Mortality Modeling and Forecasting Using Non-Gaussian Innovations

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Agenda

- Introduction
- Heavy-Tailed Distributions
- Empirical analysis
- Conclusions

Introduction

- Hainaut and Devolder (2008) were the first to apply α-stable subordinators (infinite-activity, strictly positive, Lévy processes) to model mortality hazard rates.

Introduction

- Giacometti et al. (2009) consider both the error distributions of the Lee-Carter model and its mortality index, using the NIG distribution to model mortality for different age groups.
- They observe that the NIG distributional assumption for the residuals of the Lee-Carter model is better than the Gaussian one for some age groups.

Introduction

- Wang et al. (2011; 2013) examine several non-normality of the error structures in the Lee-Carter model and Renshaw-Haberman model.
- They find that these non-Gaussian distributions indeed improve the in-sample model performance fit to the data and the out-of-sample projection errors.

Introduction

- The MBMM model (Mitchell et al.; 2013) showed the Lee-Carter model can be improved by fitting with the growth rates of mortality rates over time and age rather than the mortality rates themselves.
- Chuang and Brockett (2014) adapt the MBMM structure with NIG distribution to price the longevity derivatives.

Introduction

- Chuang and Brockett (2014) drop the error term of MBMM model because they think the error term is included in the NIG distribution.

$$m_{x,t} = m_{x,0} \exp(a_x t + N_{x,t}), N_{x,t} : NIG$$

$$\ln m(t+1, x) - \ln m(t, x) = \beta_x^{(1)} + \beta_x^{(2)} \kappa_t^{(2)} + \beta_x^{(3)} \kappa_t^{(3)} + e_{x,t}$$

- We use the normal, *t*, VG, NIG and GHST distributions into the Mitchell et al. (2013) model, to fit and forecast mortality rates.

Heavy-Tailed Distributions

- Introductions of Heavy-Tailed Distributions
- The Standardization Approaches for Heavy-Tailed Distributions
- Estimation Scheme with Standardization

Introductions of Heavy-Tailed Distributions

- GH

$$f_{GH}\left(x\middle|\alpha,\beta,\lambda,\delta,\mu\right)=\frac{\left(\frac{\sqrt{\alpha^2-\beta^2}}{\delta}\right)^{\lambda}}{\sqrt{2\pi}\left(K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)\right)}e^{\beta(x-\mu)}\frac{K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\left(\frac{\sqrt{\delta^2+(x-\mu)^2}}{\alpha}\right)^{\frac{1}{2}-\lambda}}$$

- VG

$$f_{VG}\left(x\middle|\alpha,\beta,\lambda,\mu\right)=\frac{\left(\alpha^2-\beta^2\right)^{\lambda}|x-\mu|^{\lambda-0.5}K_{\lambda-0.5}\left(\alpha|x-\mu|\right)}{\sqrt{\pi}\,(2\alpha)^{\lambda-0.5}\,\Gamma(\lambda)}\exp\left(\beta(x-\mu)\right)$$

- NIG

$$f_{NIG}\left(x\middle|\alpha,\beta,\delta,\mu\right)=\frac{\alpha\delta}{\pi}\exp\left(\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)\right)\frac{K_1\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\sqrt{\delta^2+(x-\mu)^2}}$$

- GHST

$$f_{GHST}\left(x\middle|\beta,v,\delta,\mu\right)=\frac{\delta^{v}|\beta|^{\frac{v+1}{2}}\left(\sqrt{\delta^2+(x-\mu)^2}\right)^{-\frac{v+1}{2}}\mathrm{e}^{\beta(x-\mu)}}{\sqrt{\pi}\,2^{\frac{v-1}{2}}\,\Gamma(v/2)}K_{\frac{v+1}{2}}\left(|\beta|\sqrt{\delta^2+(x-\mu)^2}\right),\ \beta\neq 0.$$

- T

$$f_{t\text{-distribution}}\left(x\middle|v,\lambda,\mu\right)=\frac{\Gamma(\frac{v+1}{2})}{\sqrt{\frac{\pi v}{\lambda}}\Gamma(\frac{v}{2})}\left[1+\frac{\lambda(x-\mu)^2}{v}\right]^{\frac{-(v+1)}{2}}$$

The Standardization Approaches for Heavy-Tailed Distributions

- We let

$$X = a + b \cdot \varepsilon$$

where ε are standardized random variables (i.e. $E(\varepsilon) = 0$ and $Var(\varepsilon) = 1$, imply $E(X) = a$ and $Var(X) = b^2$). ε can be one of the standardized Lévy Processes.

The Standardization Approaches for Heavy-Tailed Distributions

- GH

$$f_{GH}\left(y|\alpha,\beta,\lambda\right)=\frac{\left(\frac{\sqrt{\alpha^2-\beta^2}}{\delta}\right)^{\lambda}}{\sqrt{2\pi}\left(K_{\lambda}\left(\delta\sqrt{\alpha^2-\beta^2}\right)\right)}e^{\beta(y-\mu)}\frac{K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(y-\mu)^2}\right)}{\left(\frac{\sqrt{\delta^2+(y-\mu)^2}}{\alpha}\right)^{\frac{1}{2}-\lambda}},\ \mu \text{ and } \delta \text{ satisfy } E(\varepsilon)=0, V(\varepsilon)=1$$

- VG

$$f_{VG}\left(y|\alpha,\beta\right)=\frac{\left(\alpha^2-\beta^2\right)^{\lambda}\left|y-\mu\right|^{\lambda-0.5}K_{\lambda-0.5}\left(\alpha\left|y-\mu\right|\right)}{\sqrt{\pi}\left(2\alpha\right)^{\lambda-0.5}\Gamma(\lambda)}e^{\beta(y-\mu)},\mu=-\frac{2\beta\lambda}{\alpha^2-\beta^2}\ \text{and}\ \lambda=\frac{\left(\alpha^2-\beta^2\right)^2}{2\left(\alpha^2+\beta^2\right)}$$

- NIG

$$f_{NIG}\left(y|\alpha,\beta\right)=\frac{\alpha\delta}{\pi}\exp\left(\delta\sqrt{\alpha^2-\beta^2}+\beta(y-\mu)\right)\frac{K_1\left(\alpha\sqrt{\delta^2+(y-\mu)^2}\right)}{\sqrt{\delta^2+(y-\mu)^2}},\mu=-\frac{\beta\delta}{\sqrt{\alpha^2-\beta^2}}\ \text{and}\ \delta=\frac{\left(\alpha^2-\beta^2\right)^{1.5}}{\alpha^2}$$

- GHST

$$f_{GHST}\left(x|\beta,v,\delta,\mu\right)=\frac{\delta^{v}\left|\beta\right|^{\frac{v+1}{2}}\left(\sqrt{\delta^2+(x-\mu)^2}\right)^{-\frac{v+1}{2}}\mathrm{e}^{\beta(x-\mu)}}{\sqrt{\pi}\,2^{\frac{v-1}{2}}\Gamma(v/2)}K_{\frac{v+1}{2}}\left(\left|\beta\right|\sqrt{\delta^2+(x-\mu)^2}\right),\ \beta\neq 0.\ \mu \text{ and } \delta \text{ satisfy } E(\varepsilon)=0, V(\varepsilon)=1$$

- T

$$f_{t\text{-distribution}}\left(x|v\right)=\frac{\Gamma(\frac{v+1}{2})}{\sqrt{\frac{\pi v}{\lambda}}\Gamma(\frac{v}{2})}\left[1+\frac{\lambda(x-\mu)^2}{v}\right]^{\frac{-(v+1)}{2}},\mu=0,\ v>1\ \text{and}\ \lambda=\frac{v-2}{v},\ v>2$$

Estimation Scheme with Standardization

$$LLF = \sum_{t=1}^{n} \ln\left(f_X \left(\theta \middle| X_t \right)\right)$$

where $\theta \in \mathbb{R}^d$; d is the number of the unknown parameters.

- Using transformation of random variable

$$\varepsilon_t = \frac{X_t - a}{b}$$

$$LLF = \sum_{t=1}^{n} \left[\ln\left(f_\varepsilon \left(\theta \middle| \frac{X_t - a}{b} \right)\right) - \ln(b) \right]$$

Mitchell et al. (2013) model

$$\ln m(t+1,x) - \ln m(t,x) = \beta_x^{(1)} + \beta_x^{(2)}\kappa_t^{(2)} + \beta_x^{(3)}\kappa_t^{(3)} + e_{x,t}$$

The method of the estimation is the same as SVD.

