Beta stochastic volatility model

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Introduction. Quoting Jesper Andreasen from "Risk 25: No more heroes in quantitative finance?", Risk Magazine, August 2012: "Quants have hundreds of models and, even in one given asset class, a quant will have 10 models that can fit the smile. **The question is, which is the right delta? That's still an open question, even restricting it to vanilla business**. It's one reason why there's so little activity in the interest rate options markets."

Quoting Jim Gatheral from the same article: "With less trading in exotics and vanillas moving to exchanges, we **need to focus on generating realistic price dynamics for underlyings**."

I present a stochastic volatility model that:
1) can be made consistent with different volatility regimes (thus, potentially computing a correct delta)
2) is consistent with observed dynamics of the spot and its volatility
3) has very intuitive model parameters

The model is based on joint article with Piotr Karasinski:
Karasinski P and Sepp A, **Beta Stochastic Volatility Model**, Risk Magazine, pp. 66-71, October 2012

Plan of the presentation

1) Discuss existing volatility models and their limitations

2) Discuss volatility regimes observed in the market

3) Introduce beta stochastic volatility (SV) model

4) Emphasize intuitive and robust calibration of the beta SV model

5) Case study I: application of the beta SV model to model the correlation skew

6) Case study II: application of the beta SV model to model the conditional forward skew

Motivation I. Applications of volatility models

1) **Interpolators for implied volatility surface**

⊗ Represent functional forms for implied volatility at different strikes and maturities

⊗ Assume no dynamics for the underlying ("apart from the SABR model")

⊗ Applied for marking vanilla options and serve as inputs for calibration of dynamic volatility models (local vol, stochastic vol)

2) **Hedge computation for vanilla options**

⊗ Apply deterministic rules for changes in model parameters given change in the spot

⊗ The most important is the volatility backbone - the change in the ATM volatility (and its term structure) given change in the spot price

⊗ Applied for computation of hedges for vanilla and exotic books

3) **Dynamics models** (local vol, stochastic vol, local stochastic vol)

⊗ Compute the present value and hedges of exotic options given inputs from 1) and 2)

Motivation II. Missing points

1) Interpolators

⊗ Do not assume any specific dynamics

⊗ Provide a tool to compute the market observables from given snapshot of market data: at-the-money (ATM) volatility, skew, convexity, and term structures of these quantities

2) Vanilla hedge computation

⊗ Assume specific functional rules for changes in market observables given changes in market data

3) Dynamic models (local vol, Heston) for pricing exotic options

⊗ No explicit connection to market observables and their dynamics

⊗ Apply "blind" non-linear and non-intuitive fitting methods for calibration of model parameters (correlation, vol-of-vol, etc)

We need a **dynamic volatility model** that could connect all three tools in a robust and intuitive way!

Motivation III. Beta stochastic volatility model
Propose the beta stochastic volatility model that:

1) In its simplified form, the model **can be used as an interpolator**
$\otimes$ Takes market observables (ATM volatility and skew) for model calibration with model parameters easily interpreted in terms of market observable - volatility skew (with a good approximation)

2) The model **is consistent with vanilla hedge computations** - it has a model parameter to replicate the volatility backbone (with a good approximation)
$\otimes$ The model assumes the dynamics of the ATM volatility specified by the volatility backbone

3) The model **provides robust dynamics for exotic options**:
$\otimes$ It produces steep forward skews, mean-reversion
$\otimes$ The model has a mean-reversion and volatility of volatility

Implied volatility skew I

First I describe implied volatility skew and volatility regimes

For time to maturity T, the implied volatility, which is applied to value vanilla options using the Black-Scholes-Merton (BSM) formula, can be parameterized by a linear function $\sigma(K;S_0)$ of strike price K:

$$\sigma(K;S_0) = \sigma_0 + \beta\left(\frac{K}{S_0} - 1\right)$$

β, $\beta < 0$, is the slope of the volatility skew near the ATM strike

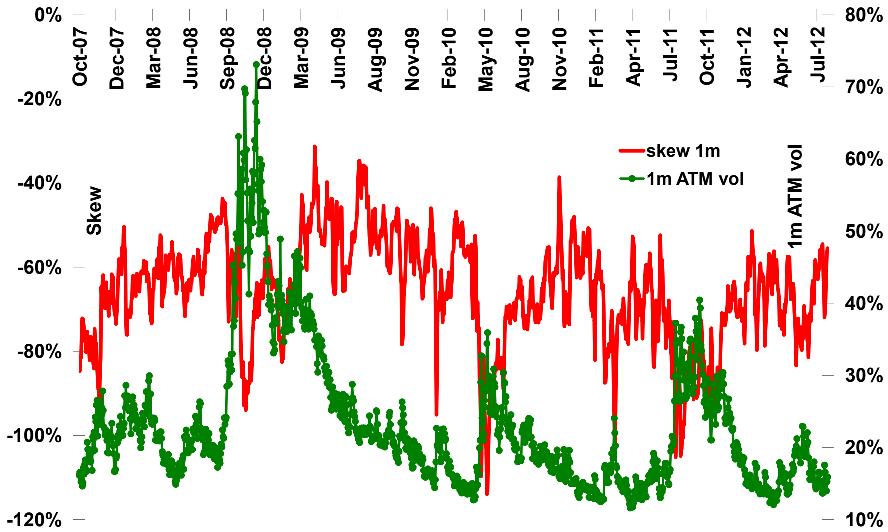
Lets take strikes at $K_{\pm} = (1 \pm \alpha)S_0$, where typically $\alpha = 5\%$:

$$\beta = \frac{1}{2\alpha}\left(\sigma((1+\alpha)S_0;S_0) - \sigma((1-\alpha)S_0;S_0)\right) \equiv \text{Skew}_\alpha$$

where Skew_α is the implied skew normalized by strike width α

The equity volatility skew is negative, as consequence of the fact that, relatively, it is more expensive to buy an OTM put option than an OTM call option.

Implied volatility skew II for 1m options on the S&P 500 index from October 2007 to July 2012



Left (red): 1m 105% − 95% skew, $\text{Skew}_{5\%}(t_n)$
Right (green): 1m ATM implied volatility

$\beta = -1.0$ means that the implied volatility of the put struck at 95% of the spot price is $-5\% \times \beta = 5\%$ higher than that of the ATM option

Sticky rules (Derman) I

1) Sticky-strike:

$$\sigma(K;S) = \sigma_0 + \beta\left(\frac{K}{S_0} - 1\right)\ ,\ \sigma_{ATM}(S) \equiv \sigma(S;S) = \sigma_0 + \beta\left(\frac{S}{S_0} - 1\right)$$

ATM vol increase as the spot declines - typical of range-bounded markets

2) Sticky-delta:

$$\sigma(K;S) = \sigma_0 + \beta\left(\frac{K-S}{S_0}\right)\ ,\ \sigma_{ATM}(S) = \sigma_0$$

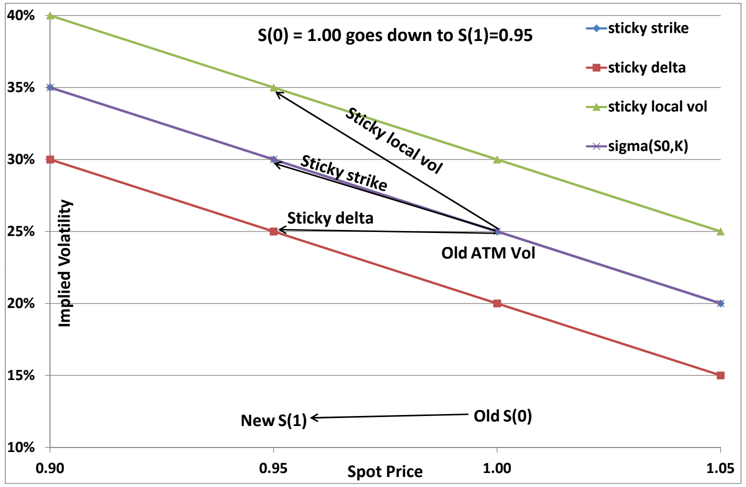
The level of the ATM volatility does not depend on spot price -typical of stable trending markets

3) Sticky local volatility:

$$\sigma(K;S) = \sigma_0 + \beta\left(\frac{K+S}{S_0} - 2\right)\ ,\ \sigma_{ATM}(S) = \sigma_0 + 2\beta\left(\frac{S}{S_0} - 1\right)$$

ATM vol increase as the spot declines twice as much as in the sticky strike case - typical of stressed markets

Sticky rules II



Given: $\beta = -1.0$ and $\sigma_{ATM}(0) = 25.00\%$
Spot change: down by -5% from $S(0) = 1.00$ to $S(1) = 0.95$
Sticky-strike regime: the ATM volatility moves along the original skew increasing by $-5\% \times \beta = 5\%$
Sticky-local regime: the ATM volatility increases by $-5\% \times 2\beta = 10\%$ and the volatility skew moves upwards
Sticky-delta regime: the ATM volatility remains unchanged with the volatility skew moving downwards

Impact on option delta

The key implication of the volatility rules is the impact on option delta Δ

We can show the following rule for call options:

$$\Delta_{Sticky-Local} \leq \Delta_{Sticky-Strike} \leq \Delta_{Sticky-Delta}$$

As a result, for hedging call options, one should be over-hedged (as compared to the BSM delta) in a trending market and under-hedged in a stressed market

