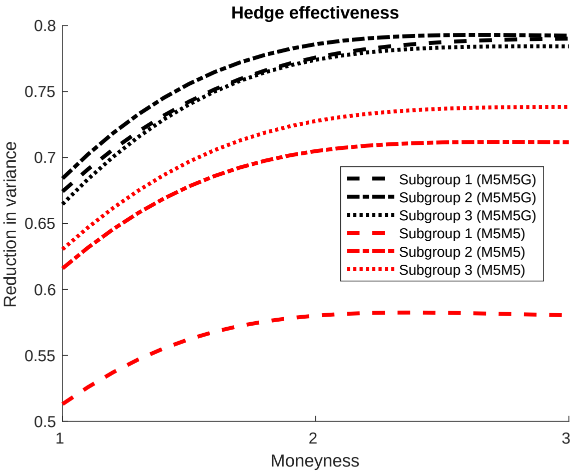
To investigate the impact of population basis risk and socioeconomic mortality differentials, we consider the following scenarios:

- The mortality experience of the liability being hedged is identical to the population linked to the CBD mortality indexes.
- The static delta hedge is calibrated using the M5M5 model, in which the cointegration relationship is not utilized.

The impact of population basis risk



The impact of socioeconomic differentials



Conclusion

In this study, we have

- Examined the co-integrated relationship between the CBD mortality indexes and populations with socioeconomic mortality differentials.
- Developed a generalized CBD model that is suitable for modeling socioeconomic differentials using the CBD mortality indexes.
- Investigated the impact of population basis risk and socioeconomic differentials on CBD index-based longevity hedges.

Thank You!

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