Forecasting of Mitchell et al. (2013) model

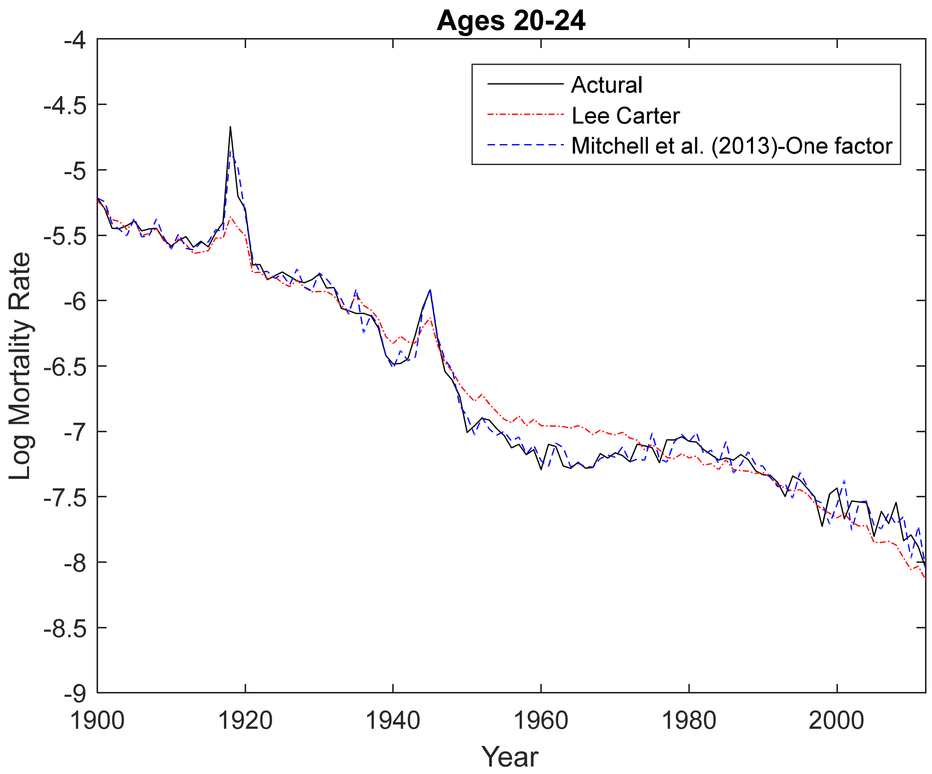
$$\kappa_t^{(2)} \sim NIG(\mu^{(2)}, \delta^{(2)}, \theta^{(2)}, \lambda^{(2)})$$

$$\kappa_t^{(3)} \sim NIG(\mu^{(3)}, \delta^{(3)}, \theta^{(3)}, \lambda^{(3)})$$

Mortality Data

- Denmark, Finland, France, Italy, Netherlands, Norway, Sweden, and England and Wales from 1900 to 2012
- Human Mortality Database (HMD)

Denmark



Ages 20-24
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1900
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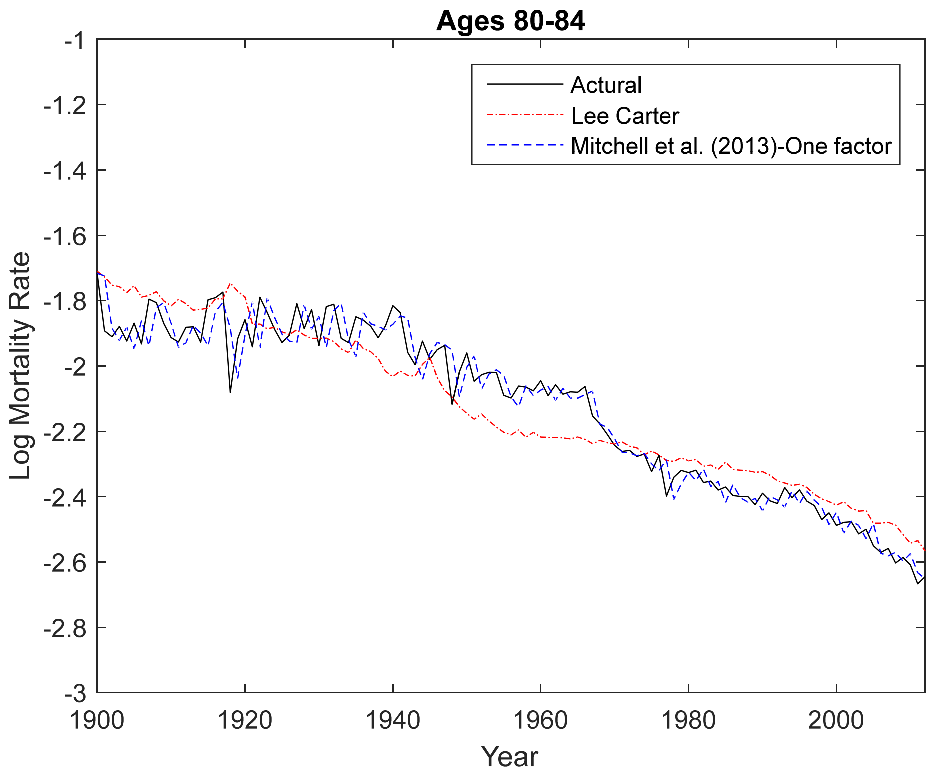
Year
Log Mortality Rate
Actural
Lee Carter
Mitchell et al. (2013)-One factor