Thus, the identification of market regimes plays an important role to compute option hedges

While computation of hedges is relatively easy for vanilla options and can be implemented using the BSM model, for path-dependent exotic options, we need a dynamic model consistent with different volatility regimes

Stickiness ratio I

To identify volatility regimes we introduce the stickiness ratio

Given price return from time t_{n-1} to t_n:

$$X(t_n) = \frac{S(t_n) - S(t_{n-1})}{S(t_{n-1})}$$

We make prediction for change in the ATM volatility:

$$\sigma_{ATM}(t_n) = \sigma_{ATM}(t_{n-1}) + \beta R(t_n) X(t_n)$$

where the stickiness ratio $R(t_n)$ indicates the rate of change in the ATM volatility predicted by the skew and price return

Stickiness ratio R is a model-dependent quantity, informally:

$$R \approx \frac{1}{\beta}\frac{\partial}{\partial S}\sigma_{ATM}(S)$$

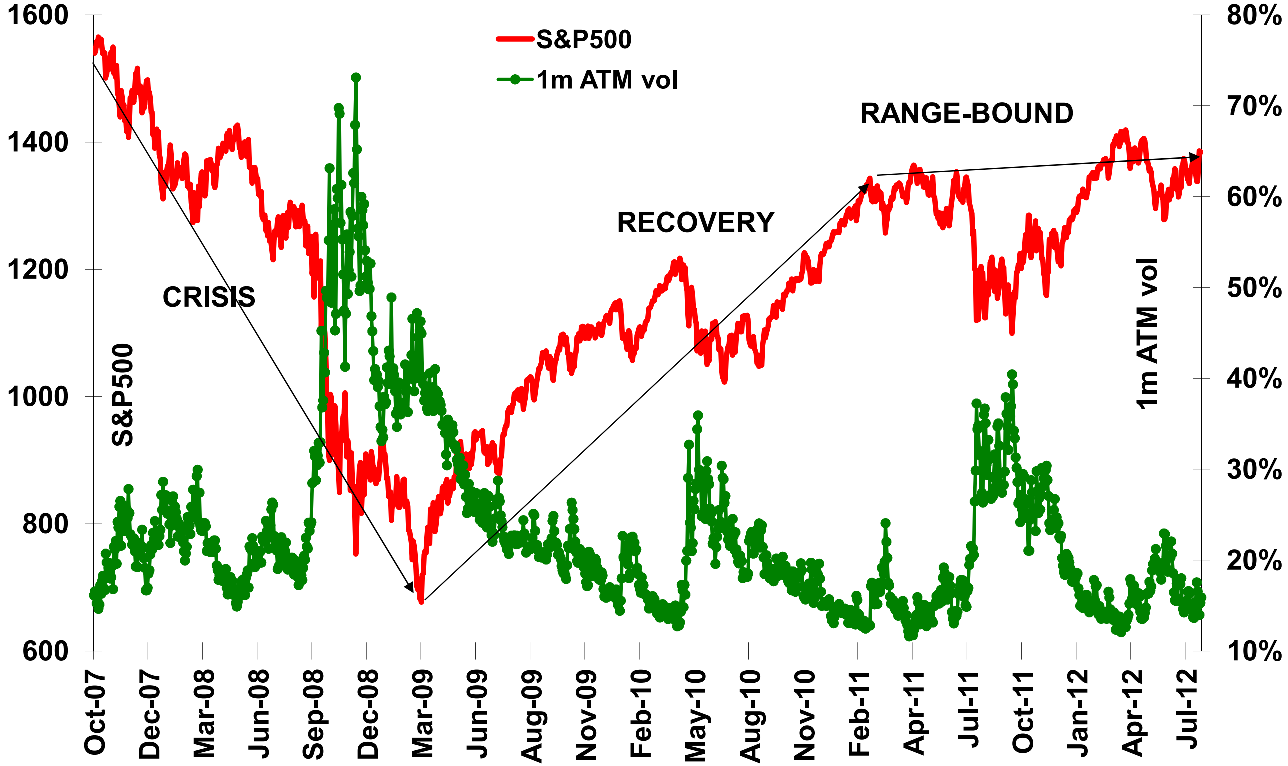
We obtain that:
$R = 1$ under sticky-strike
$R = 0$ under sticky-delta
$R = 2$ under sticky-local vol

Stickiness ratio II

Empirical test is based on using market data for S&P500 (SPX) options from 9-Oct-07 to 1-Jul-12 divided into three zones

	crisis	**recovery**	**range-bound**
start date	9-Oct-07	5-Mar-09	18-Feb-11
end date	5-Mar-09	18-Feb-11	31-Jul-12
number days	354	501	365
start SPX	1565.15	682.55	1343.01
end SPX	682.55	1343.01	1384.06
return	**-56.39%**	**96.76%**	**3.06%**
start ATM 1m	14.65%	45.28%	12.81%
end ATM 1m	45.28%	12.81%	15.90%
vol change	**30.63%**	**-32.47%**	**3.09%**
start Skew 1m	-72.20%	-61.30%	-69.50%
end Skew 1m	-57.80%	-69.50%	-55.50%
skew change	**14.40%**	**-8.20%**	**14.00%**

Stickiness ratio III



Stickiness ratio IV

To test Stickiness empirically, we apply the regression model for parameter $\overline{R}$ within each zone using daily changes:

$$\sigma_{ATM}(t_n) - \sigma_{ATM}(t_{n-1}) = \overline{R} \times \text{Skew}_{5\%}(t_{n-1}) X(t_n) + \epsilon_n$$

where $X(t_n)$ is realized return for day n

$\sigma_{ATM}(t_n)$ and $\text{Skew}_\alpha(t_n)$ are the ATM volatility and skew observed at the end of the n-th day

ϵ_n is iid normal residuals

Informal definition of the stickiness ratio:

$$R(t_n) = \frac{\sigma_{ATM}(t_n) - \sigma_{ATM}(t_{n-1})}{X(t_n)\text{Skew}_{5\%}(t_{n-1})}$$

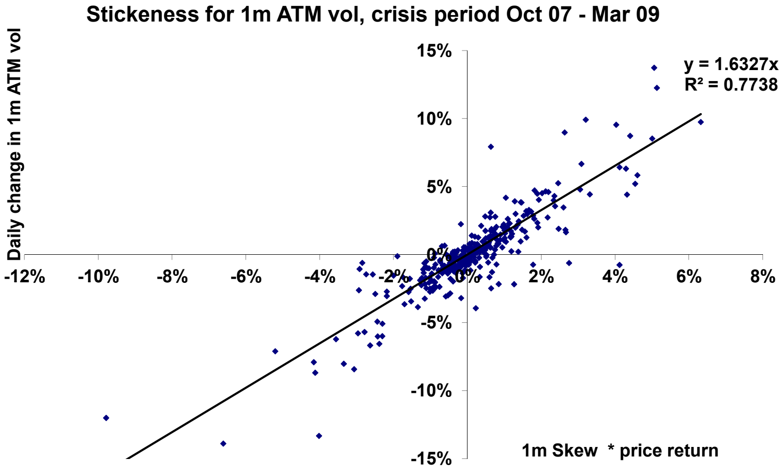
We expect that the average value of R, $\overline{R}$, as follows:

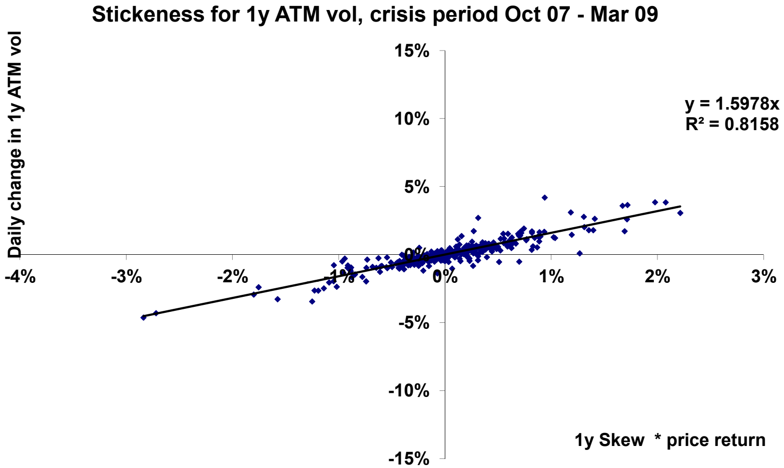
$\overline{R} = 1$ under the sticky-strike regime

$\overline{R} = 0$ under the sticky-delta regime

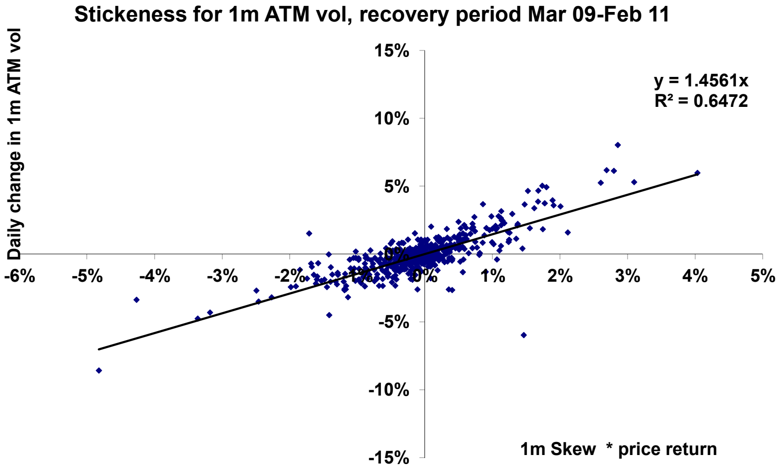
$\overline{R} = 2$ under the sticky-local regime

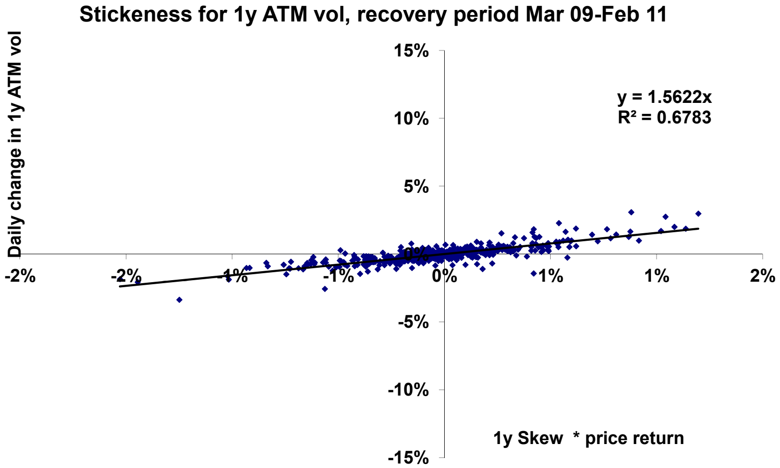
Stickiness ratio (crisis) for 1m and 1y ATM vols



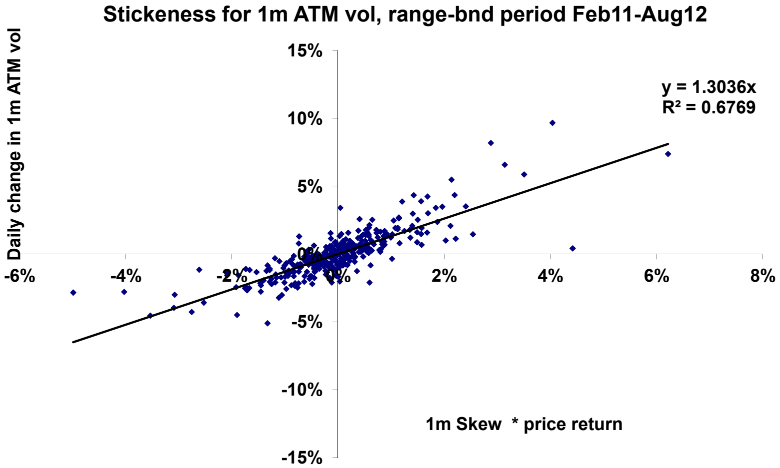


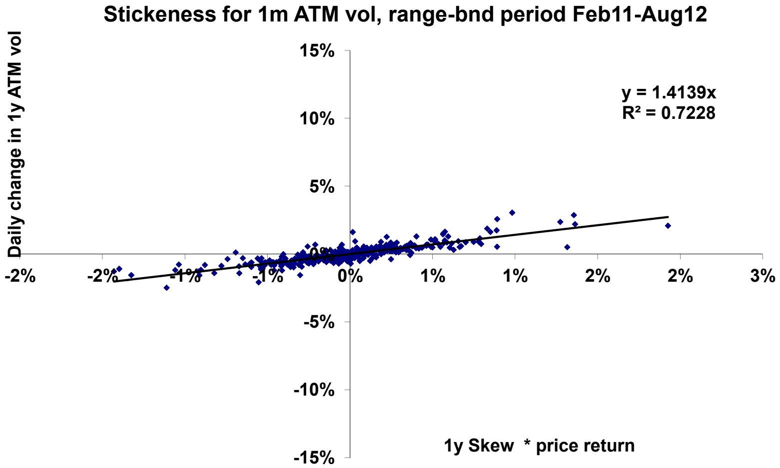
Stickiness ratio (recovery) for 1m and 1y ATM vols





Stickiness ratio (range) for 1m and 1y ATM vols





Stickiness ratio V. Conclusions

Summary of the regression model:

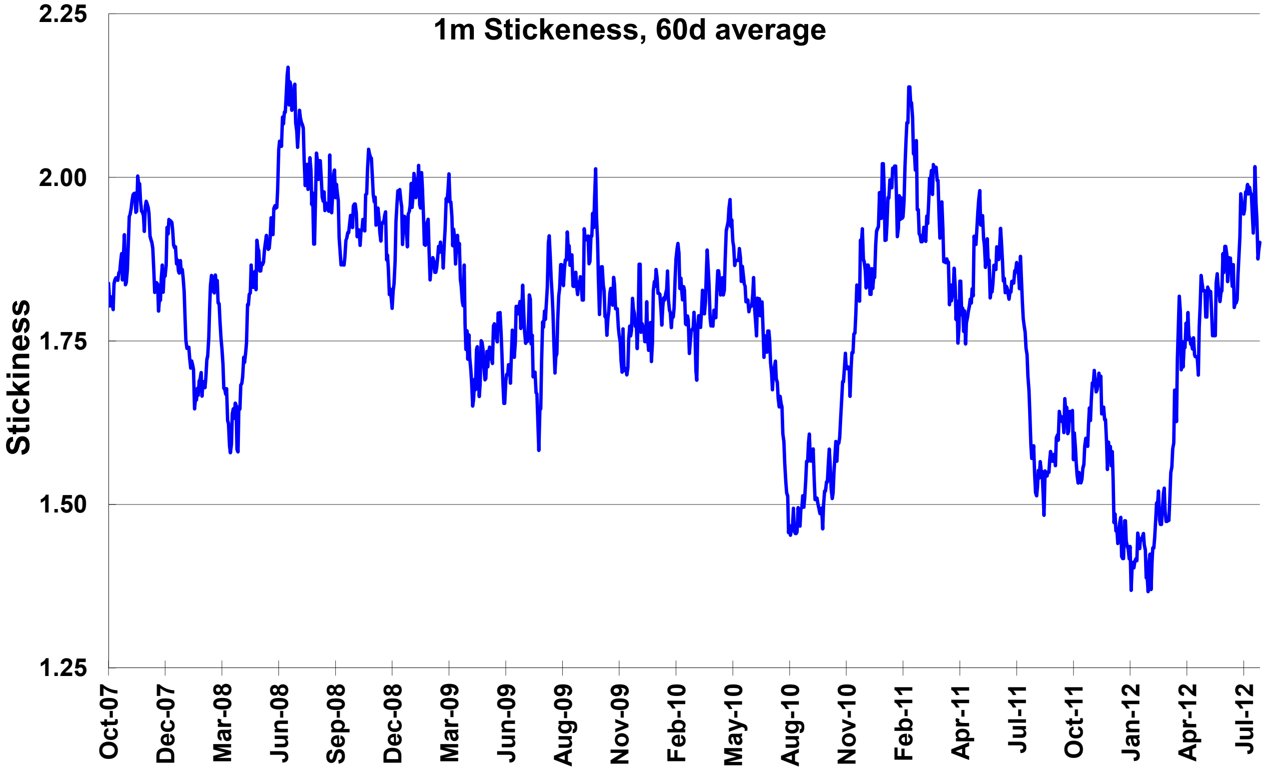
	crisis	recovery	range-bound
Stickiness, 1m	1.63	1.46	1.30
Stickiness, 1y	1.60	1.56	1.41
R^2, 1m	77%	65%	68%
R^2, 1y	82%	68%	72%

1) The concept of the stickiness is **statistically significant** explaining about 80% of the variation in ATM volatility during crisis period and about 70% of the variation during recovery and range-bound periods

2) Stickiness ratio is
stronger during crisis period, $\overline{R} \approx 1.6$ (closer to sticky local vol)
less strong during recovery period, $\overline{R} \approx 1.5$
weaker during range-bound period, $\overline{R} \approx 1.35$ (closer to sticky-strike)

3) The volatility regime is typically neither sticky-local nor sticky-strike but rather a combination of both

Stickiness ratio VI. Time series



Stickiness ratio VII. Dynamic models A
Now we consider how to model the stickiness ratio within the dynamic SV models

The primary driver is change in the spot price, $\Delta S/S$

The key in this analysis is what happens to the level of model volatility given change in the spot price (for a very nice discussion see "A Note on Hedging with Local and Stochastic Volatility Models" by Mercurio-Morini, on ssrn.com)

The model-consistent hedge:
The level of volatility changes by (approximately): Skew $\times$ $\Delta S/S$

The model-inconsistent hedge:
The level of volatility remains unchanged

Implication for the stickiness under pure SV models:
$\overline{R} = 2$ under the model-consistent hedge
$\overline{R} = 0$ under the model-inconsistent hedge

Stickiness ratio VII. Dynamic models B
How to make $\overline{R} = 1.5$ using SV models?
Under the model-consistent hedge: impossible?
Under the model-inconsistent hedge: mix SV with local volatility