Ages 80-84
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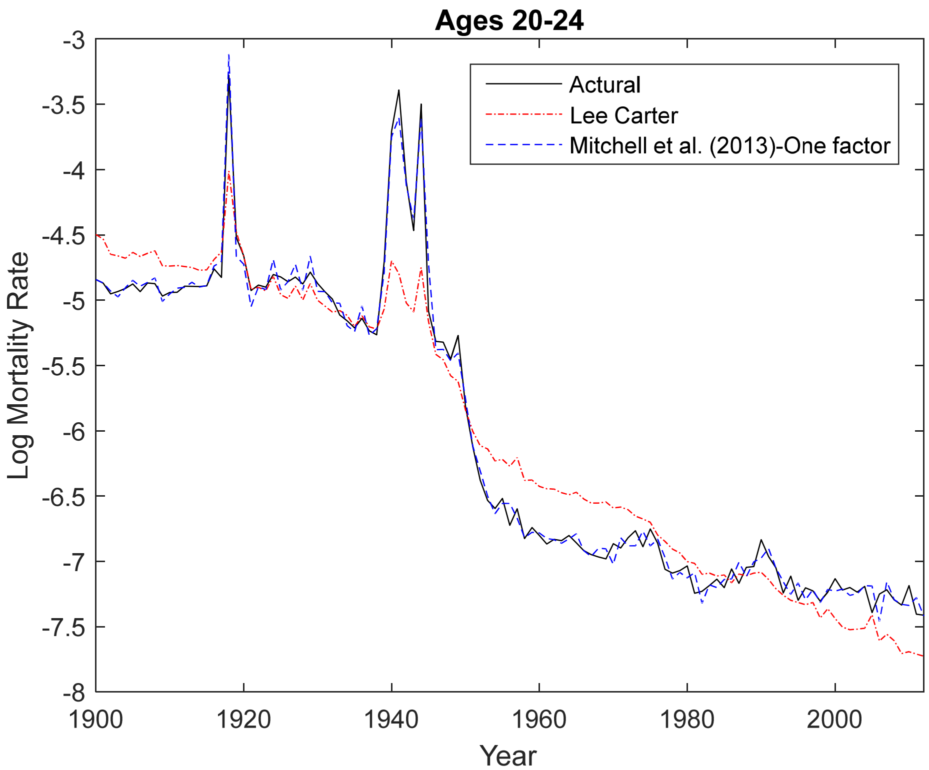
Finland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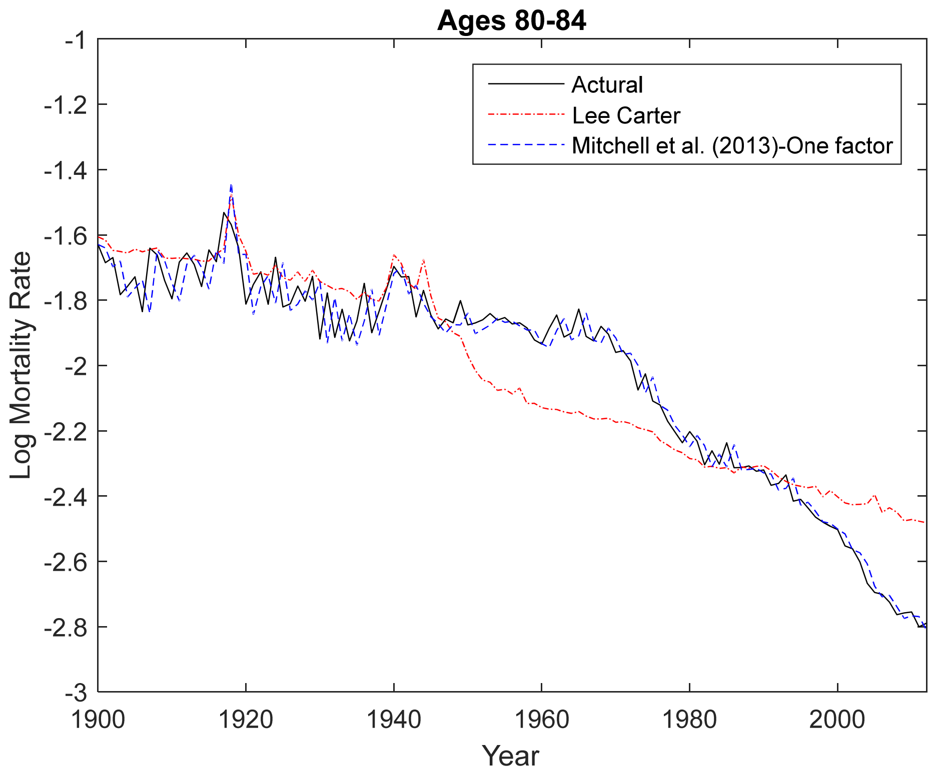
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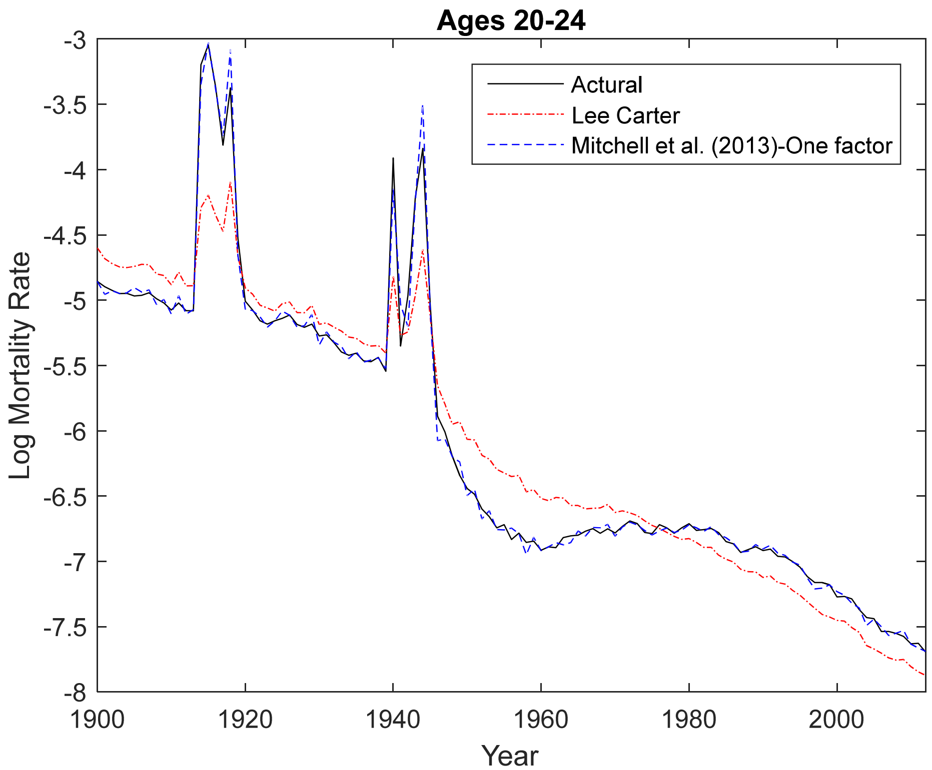
France



Ages 20-24
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Mitchell et al. (2013)-One factor
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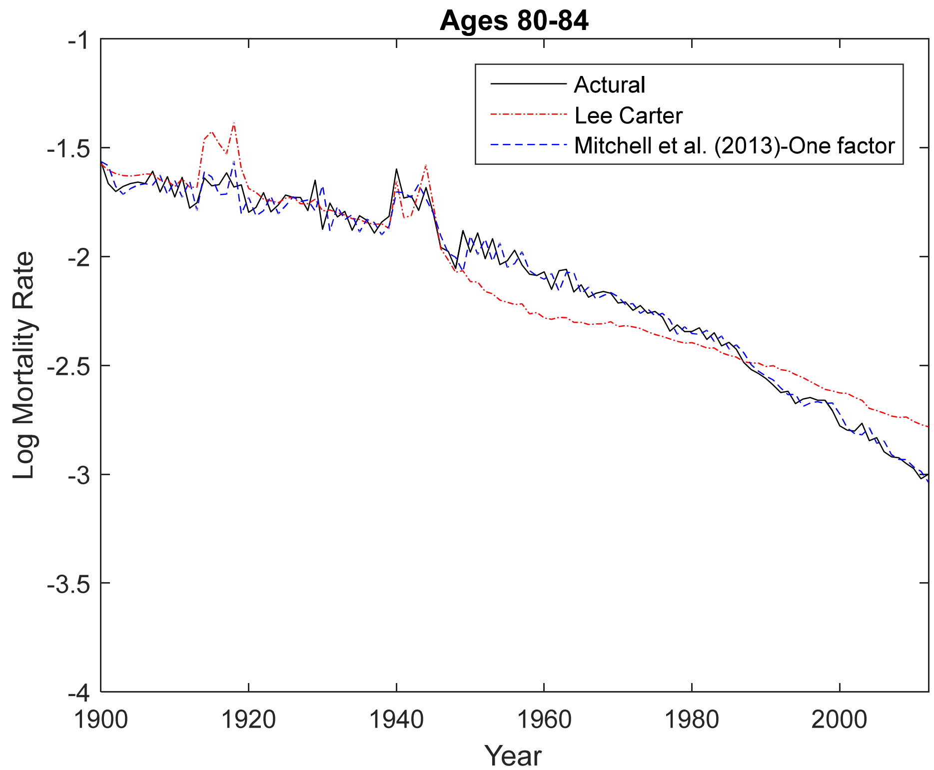