Remedy: add jump process
Under any spot-homogeneous jump model, $\overline{R} = 0$

The only way to have a model-consistent hedging that fits the desired stickiness ratio is to **mix stochastic volatility with jumps**:
the higher is the stickiness ratio, the lower is the jump premium
the lower is the stickiness ratio, the higher is the jump premium

Jump premium is lower during crisis periods (after a big crash or excessive market panic, the probability of a second one is lower because of realized de-leveraging and de-risking of investment portfolios, central banks interventions)

Jump premium is higher during recovery and range-bound periods (renewed fear of tail events, increased leverage and risk-taking given small levels of realized volatility and related hedging)

Stickiness ratio VII. Dynamic models C
The above consideration explain that the stickiness ratio is stronger during crisis period, $\overline{R} \approx 1.6$ (closer to sticky local vol) weaker during range-bound and recovery periods, $\overline{R} \approx 1.35$ (closer to sticky-strike)

To model this feature within an SV model, we need to specify a proportion of the skew attributed to jumps (see my 2011 presentation for Risk Quant congress and 2012 presentation for Global derivatives)

During crisis periods, the weight of jumps is about 20%
During range-bound and recovery periods, the weight of jumps is about 40%

Beta stochastic volatility I

First, I present a simplified version of the beta stochastic volatility model introduced in Karasinski and Sepp (2012) with no mean-reversion and volatility-of-volatility:

$$
\begin{aligned}
\frac{dS(t)}{S(t)} &= \sigma(t)(S(t))^{\beta_S} dW(t),\ S(0) = S_0 \\
d\sigma(t) &= \beta_V \frac{dS(t)}{S(t)},\ \sigma(0) = \sigma_0
\end{aligned} \tag{1}
$$

where $S(t)$ is the spot price
$\sigma(t)$ is instantaneous volatility and σ_0 is initial level of ATM volatility
$W(t)$ is a Brownian motion - the only source of randomness

To produce the volatility skew and the dependence between the price and implied volatility, the model relies on the two parameters:
β_S is the backbone beta
β_V is the volatility beta

Estimates of β_S and β_V are easily inferred from implied/historical data

Volatility beta

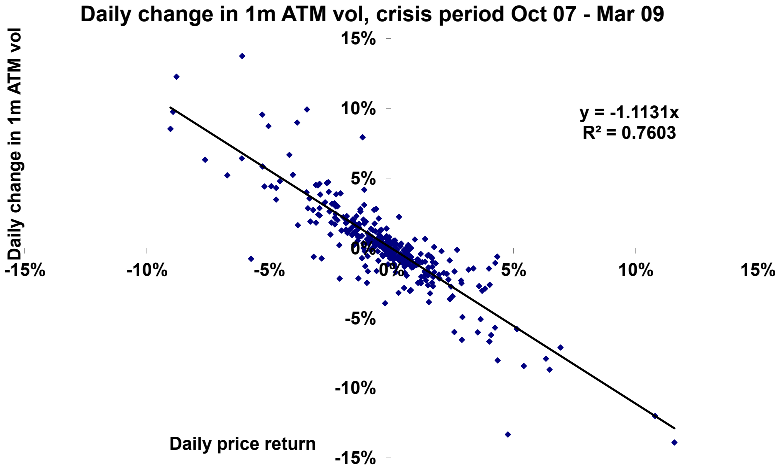
We replicate $\sigma(t)$ by short-term ATM volatility, $\sigma(t) = \sigma_{ATM}(S(t))$ to estimate model parameters by the regression model

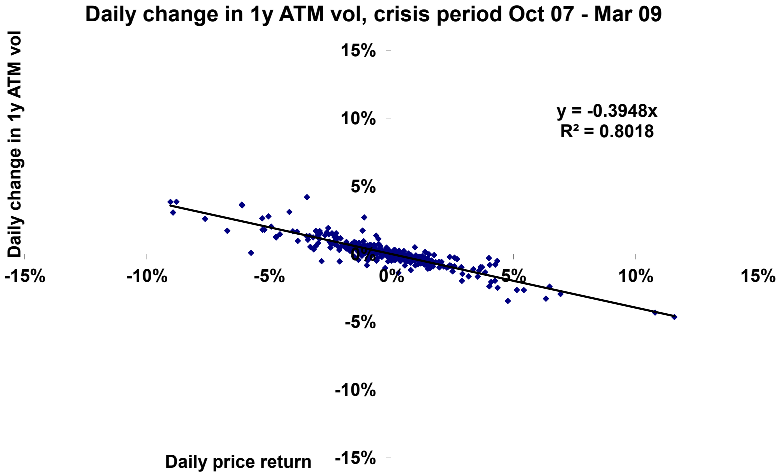
Volatility beta β_V is a measure of linear dependence between daily returns and changes in the ATM volatility:

$$\sigma_{ATM}(S(t_n)) - \sigma_{ATM}(S(t_{n-1})) = \beta_V \frac{S(t_n) - S(t_{n-1})}{S(t_{n-1})}$$

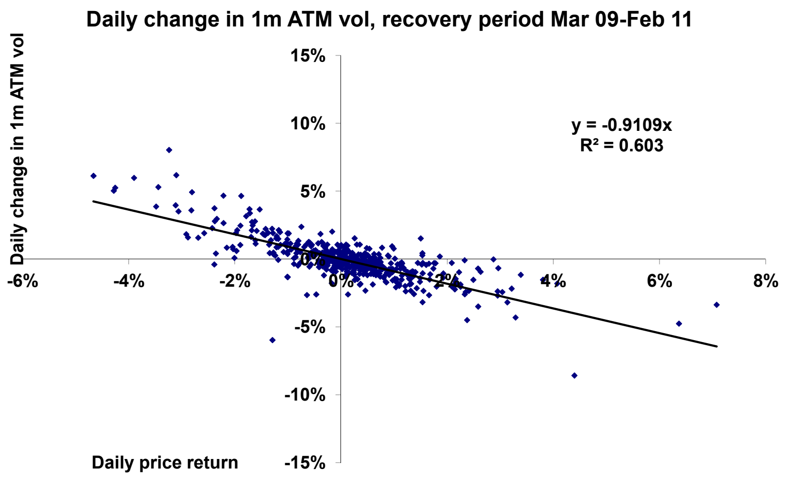
Next we examine this regression model empirically

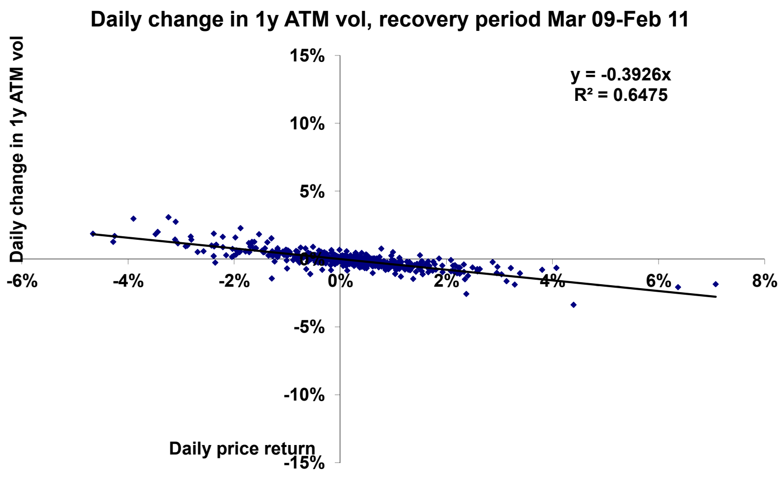
Volatility beta (crisis) for 1m and 1y ATM vols



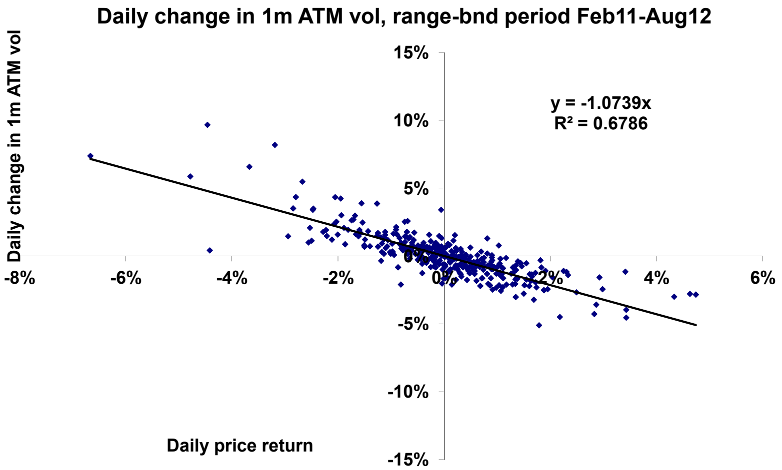


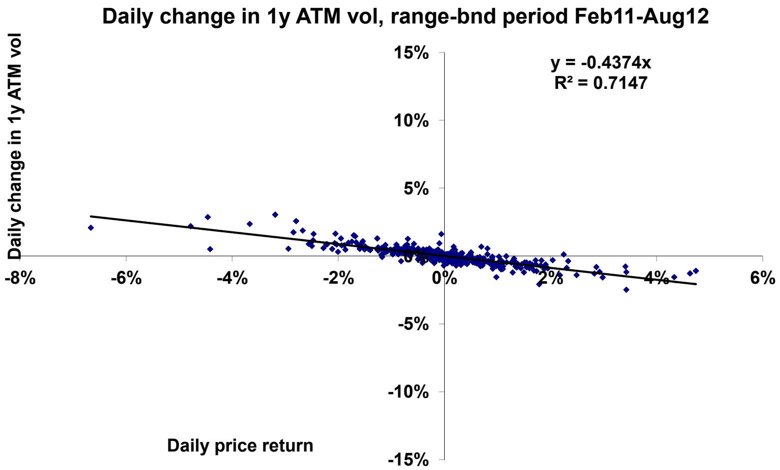
Volatility beta (recovery) for 1m and 1y ATM vols





Volatility beta (range) for 1m and 1y ATM vols





Volatility beta. Summary

	crisis	recovery	range-bound
Volatility beta 1m	-1.11	-0.91	-1.07
Volatility beta 1y	-0.39	-0.39	-0.44
R^2 1m	76%	60%	68%
R^2 1y	80%	65%	71%

The volatility beta is pretty stable across different market regimes

The longer term ATM volatility is less sensitive to changes in the spot

Changes in the spot price explain about:
80% in changes in the ATM volatility during crisis period
60% in changes in the ATM volatility during recovery period (ATM volatility reacts slower to increases in the spot price)
70% in changes in the ATM volatility during range-bound period (jump premium start to play bigger role n recovery and range-bound periods)

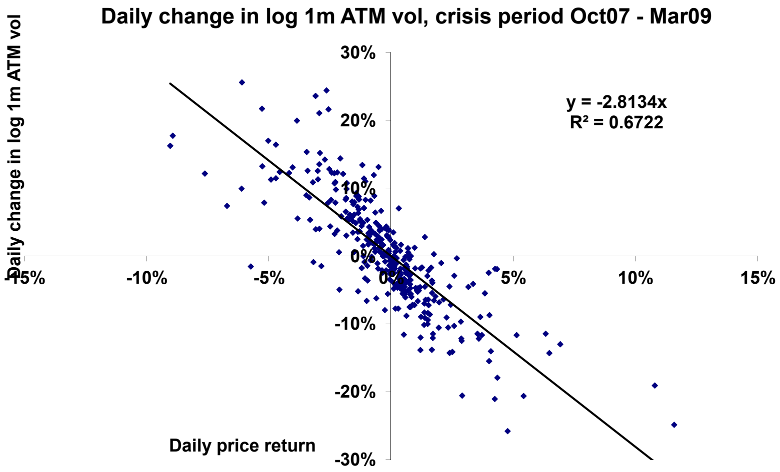
The backbone beta

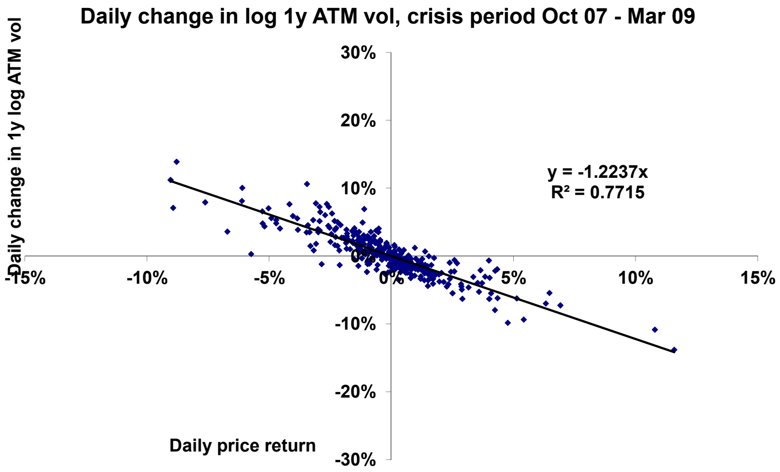
The backbone beta β_S is a measure of daily changes in the logarithm of the ATM volatility to daily returns on the stock

$$\ln\left[\sigma_{ATM}(S(t_n))\right] - \ln\left[\sigma_{ATM}(S(t_{n-1}))\right] = \beta_S \frac{S(t_n) - S(t_{n-1})}{S(t_{n-1})}$$

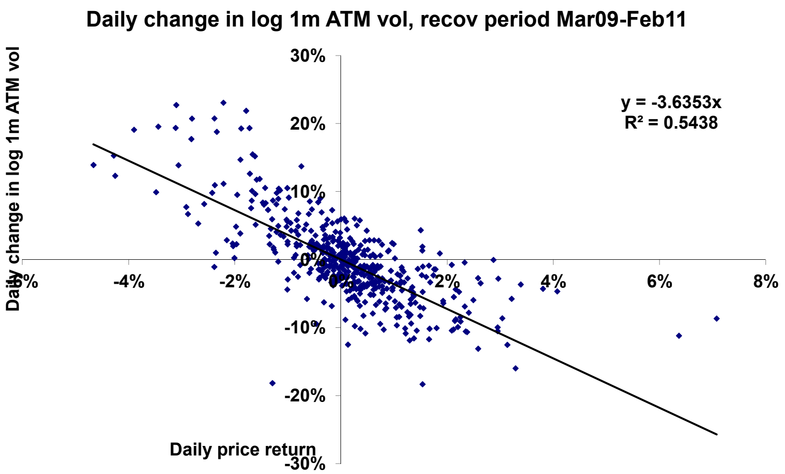
Next we examine this regression model empirically

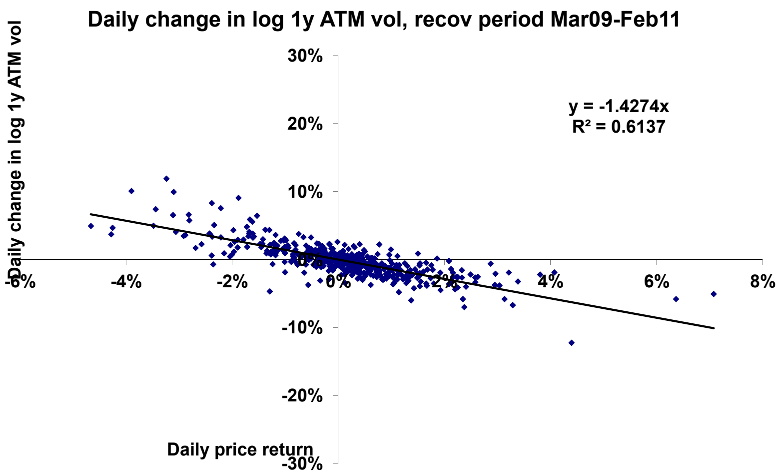
Backbone beta (crisis) for 1m and 1y ATM vols



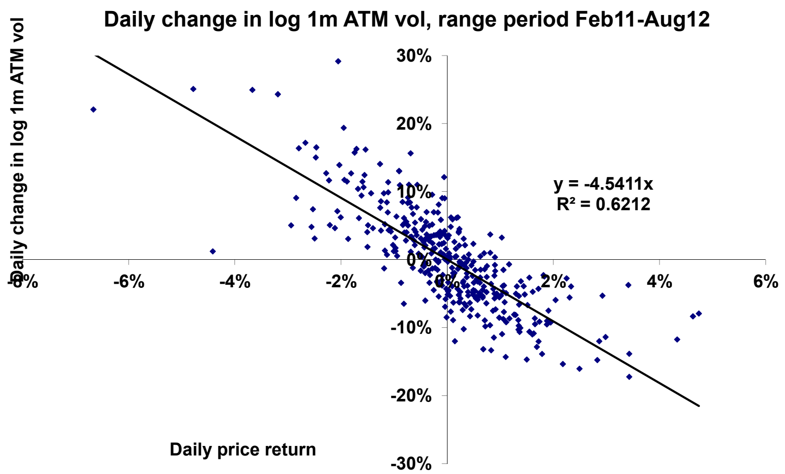


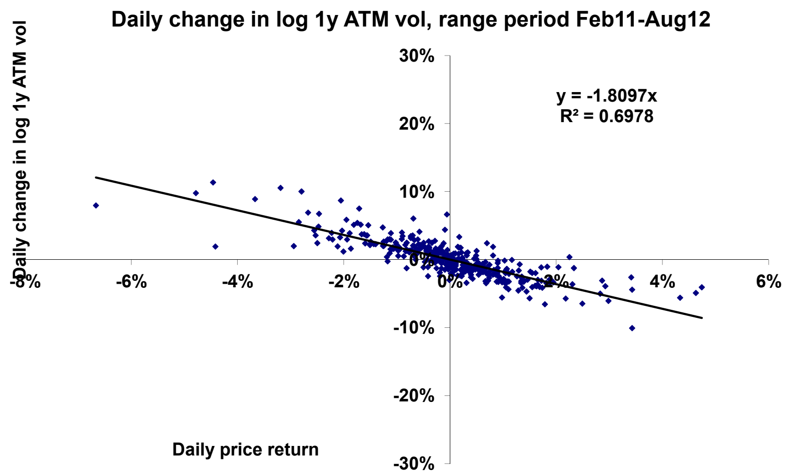
Backbone beta (recovery) for 1m and 1y ATM vols