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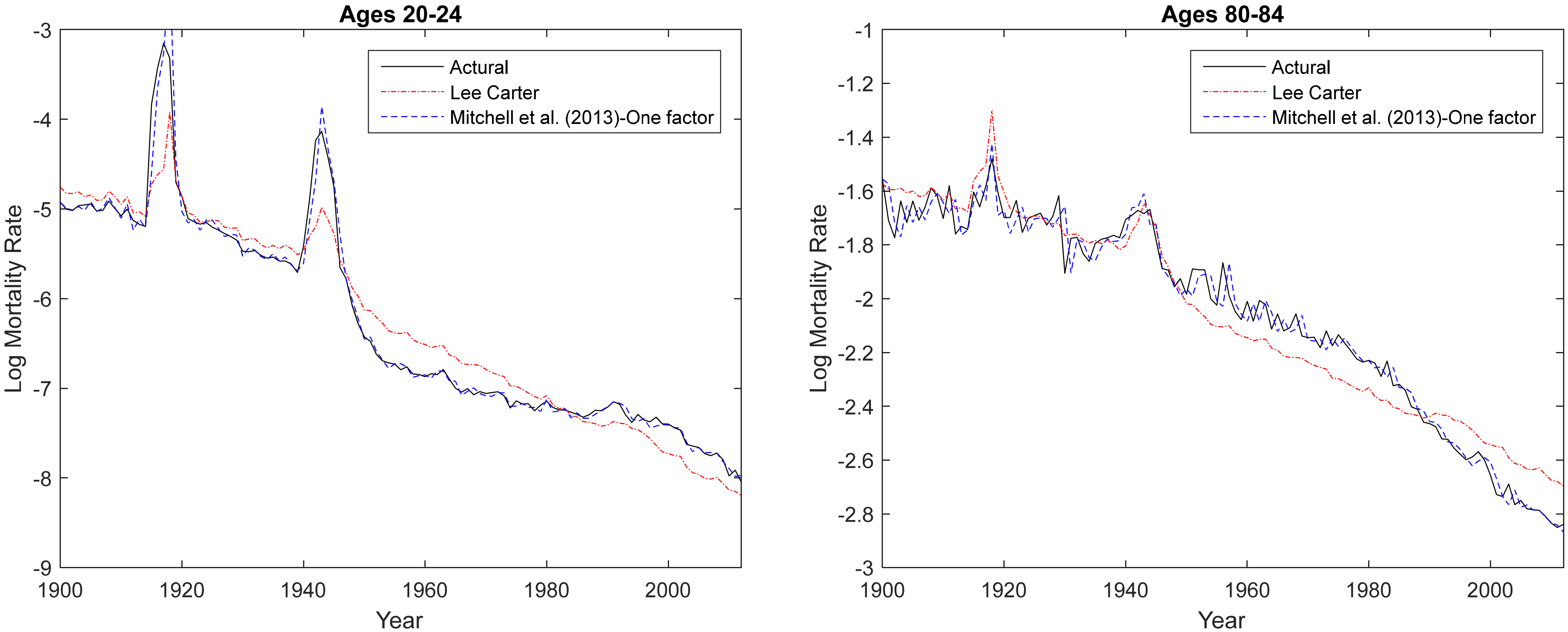
Italy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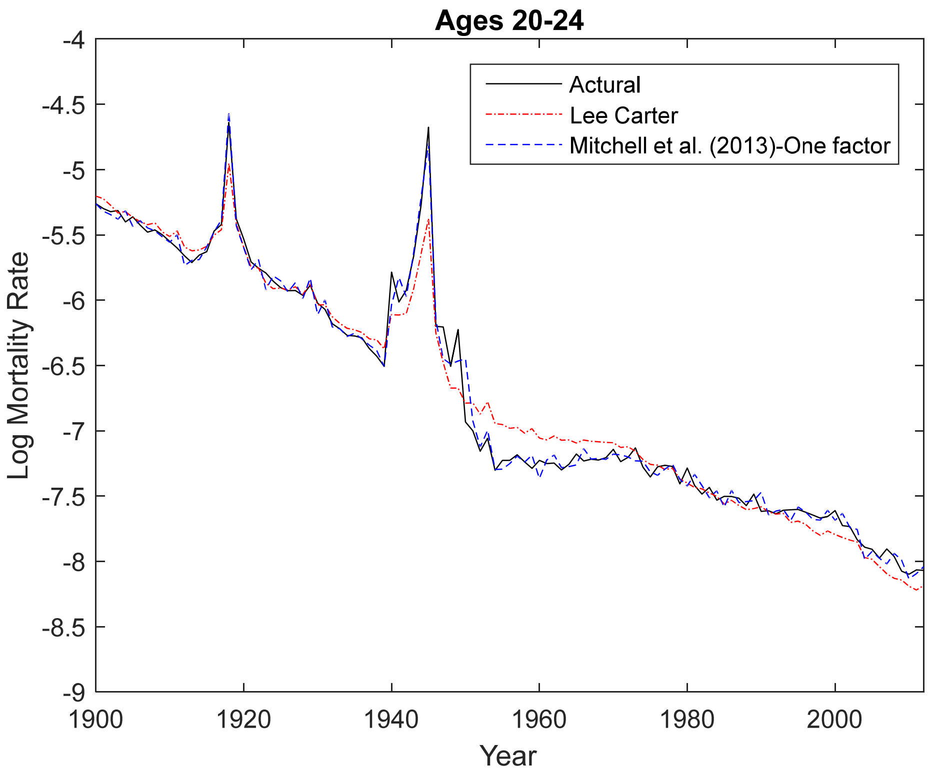
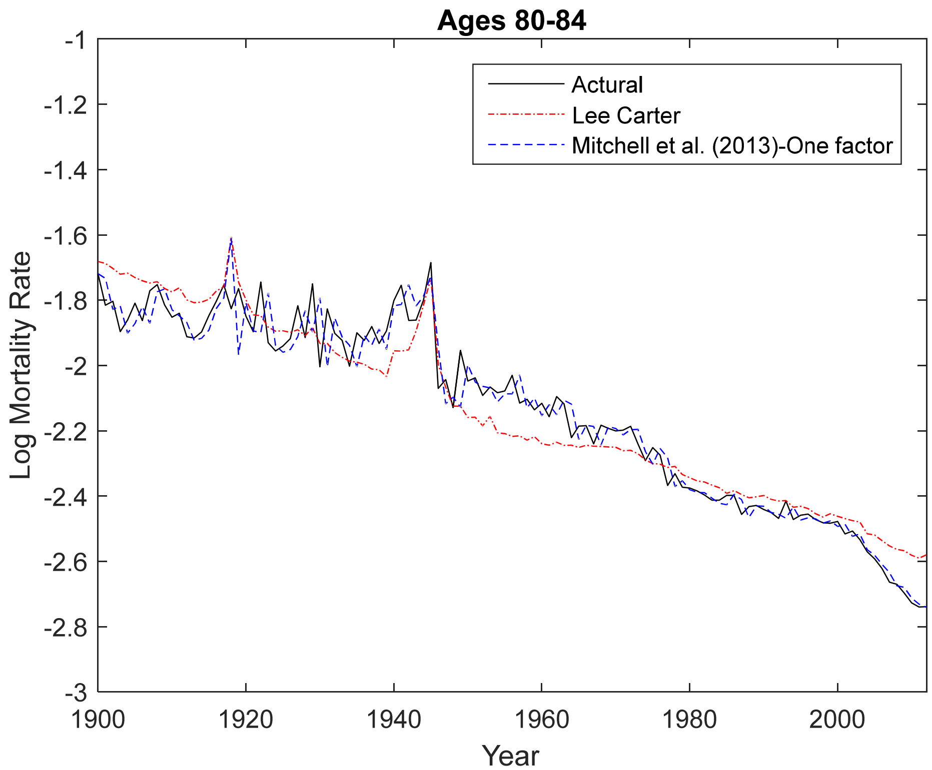
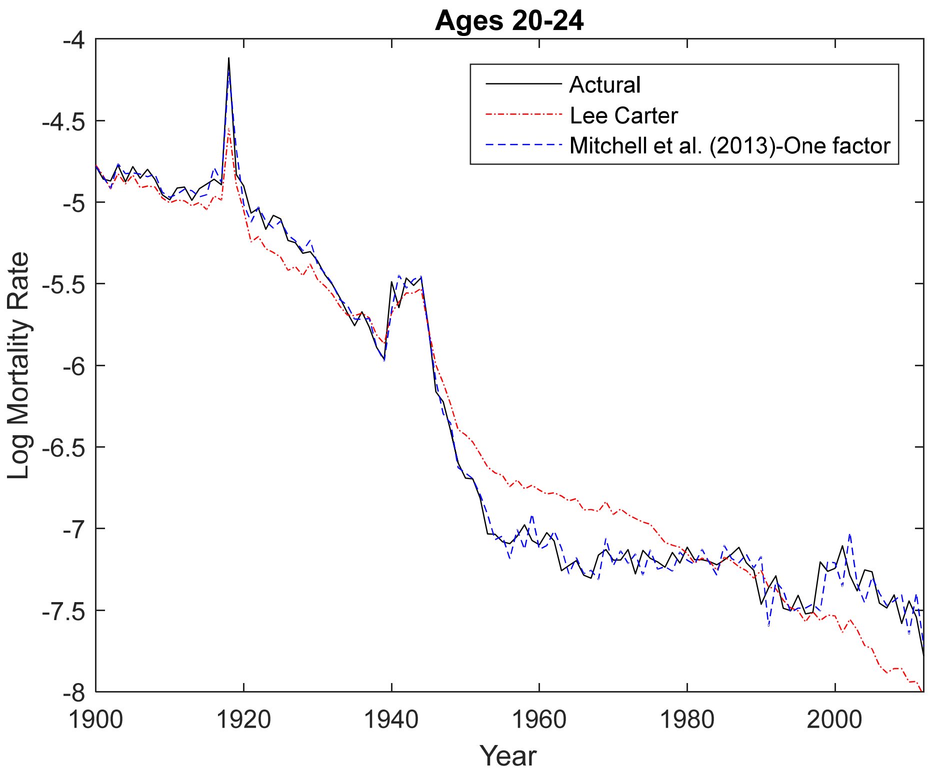
Netherlands