Backbone beta (range) for 1m and 1y ATM vols





The backbone beta. Summary A

	crisis	recovery	range-bound
Backbone beta 1m	-2.81	-3.64	-4.54
Backbone beta 1y	-1.22	-1.43	-1.81
R^2 1m	67%	54%	62%
R^2 1y	77%	61%	70%

The value of the backbone beta appears to be less stable across different market regimes (compared to volatility beta)

Explanatory power is somewhat less (by 5-7%) for 1m ATM vols (compared to volatility beta)

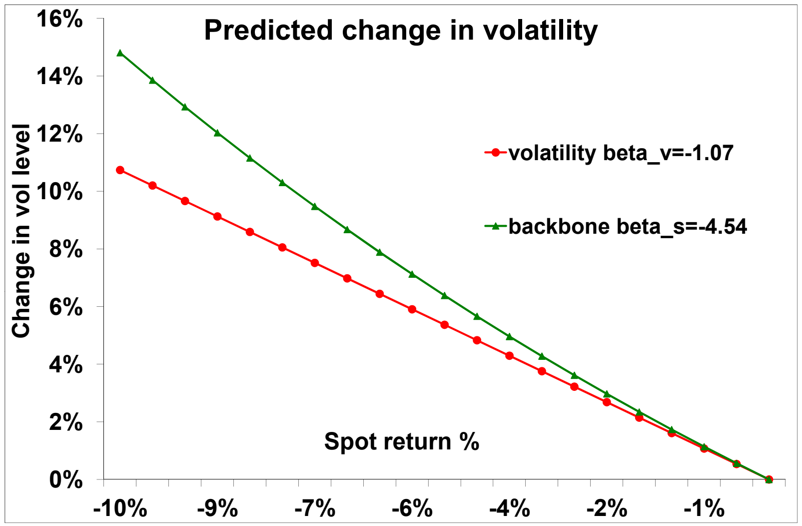
Similar explanatory power for 1y volatilities

The backbone beta. Summary B

Change in the level of the ATM volatility implied by backbone beta β_S is proportional to initial value of the ATM volatility

High negative value of β_S implies a big spike in volatility given a modest drop in the price - a feature of sticky local volatility model

In the figure, using estimated parameters $\beta_V = -1.07$, $\beta_S - 4.54$ in range-bound period, $\sigma(0) = 20\%$



Connection to the SABR model (Hagan et al (2002))

Model parameters are related to the SABR model as follows:

$$\hat{a} = \sigma_0 \ , \ \rho = -1 \ , \ \nu = -\beta_V \ , \ \beta = \beta_S + 1$$

Using formula (3.1a) in Hagan *et al* (2002) for a short maturity and small log-moneynes k, $k = \ln(K/S_0)$, we obtain the following relationship for the BSM implied volatility $\sigma_{IMP}(k)$:

$$\sigma_{IMP}(k) = \frac{\sigma_0}{S^{\beta_S}} \left[1 + \frac{1}{2} \left(\beta_S + \frac{\beta_V}{\sigma_0} \right) k + \frac{1}{12} \left(\beta_S^2 - \left(\frac{\beta_V}{\sigma_0} \right)^2 \right) k^2 \right]$$

Thus, in a simple case, the model can be directly linked to the implied volatility interpolator represented by the SABR model

Model implied skew

We obtain the following approximate but accurate relationship between the model parameters and short-term implied ATM volatility, $\sigma_{ATM}(S)$, and skew Skew_α:

$$\sigma_0 S^{\beta_S} = \sigma_{ATM}(S)$$

$$\beta_S + \frac{\beta_V}{\sigma_0} = \frac{2\text{Skew}_\alpha}{\sigma_{ATM}(S)} \equiv \Lambda$$

The first equation is known as the backbone that defines the trajectory of the ATM volatility given a change in the spot price:

$$\frac{\sigma_{ATM}(S) - \sigma_{ATM}(S_0)}{\sigma_{ATM}(S_0)} \approx \beta_S \frac{S - S_0}{S_0} \quad (2)$$

Model implied stickiness and volatility regimes
If we insist on model-inconsistent delta (change in spot with volatility level unchanged):
fit backbone beta β_S to reproduce specified stickiness ratio
adjust β_V so that the model fits the market skew

Using stickiness ratio $R(t_n)$ along with (2), we obtain that empirically:

$$\beta_S(t_n) = \frac{\text{Skew}_\alpha(t_{n-1})}{\sigma_{ATM}(t_{n-1})} R(t_n)$$

Thus, given an estimated value of the stickiness rate we imply β_S

Finally, by mixing parameters β_S and β_V we can produce different volatility regimes:
sticky-delta with $\beta_S = 0$ and $\beta_V \approx 2\text{Skew}_\alpha$
sticky-local volatility with $\beta_V = 0$ and $\beta_S \approx \Lambda$

From the empirical data we infer that, approximately,
$\beta_S \approx 70\%\Lambda$ and $\beta_V = 30\% \times 2\text{Skew}_\alpha$

Beta stochastic volatility model
Let me consider pure SV beta expressed in terms of normalized volatility factor $Y(t)$ (this version is applied in practice for beta SV with local volatility):

$$\begin{aligned} \frac{dS(t)}{S(t)} &= (1+Y(t))\sigma dW(t),\ S(0)=S_0 \\ dY(t) &= \tilde{\beta}_V \frac{dS(t)}{S(t)},\ Y(0)=0 \end{aligned} \tag{3}$$

where σ is the overall level of the volatility (can be deterministic or local $\sigma(t,S)$)
$Y(t)$ is the normalized volatility factor fluctuation around zero

Volatility parameter $\tilde{\beta}_V$ can be implied from short term ATM volatility σ_{ATM} and skew Skew_α:

$$\tilde{\beta}_V = \frac{2\text{Skew}_\alpha}{\sigma_{ATM}} \tag{4}$$

The goal now is to investigate the dynamics of the skew

Beta stochastic volatility model. Skew

Inverting the above equation:

$$\text{Skew}(t) = \frac{1}{2}\tilde{\beta}_V \sigma_{ATM}(t) \tag{5}$$

Dynamically, using (4):

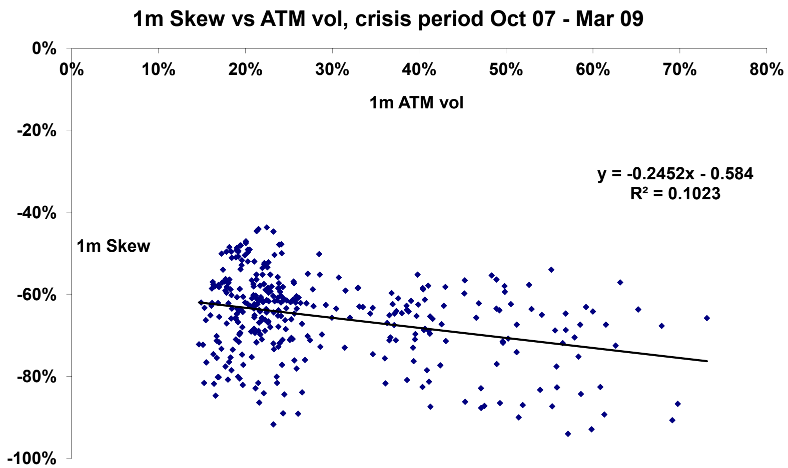
$$\begin{aligned} d\text{Skew}(t) &= \frac{1}{2}\tilde{\beta}_V d\sigma_{ATM}(t) \propto \frac{1}{2}\tilde{\beta}_V \sigma_{ATM}(t) dY(t) \\ &\propto \frac{1}{2}\sigma_{ATM}(t)\left(\tilde{\beta}_V\right)^2 \frac{dS(t)}{S(t)} \end{aligned}$$

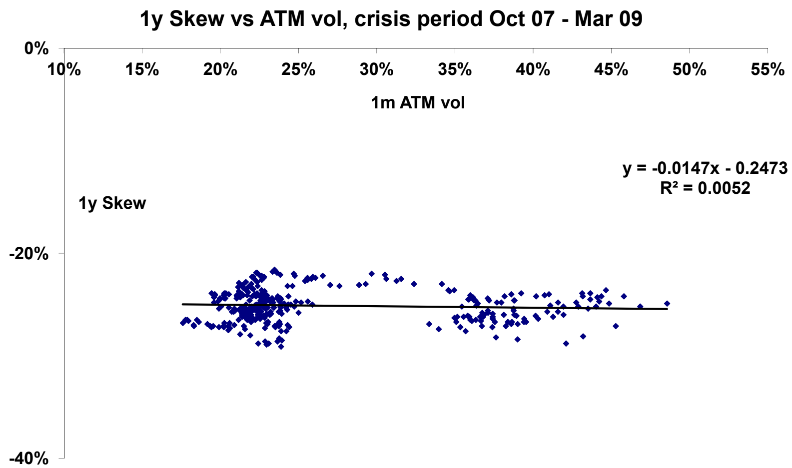
To test the above equation empirically, we apply the regression model for coefficient q:

$$\text{Skew}(t_n) - \text{Skew}(t_{n-1}) = q\left[2\left(\text{Skew}(t_{n-1})\right)^2 \frac{1}{\sigma_{ATM}(t_{n-1})}\frac{S(t_n) - S(t_{n-1})}{S(t_{n-1})}\right] \tag{6}$$

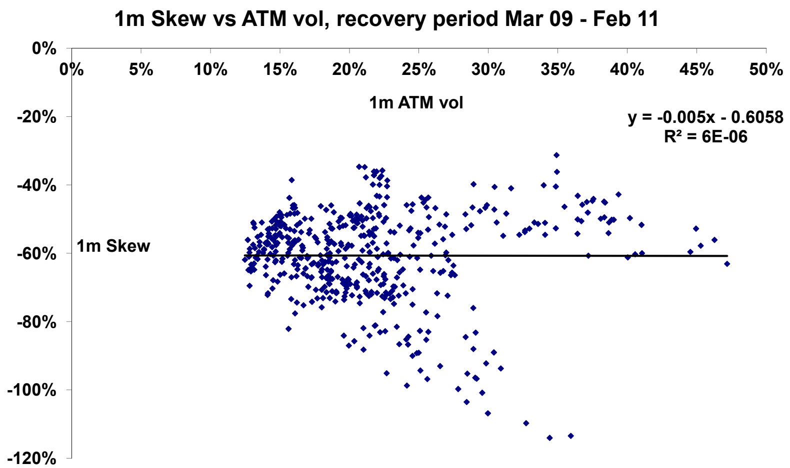
First, we test (5)

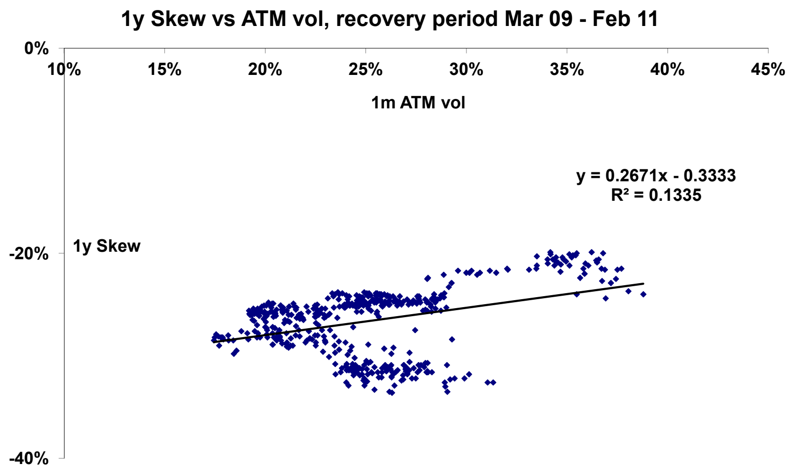
Skew vs ATM volatility (crisis) for 1m and 1y ATM vols



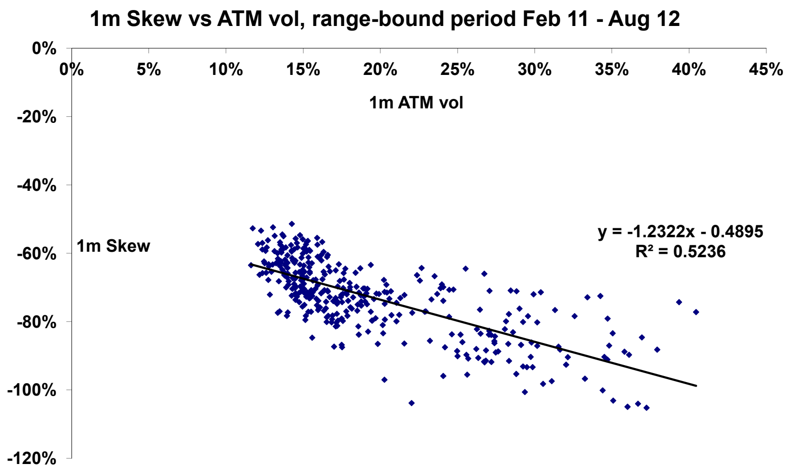


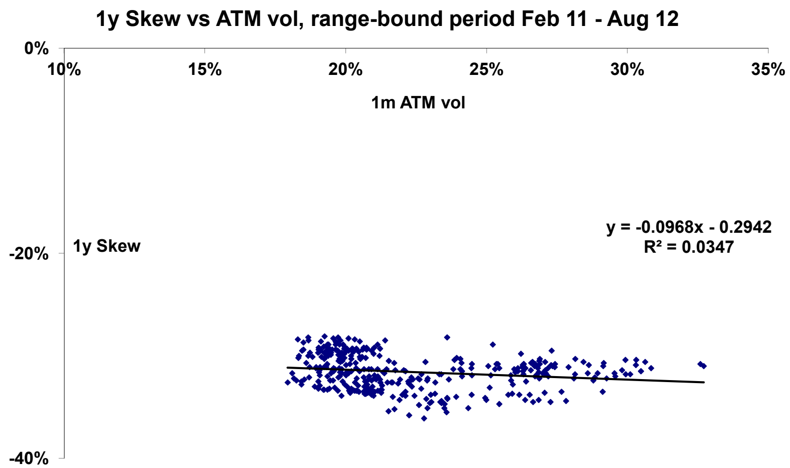
Skew vs ATM volatility (recovery) for 1m and 1y ATM vols





Skew vs ATM volatility (range) for 1m and 1y ATM vols





Skew vs ATM volatility. Summary

	crisis	recovery	range-bound
Skew-vol beta 1m	-0.25	-0.01	-1.23
Skew-vol beta 1y	-0.01	-0.27	-0.10
R^2 1m	10%	0%	52%
R^2 1y	1%	13%	3%

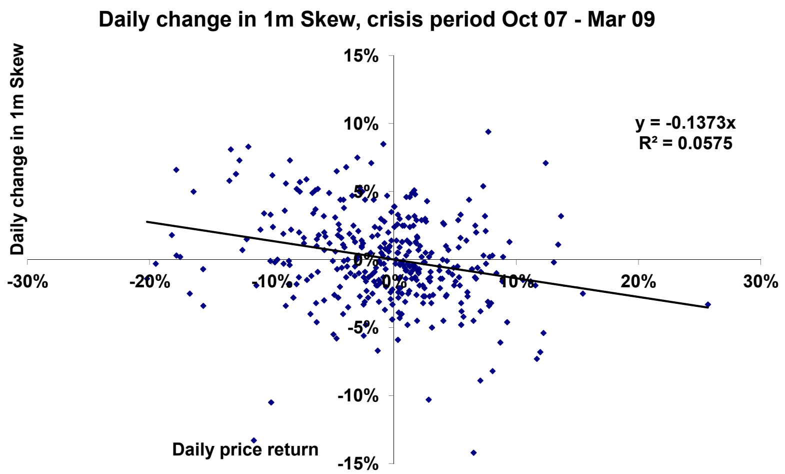
Empirically, in general, a high level of ATM volatility implies a higher level of the skew but the relationship is not strong and is mixed

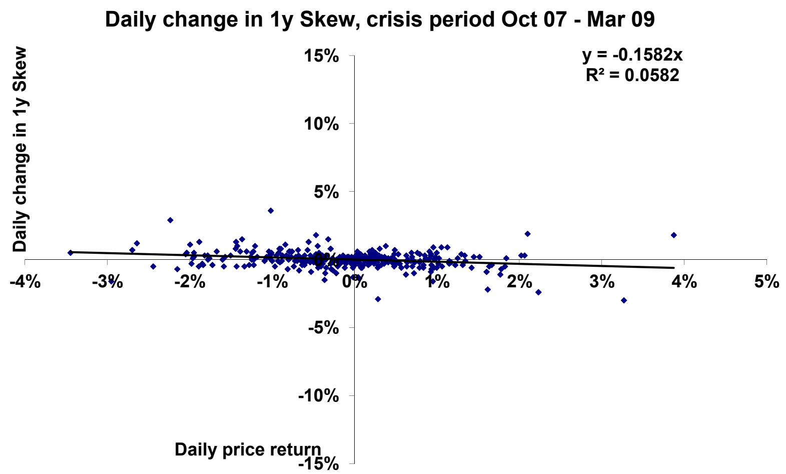
For short-term skew, the relationship is stronger in crisis and range-bound periods

For longer-term skew, the relationship is stronger in recovery periods

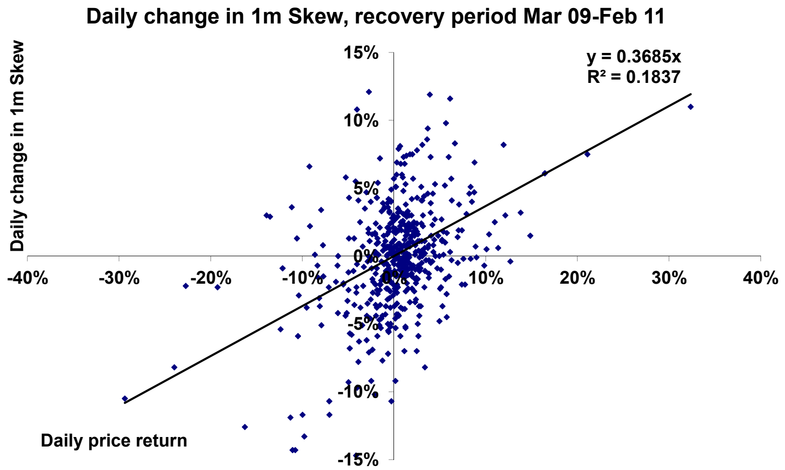
Next, we test (6) for relationship between changes in the skew and spot returns