Norway



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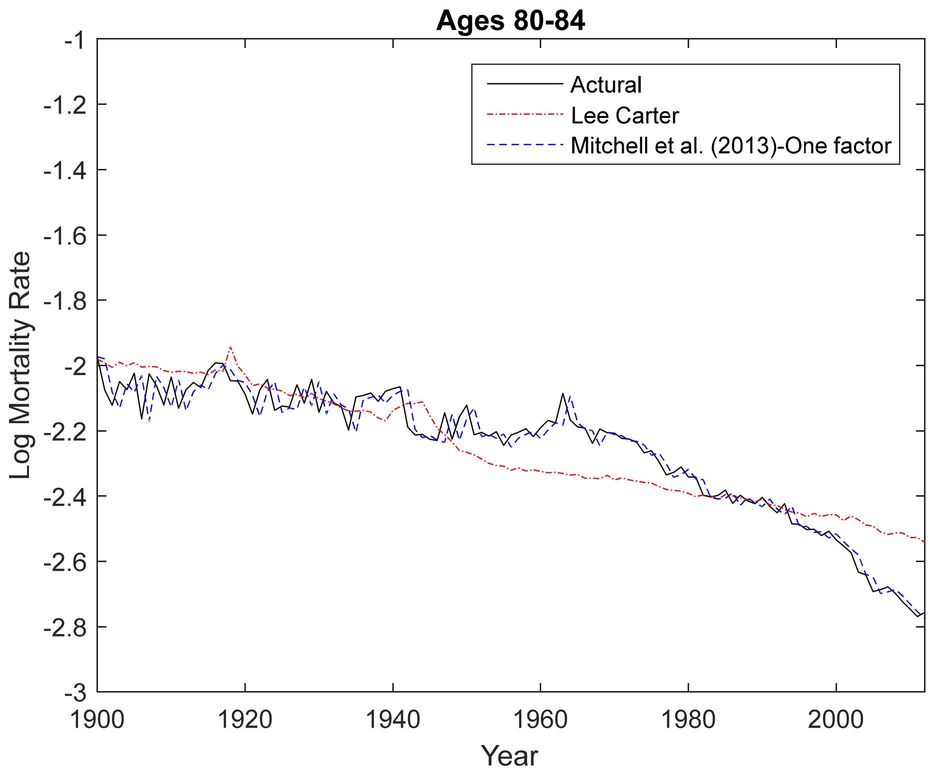

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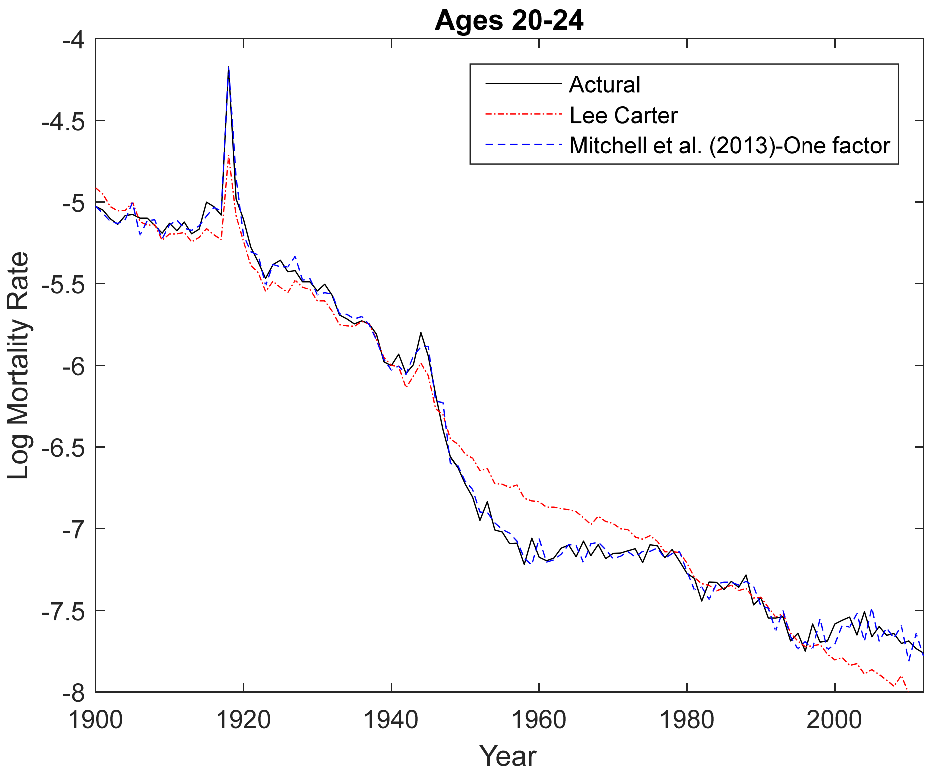
Sweden



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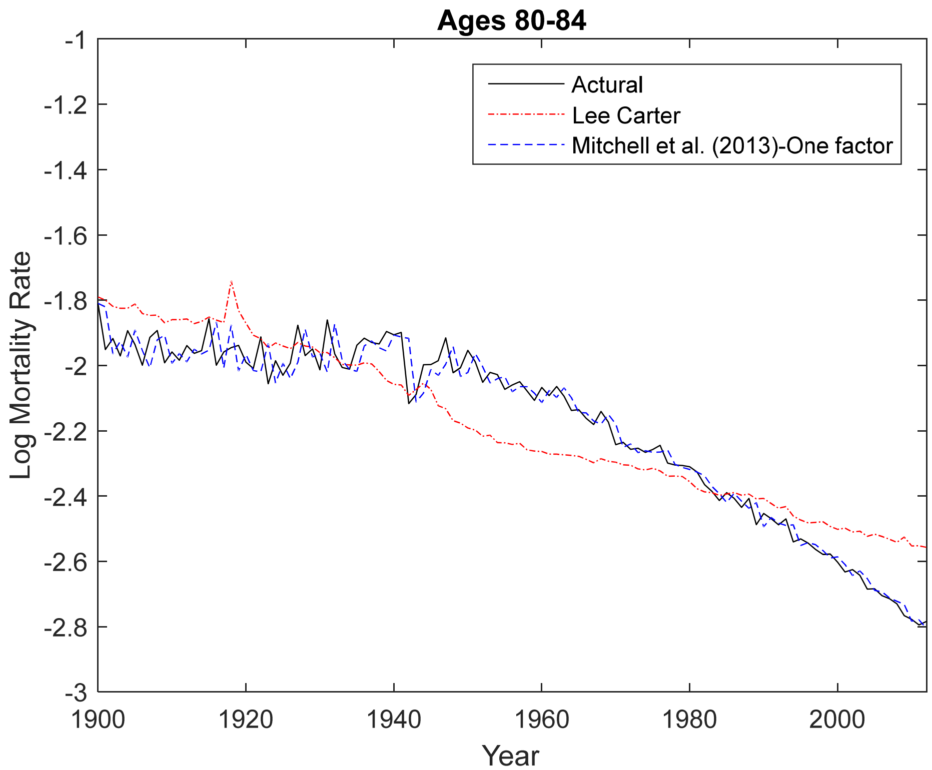

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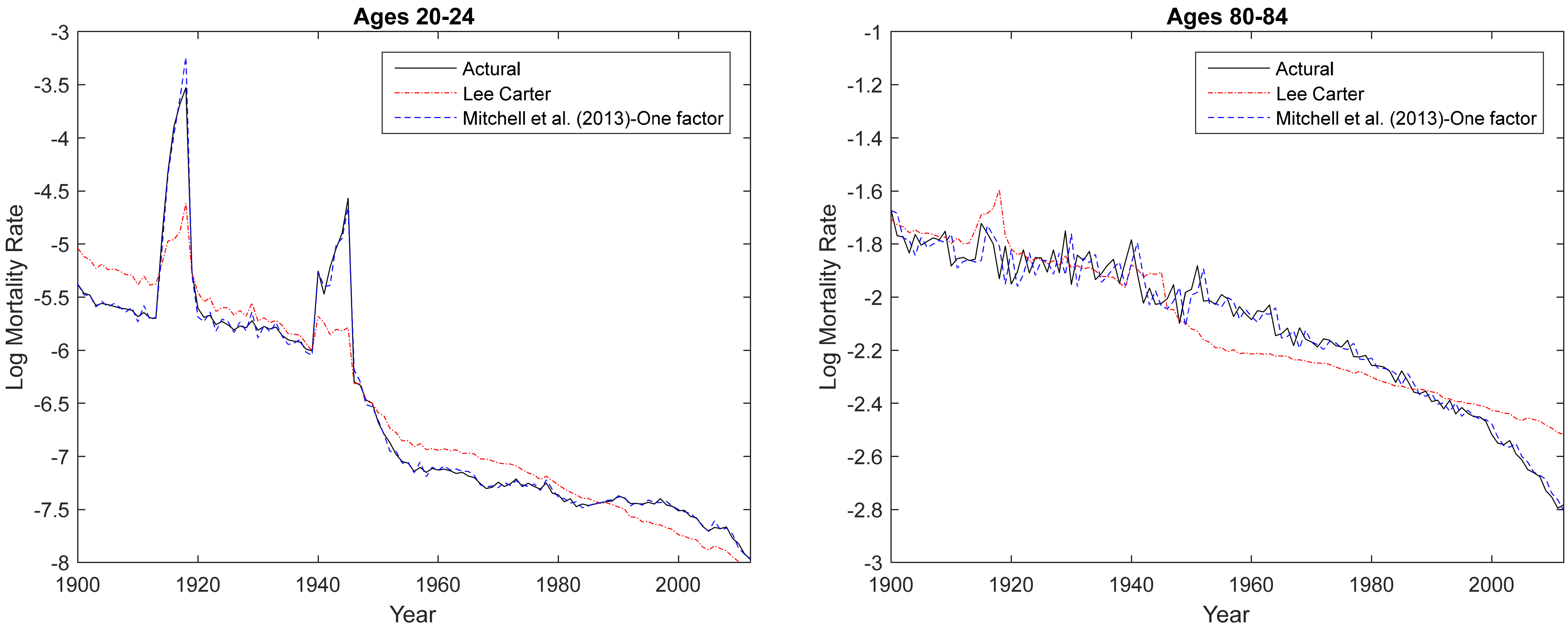
England and Wales



Normality Tests for the Residuals and Mortality Indices

- JB test statistic

$$JB = n\left[\frac{s^2}{6} + \frac{(k-3)^2}{24}\right]$$

where n is the sample size, s is sample skewness, and k is sample kurtosis.

Skewness, Excess Kurtosis and the Jarque-Bera Test $e_{x,t}$

Country	Denmark	Finland	France	Italy	Netherlands	Norway	Sweden	England and Wales
Panel A : $e_{x,t}$								
Skewness	-0.0558	-0.0066	0.1042	-0.8910	-0.3555	0.1053	0.4257	-0.1579
	(0.0505)	(0.0505)	(0.0505)	(0.0505)	(0.0505)	(0.0505)	(0.0505)	(0.0505)
Excess Kurtosis	3.2496	9.8102	9.7901	31.2115	5.5555	7.9468	6.2412	7.5120
	(0.1010)	(0.1010)	(0.1010)	(0.1010)	(0.1010)	(0.1010)	(0.1010)	(0.1010)
JB test	1036	9432	9397	95779	3074	6193	3888	5540
	[<0.001]	[<0.001]	[<0.001]	[<0.001]	[<0.001]	[<0.001]	[<0.001]	[<0.001

Skewness, Excess Kurtosis and the Jarque-Bera Test $\kappa_t^{(2)}$

Country	Denmark	Finland	France	Italy	Netherlands	Norway	Sweden	England and Wales
Panel B : $\kappa_t^{(2)}$								
Skewness	0.6872	0.7250	0.7402	-0.4770	-2.3937	1.0704	2.2840	-3.7348
	(0.2315)	(0.2315)	(0.2315)	(0.2315)	(0.2315)	(0.2315)	(0.2315)	(0.2315)
Excess Kurtosis	7.2245	15.4151	11.5483	16.3065	22.7912	13.8665	30.2867	24.8346
	(0.4629)	(0.4629)	(0.4629)	(0.4629)	(0.4629)	(0.4629)	(0.4629)	(0.4629)
JB test	252	1119	633	1245	2531	919	4378	3139
	[<0.001]	[<0.001]	[<0.001]	[<0.001]	[<0.001]	[<0.001]	[<0.001]	[<0.001]

Skewness, Excess Kurtosis and the Jarque-Bera Test $\kappa_t^{(3)}$

Country	Denmark	Finland	France	Italy	Netherlands	Norway	Sweden	England and Wales
Panel C : $\kappa_t^{(3)}$								
Skewness	-0.3203	-0.4373	-0.5526	2.0900	-0.3269	0.6492	0.5780	-0.3810
	(0.2315)	(0.2315)	(0.2315)	(0.2315)	(0.2315)	(0.2315)	(0.2315)	(0.2315)
Excess Kurtosis	5.3860	1.4972	3.1457	29.1174	2.4523	2.3760	1.9093	2.0517
	(0.4629)	(0.4629)	(0.4629)	(0.4629)	(0.4629)	(0.4629)	(0.4629)	(0.4629)
JB test	137	14	52	4038	30	34	23	22
	[<0.001]	0.0076	[<0.001]	[<0.001]	0.0011	[<0.001]	0.0022	0.0025

MBMM Model with Non-Gaussian Distributions

$$\ln m(t+1,x) - \ln m(t,x) = \beta_x^{(1)} + \beta_x^{(2)}\kappa_t^{(2)} + e_{x,t}$$

We model the error term $e_{x,t}$ and $\kappa_t^{(2)}$ using the four heavy-tailed distributions: t, VG, NIG and GHST.

Empirical Analysis

- Model Comparison

- In-Sample Goodness of Fit
  Period: 1900-2004
- Mortality Projection
  Period: 2005-2012

Model Comparison

- Model Criteria:
  - LLF: Log Likelihood Function
  - AIC: Akaike Information Criterion
  - BIC: Bayesian Information Criterion
- Goodness-of-fit Tests:
  - KS: Kolmogorov-Smirnov test
  - AD: Anderson-Darling test
  - CvM: Cramér-von-Mises test

Model Criteria

- LLF

- AIC $$AIC = -LLF + NPS$$

  where *NPS* is the effective number of parameters being estimated.

- BIC $$BIC = -LLF + 0.5 \times NPS \times \log(NOS)$$

  where *NOS* is the number of observations.

Goodness-of-fit Tests

$H_0 : G(x) = F(x;\Theta)$, *G*: empirical CDF; *F*: hypothetic CDF.

- KS: $KS = \sup_{\{x\}} \left| F(x;\Theta) - G(x) \right|$
- AD: $AD^2 = -NOS - S$

  $$S = \sum_{i=1}^{NOS} \frac{(2i-1)}{NOS} \left[\ln F(y_i;\Theta) + \ln\left(1 - F(y_{NOS+1-i};\Theta)\right) \right]$$

  y_i are the observed values in increasing order.
- CvM: $CvM = \frac{1}{12NOS} + \sum_{i=1}^{NOS} \left[\frac{2i-1}{2NOS} - F(y_i;\Theta) \right]^2$
- A lower value of this test statistic indicates a higher possibility that the mortality data come from the distribution *F*.

In-Sample Goodness of Fit

- Goodness-of-fit Measures for the Residuals of Our Model
- Goodness-of-fit Tests for the Residuals of Our Model
- Goodness-of-fit Measures for the Mortality Indices
- Goodness-of-fit Tests for the Mortality Indices

Goodness-of-fit Measures for the Residuals of MBMM Model

Denmark						
Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	2352.27	-2225.27	-2056.75	5	5	5
T	2575.58	-2447.58	-2277.73	3	3	3
VG	2600.08	-2471.08	-2299.89	2	2	2
NIG	2606.28	-2477.28	-2306.09	1	1	1
GHST	2575.58	-2446.58	-2275.40	4	4	4

Netherlands						
Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	2710.41	-2583.41	-2414.88	5	5	5
T	3020.03	-2892.03	-2722.18	3	2	2
VG	2997.16	-2868.16	-2696.97	4	4	4
NIG	3070.70	-2941.70	-2770.52	1	1	1
GHST	3020.09	-2891.09	-2719.91	2	3	3

Finland						
Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	1799.86	-1672.86	-1504.33	5	5	5
T	2222.61	-2094.61	-1924.76	3	3	3
VG	2235.79	-2106.79	-1935.61	2	2	2
NIG	2281.54	-2152.54	-1981.36	1	1	1
GHST	2222.61	-2093.61	-1922.43	4	4	4

Norway						
Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	2596.46	-2469.46	-2300.94	5	5	5
T	2811.41	-2683.41	-2513.55	4	3	3
VG	2824.95	-2695.95	-2524.77	1	1	1
NIG	2818.82	-2689.82	-2518.64	2	2	2
GHST	2811.56	-2682.56	-2511.37	3	4	4

France						
Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	2809.77	-2682.77	-2514.25	5	5	5
T	3211.47	-3083.47	-2913.61	2	2	2
VG	3130.04	-3001.04	-2829.86	4	4	4
NIG	3249.80	-3120.80	-2949.62	1	1	1
GHST	3210.18	-3081.18	-2910.00	3	3	3

Sweden						
Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	2823.13	-2696.13	-2527.61	5	5	5
T	3050.55	-2922.55	-2752.70	2	2	2
VG	3004.81	-2875.81	-2704.63	4	4	4
NIG	3083.60	-2954.60	-2783.41	1	1	1
GHST	3050.55	-2921.55	-2750.37	3	3	3

Italy						
Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	2532.45	-2405.45	-2236.93	5	5	5
T	3191.09	-3063.09	-2893.23	2	2	2
VG	2996.44	-2867.44	-2696.26	4	4	4
NIG	3221.91	-3092.91	-2921.73	1	1	1
GHST	3191.00	-3062.00	-2890.82	3	3	3

England and Wales						
Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	2978.86	-2851.86	-2683.33	5	5	5
T	3005.86	-2877.86	-2708.00	4	4	4
VG	3266.63	-3137.63	-2966.45	2	2	2
NIG	3294.39	-3165.39	-2994.21	1	1	1
GHST	3257.47	-3128.47	-2957.29	3	3	3

Goodness-of-fit Tests for the Residuals of MBMM Model

Denmark									
Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.064**	0.029	0.034	20.287**	2.461	3.867	3.516*	0.458	0.748
T	0.022	0.028	0.034	2.181	2.468	3.786	0.308	0.454	0.735
VG	0.021	0.029	0.034	0.942	2.492	3.852	0.148	0.462	0.731
NIG	0.010	0.028	0.034	0.175	2.485	4.083	0.028	0.463	0.758
GHST	0.022	0.029	0.035	2.186	2.558	3.961	0.311	0.477	0.765

Finland									
Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.109*	0.029	0.035	60.464**	2.510	3.916	10.523**	0.461	0.746
T	0.027	0.029	0.034	4.407**	2.503	3.797	0.454	0.460	0.738
VG	0.018	0.029	0.034	1.342	2.447	3.754	0.174	0.452	0.704
NIG	0.011	0.029	0.034	0.354	2.510	4.051	0.042	0.461	0.761
GHST	0.027	0.029	0.035	4.416**	2.484	3.929	0.457	0.463	0.752

France									
Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.097**	0.029	0.035	49.163**	2.510	3.916	8.559**	0.461	0.746
T	0.028	0.029	0.035	3.913	2.530	3.917	0.480*	0.465	0.755
VG	0.036*	0.029	0.035	4.673**	2.432	3.763	0.804**	0.455	0.712
NIG	0.016	0.029	0.034	0.328	2.534	3.983	0.043	0.467	0.743
GHST	0.028	0.029	0.035	3.862	2.476	3.986	0.483*	0.461	0.759

Italy									
Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.121**	0.029	0.035	Inf**	2.510	3.916	13.239**	0.461	0.746
T	0.026	0.029	0.034	2.594*	2.504	4.019	0.309	0.465	0.765
VG	0.051**	0.029	0.034	8.785**	2.401	3.752	1.428**	0.442	0.719
NIG	0.014	0.029	0.034	0.518	2.513	4.019	0.056	0.454	0.719
GHST	0.025	0.029	0.034	2.599*	2.573	3.972	0.309	0.478	0.748

Note: * and ** denote significance at the 5% and 1% level, respectively.

Goodness-of-fit Tests for the Residuals of MBMM Model

Netherlands									
Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.082**	0.029	0.035	34.179**	2.510	3.916	5.918**	0.461	0.746
T	0.025	0.029	0.034	3.410*	2.490	3.851	0.415	0.462	0.745
VG	0.025	0.029	0.034	1.215	2.447	3.754	0.186	0.452	0.704
NIG	0.012	0.029	0.034	0.419	2.511	4.051	0.061	0.461	0.761
GHST	0.024	0.029	0.035	3.367*	2.484	3.929	0.404	0.463	0.752
Norway									
Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.062**	0.029	0.035	20.611**	2.510	3.916	3.420**	0.461	0.746
T	0.016	0.029	0.034	0.842	2.481	3.913	0.097	0.461	0.739
VG	0.031*	0.028	0.034	2.155	2.442	3.763	0.354	0.451	0.714
NIG	0.010	0.029	0.034	0.360	2.527	3.794	0.049	0.470	0.742
GHST	0.015	0.029	0.034	0.822	2.547	3.954	0.093	0.469	0.748
Sweden									
Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.073**	0.029	0.035	26.094**	2.510	3.916	4.582**	0.461	0.746
T	0.026	0.029	0.034	2.638*	2.473	3.912	0.369	0.462	0.738
VG	0.016	0.028	0.034	0.561	2.437	3.798	0.074	0.450	0.704
NIG	0.012	0.029	0.034	0.409	2.527	3.868	0.063	0.467	0.747
GHST	0.026	0.029	0.034	2.659*	2.557	3.947	0.374	0.467	0.754
England and Wales									
Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.084**	0.029	0.035	34.114**	2.510	3.916	6.161**	0.461	0.746
T	0.050**	0.029	0.034	7.421**	2.466	3.818	1.389**	0.456	0.740
VG	0.019	0.029	0.034	0.684	2.452	3.787	0.125	0.455	0.722
NIG	0.009	0.029	0.034	0.228	2.508	3.998	0.024	0.461	0.749
GHST	0.032*	0.029	0.034	3.487*	2.464	3.954	0.483	0.458	0.745

Note: * and ** denote significance at the 5% and 1% level, respectively.

Goodness-of-fit Measures for the Mortality Indices

Denmark							Netherlands						
Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank	Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-121.58	123.58	126.24	5	5	5	Normal	-227.54	229.54	232.20	5	5	5
T	-100.77	103.77	107.76	3	3	1	T	-184.46	187.46	191.44	4	4	4
VG	-101.55	105.55	110.86	4	4	4	VG	-163.39	167.39	172.70	3	3	3
NIG	-99.67	103.67	108.98	2	2	3	NIG	-148.52	152.52	157.82	1	1	1
GHST	-99.13	103.13	108.44	1	1	2	GHST	-162.06	166.06	171.37	2	2	2
Finland							Norway						
Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank	Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-213.51	215.51	218.16	5	5	5	Normal	-147.11	149.11	151.76	5	5	5
T	-173.00	176.00	179.98	4	4	4	T	-115.40	118.40	122.38	4	3	3
VG	-150.30	154.30	159.60	2	2	2	VG	-115.38	119.38	124.69	3	4	4
NIG	-135.82	139.82	145.13	1	1	1	NIG	-110.77	114.77	120.08	2	2	2
GHST	-150.40	154.40	159.71	3	3	3	GHST	-109.77	113.77	119.08	1	1	1
France							Sweden						
Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank	Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-237.21	239.21	241.87	5	5	5	Normal	-167.81	169.81	172.46	5	5	5
T	-198.93	201.93	205.91	4	4	4	T	-124.96	127.96	131.94	4	4	4
VG	-149.20	153.20	158.51	2	2	2	VG	-118.61	122.61	127.92	3	3	3
NIG	-134.89	138.89	144.19	1	1	1	NIG	-114.13	118.13	123.44	1	1	1
GHST	-163.08	167.08	172.39	3	3	3	GHST	-115.52	119.52	124.83	2	2	2
Italy							England and Wales						
Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank	Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-222.87	224.87	227.52	5	5	5	Normal	-181.73	183.73	186.38	5	5	5
T	-181.48	184.48	188.47	4	4	4	T	-133.19	136.19	140.17	4	4	4
VG	-161.80	165.80	171.11	2	2	2	VG	-115.11	119.11	124.42	3	3	3
NIG	-153.34	157.34	162.65	1	1	1	NIG	-78.69	82.69	88.00	1	1	1
GHST	-164.05	168.05	173.36	3	3	3	GHST	-98.98	102.98	108.29	2	2	2

Goodness-of-fit Tests for the Mortality Indices

Denmark									
Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.133*	0.127	0.154	3.404**	2.472	3.865	0.539*	0.457	0.746
T	0.057	0.125	0.152	0.392	2.454	3.814	0.049	0.457	0.730
VG	0.078	0.127	0.153	0.507	2.519	3.964	0.077	0.461	0.761
NIG	0.060	0.127	0.153	0.292	2.493	3.801	0.045	0.459	0.731
GHST	0.057	0.127	0.154	0.234	2.471	3.808	0.038	0.455	0.719
Finland									
Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.256**	0.127	0.154	13.102**	2.472	3.865	2.506**	0.457	0.746
T	0.213**	0.126	0.152	8.423**	2.471	3.814	1.583**	0.461	0.730
VG	0.118	0.128	0.155	2.630*	2.513	3.961	0.432	0.463	0.757
NIG	0.066	0.127	0.153	0.465	2.510	3.930	0.062	0.458	0.730
GHST	0.080	0.127	0.154	1.277	2.466	3.798	0.084	0.454	0.718
France									
Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.322**	0.127	0.154	17.238**	2.472	3.865	3.491**	0.457	0.746
T	0.287**	0.126	0.152	13.468**	2.469	3.818	2.746**	0.462	0.726
VG	0.173**	0.128	0.155	4.328**	2.533	3.975	0.763**	0.462	0.756
NIG	0.127	0.127	0.153	2.480	2.526	4.030	0.398	0.458	0.728
GHST	0.099	0.127	0.153	3.552*	2.453	3.783	0.231	0.454	0.717
Italy									
Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.235**	0.127	0.154	10.930**	2.472	3.865	2.086**	0.457	0.746
T	0.186**	0.126	0.152	6.538**	2.473	3.812	1.232**	0.462	0.726
VG	0.136*	0.128	0.155	2.148	2.519	3.955	0.400	0.463	0.757
NIG	0.049	0.127	0.153	0.193	2.496	3.879	0.029	0.458	0.729
GHST	0.070	0.127	0.154	0.966	2.466	3.802	0.093	0.454	0.718

Note: * and ** denote significance at the 5% and 1% level, respectively.

Goodness-of-fit Tests for the Mortality Indices

Netherlands									
Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.241**	0.127	0.154	12.652**	2.472	3.865	2.421***	0.457	0.746
T	0.205**	0.126	0.152	8.082**	2.473	3.812	1.534**	0.462	0.726
VG	0.195**	0.128	0.155	4.691**	2.514	3.955	0.952*	0.462	0.758
NIG	0.048	0.127	0.153	0.275	2.520	3.991	0.037	0.458	0.729
GHST	0.090	0.127	0.154	1.276	2.466	3.802	0.132	0.454	0.718
Norway									
Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.163**	0.127	0.154	5.623**	2.472	3.865	0.936**	0.457	0.746
T	0.091	0.126	0.152	1.451	2.471	3.818	0.218	0.461	0.730
VG	0.082	0.127	0.154	1.007	2.520	3.964	0.150	0.463	0.761
NIG	0.052	0.127	0.153	0.313	2.493	3.822	0.033	0.459	0.730
GHST	0.040	0.127	0.154	0.237	2.470	3.808	0.026	0.455	0.719
Sweden									
Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.190**	0.127	0.154	8.338**	2.472	3.865	1.417**	0.457	0.746
T	0.136*	0.126	0.153	3.403*	2.494	3.871	0.546*	0.464	0.734
VG	0.071	0.127	0.155	1.119	2.518	3.965	0.140	0.461	0.759
NIG	0.046	0.127	0.153	0.397	2.503	3.880	0.041	0.459	0.730
GHST	0.056	0.127	0.154	0.486	2.469	3.805	0.045	0.454	0.719
England and Wales									
Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.300**	0.127	0.154	16.621**	2.472	3.865	3.292**	0.457	0.746
T	0.258**	0.126	0.152	11.829**	2.473	3.812	2.353**	0.462	0.726
VG	0.286**	0.128	0.155	12.038**	2.518	3.952	2.567**	0.463	0.759
NIG	0.079	0.127	0.153	1.237	2.534	4.117	0.153	0.459	0.729
GHST	0.081	0.127	0.153	2.467*	2.458	3.787	0.204	0.454	0.717

Note: * and ** denote significance at the 5% and 1% level, respectively.

Mortality Projection

- Fitting period: 1900-2004
- Forecasting period: 2005-2012
- Simulation Paths: 1,000,000
- Mean Absolute Percentage Error

$$MAPE = 100\% \times \frac{1}{n}\sum_{i=1}^{n}\left|\frac{A_i - F_i}{A_i}\right|$$

where A_i is the logarithm of the historical mortality rate; F_i is the natural logarithm of the forecast mortality rate; and *n* is the number of observations.

Percentile of MAPE of Mortality Projection

Denmark							
Model	Mean	90%	95%	Mean Rank	90% Rank	95% Rank	Average Rank
NIG-Normal	3.432	3.642	3.715	5	5	3	4.33
NIG-T	3.360	3.545	3.629	1	1	1	1.00
NIG-VG	3.382	3.613	3.710	2	3	2	2.33
NIG-NIG	3.382	3.616	3.730	3	4	4	3.67
NIG-GHST	3.383	3.603	3.743	4	2	5	3.67
Finland							
Model	Mean	90%	95%	Mean Rank	90% Rank	95% Rank	Average Rank
NIG-Normal	2.659	3.131	3.283	5	5	5	5.00
NIG-T	2.579	3.040	3.221	4	4	2	3.33
NIG-VG	2.556	3.037	3.242	3	3	4	3.33
NIG-NIG	2.451	2.922	3.229	2	2	3	2.33
NIG-GHST	2.292	2.513	2.595	1	1	1	1.00
France							
Model	Mean	90%	95%	Mean Rank	90% Rank	95% Rank	Average Rank
NIG-Normal	2.478	3.046	3.231	5	5	3	4.33
NIG-T	2.370	2.940	3.175	4	3	2	3.00
NIG-VG	2.322	3.005	3.311	3	4	5	4.00
NIG-NIG	2.185	2.787	3.233	2	2	4	2.67
NIG-GHST	1.921	2.107	2.175	1	1	1	1.00
Italy							
Model	Mean	90%	95%	Mean Rank	90% Rank	95% Rank	Average Rank
NIG-Normal	2.351	2.898	3.071	5	5	4	4.67
NIG-T	2.289	2.817	3.022	4	4	2	3.33
NIG-VG	2.286	2.809	3.032	3	3	3	3.00
NIG-NIG	2.223	2.766	3.114	2	2	5	3.00
NIG-GHST	2.040	2.320	2.421	1	1	1	1.00

Percentile of MAPE of Mortality Projection

Netherlands							
Model	Mean	90%	95%	Mean Rank	90% Rank	95% Rank	Average Rank
NIG-Normal	2.400	2.991	3.180	5	5	5	5.00
NIG-T	2.266	2.845	3.080	4	4	2	3.33
NIG-VG	2.249	2.842	3.094	3	3	3	3.00
NIG-NIG	2.104	2.718	3.131	2	2	4	2.67
NIG-GHST	1.888	2.164	2.273	1	1	1	1.00
Norway							
Model	Mean	90%	95%	Mean Rank	90% Rank	95% Rank	Average Rank
VG-Normal	2.440	2.689	2.768	5	5	5	5.00
VG-T	2.409	2.630	2.717	2	3	2	2.33
VG-VG	2.413	2.624	2.704	4	2	1	2.33
VG-NIG	2.413	2.640	2.748	3	4	4	3.67
VG-GHST	2.405	2.605	2.720	1	1	3	1.67
Sweden							
Model	Mean	90%	95%	Mean Rank	90% Rank	95% Rank	Average Rank
NIG-Normal	1.897	2.153	2.237	5	5	3	4.33
NIG-T	1.836	2.077	2.180	4	2	2	2.67
NIG-VG	1.832	2.132	2.265	3	4	5	4.00
NIG-NIG	1.802	2.081	2.241	2	3	4	3.00
NIG-GHST	1.726	1.879	1.941	1	1	1	1.00
England and Wales							
Model	Mean	90%	95%	Mean Rank	90% Rank	95% Rank	Average Rank
NIG-Normal	2.572	2.884	2.981	5	5	4	4.67
NIG-T	2.509	2.797	2.905	3	3	2	2.67
NIG-VG	2.560	2.825	2.933	4	4	3	3.67
NIG-NIG	2.497	2.779	2.990	2	2	5	3.00
NIG-GHST	2.385	2.500	2.540	1	1	1	1.00

Conclusions

- We attempt to incorporate four heavy-tailed distributions—*t*, VG, NIG, and GHST—into the MBMM model.

- Using mortality data from eight countries, Denmark Finland, France, Italy, Netherlands, Norway, Sweden, and England and Wales, we apply the BIC and KS, AD, and CvM tests and find consistent support for the non-Gaussian residuals of the MBMM model.

Conclusions

- When we calibrate the parameters of the MBMM model, the NIG-*t* model is the best one for Denmark, the VG-GHST model offers the best goodness of fit for Norway mortality data, the NIG-NIG model is the best one for the mortality data from Finland, France, Italy, Netherlands, Sweden and England and Wales.

Conclusions

- The normal distribution provides weak mortality projection performance, whereas *t* and its skew extension provide good mortality projections.
- The MAPE of the paper is significantly less the results of Wang et al. (2011).
- Check the distributions of mortality indices and error terms in the beginning, do not drop the error terms arbitrarily.

Extension

- Update the data to latest.
- Capture Covid-19 effects.
- Compare with the model proposed by Wang et al. (2011).

Thanks for your attention