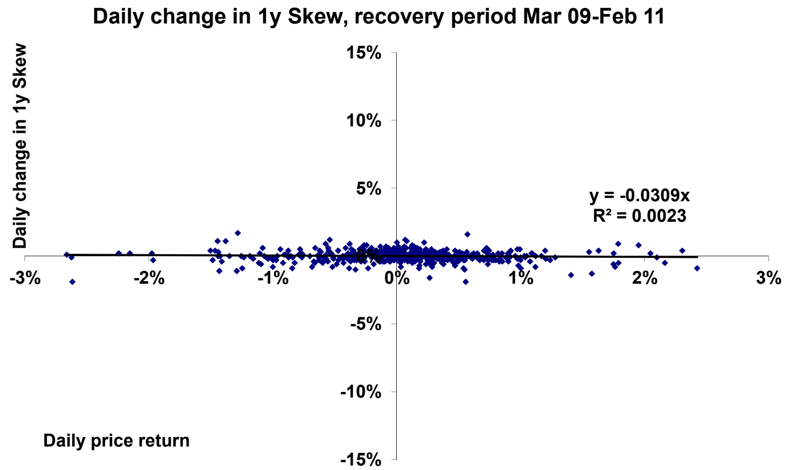
Skew vs price return (crisis) for 1m and 1y skews



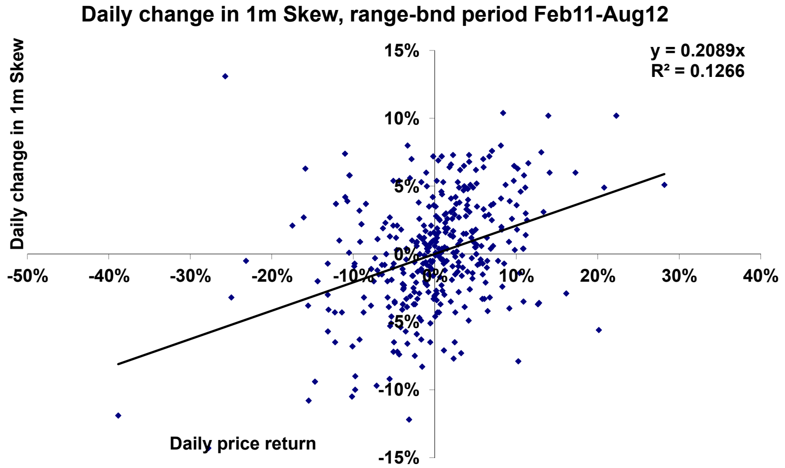


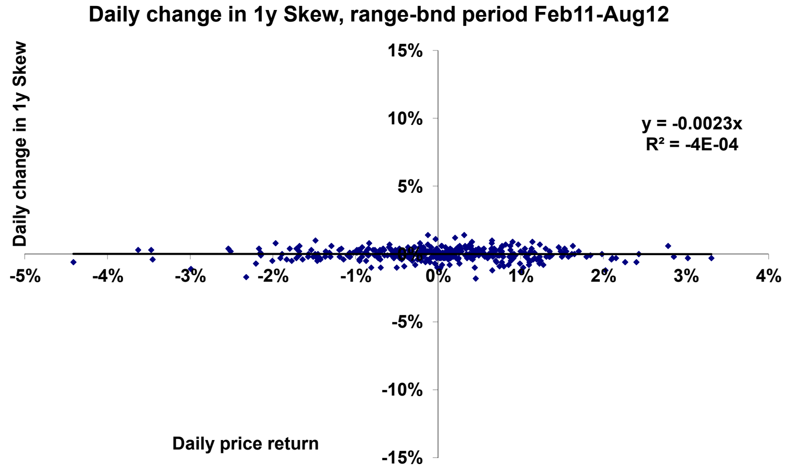
Skew vs price return (recovery) for 1m and 1y skews





Skew vs price return (range) for 1m and 1y skews





Skew vs price return. Summary

	crisis	recovery	range-bound
Skew-return beta 1m	-0.14	0.37	0.21
Skew-return beta 1y	-0.16	-0.03	0.00
R^2 1m	6%	18%	13%
R^2 1y	6%	0%	0%

The short-term skew appears to be somewhat dependent on spot changes:
during crisis periods, negative returns decrease the skew (de-leveraging reduces need for downside protection)
during recovery and range-bound periods, negative returns increase the skew (risk-aversion is high especially during recovery period)

The long-term skew does not appear to depend on spot returns

Skew vs price return. Conclusions

The skew does not seem to depend on either volatility or spot dynamics (especially for longer maturities)

About 20% of variations in the short-term skew can be attributed to changes in the spot

Only jumps appear to have a reasonable explanation for the skew (the fear of a crash does not (or little) depend on current values of variables)

Full beta stochastic volatility model I

The pricing version of the beta model is specified as follows:

$$\frac{dS(t)}{S(t)} = \mu(t)dt + (1 + Y(t))\sigma dW^{(0)}(t), \ S(0) = S$$
$$dY(t) = -\kappa Y(t)dt + \beta_V(1 + Y(t))\sigma dW^{(0)}(t) + \varepsilon dW^{(1)}(t), \ Y(0) = 0 \quad (7)$$

where:

β_V ($\beta_V < 0$) is the rate of change in the volatility corresponding to change in the spot price

ε is idiosyncratic volatility of volatility

κ is the mean-reversion rate

$W^{(0)}(t)$ and $W^{(1)}(t)$ are two Brownians with $dW^{(0)}(t)dW^{(1)}(t) = 0$

$\mu(t)$ is the risk-neutral drift

σ is the overall level of volatility

σ is set to either constant volatility σ_{CV} or deterministic volatility $\sigma_{DV}(t)$, or local stochastic volatility $\sigma_{LSV}(t, S)$

$\sigma = \{\sigma_{CV}, \sigma_{DV}(t), \sigma_{LSV}(t, S)\}$

Beta stochastic volatility model. II

Using dynamics (7), for the log-spot, $X(t) = \ln\left(\frac{S(t)}{S(0)}\right)$ we obtain:

$$
\begin{aligned}
dX(t) &= \mu(t)dt - \frac{1}{2}\sigma^2(1+Y(t))^2 dt + \sigma(1+Y(t))dW^{(0)}(t),\ X(0)=0 \\
dY(t) &= \beta\sigma(1+Y(t))dW^{(0)}(t) - \kappa Y(t)dt + \varepsilon dW^{(1)}(t),\ Y(0)=0
\end{aligned}
\tag{8}
$$

with

$$
\begin{aligned}
dY(t)dY(t) &= \left(\varepsilon^2 + \beta^2\sigma^2(1+Y(t))^2\right)dt \\
dX(t)dY(t) &= \beta\sigma^2(1+Y(t))^2 dt
\end{aligned}
$$

The pricing equation for value function $U(t,T,X,Y)$ has the form:

$$
\begin{aligned}
&U_t + \frac{1}{2}\sigma^2(1+2Y+Y^2)\left[U_{XX} - U_X\right] + \mu(t)U_X \\
&+ \frac{1}{2}\left(\varepsilon^2 + \beta^2\sigma^2\left(1+2Y+Y^2\right)\right)U_{YY} - \kappa Y U_Y \\
&+ \beta\sigma^2\left(1+2Y+Y^2\right)U_{XY} - r(t)U = 0
\end{aligned}
\tag{9}
$$

where $r(t)$ is the discount rate and subscripts denote partial derivatives

Beta stochastic volatility model. III
The parameters of the stochastic volatility, β, ε and κ are specified before the calibration

We calibrate the local volatility $\sigma \equiv \sigma_{LSV}(t,S)$, using either a parametric local volatility (CEV) or non-parametric local volatility, so that the vanilla surface is matched by construction

For calibration of $\sigma_{LSV}(t,S)$ we apply the conditional expectation (Lipton A, The vol smile problem, *Risk, February* 2002):

$$\sigma^2_{LSV}(T,K)\mathbb{E}\left[(1+Y(T))^2 \,|\, S(T)=K\right] = \sigma^2_{LV}(T,K)$$

where $\sigma^2_{LV}(T,K)$ is the local Dupire volatility

The above expectation is computed by solving the forward PDE corresponding to pricing PDE (9) using finite-difference methods and computing $\sigma^2_{LSV}(T,K)$ stepping forward in time

Once $\sigma_{LSV}(t,S)$ is calibrated we use either backward PDE-s or MC simulation for valuation of exotic options

Beta SV model. Approximation for call price I

I propose an affine approximation for pricing equation (9) with constant or deterministic volatility σ:

$$G(t, T, X, Y; \Phi) = \exp\left\{-\Phi X + A^{(0)} + A^{(1)}Y + A^{(2)}Y^2\right\}$$

with $A^{(n)}(T;T) = 0$, $n = 0, 1, 2$

By substitution this into PDE (9) and collecting terms proportional to Y and Y^2 only, we obtain a system of ODE-s for $A^{(n)}(t)$:

$$A_t^{(0)} + v_0 A^{(2)} + \frac{1}{2} v_0 (A^{(1)})^2 - \Phi A^{(1)} c_0 + \frac{1}{2} q = 0$$

$$A_t^{(1)} + \frac{1}{2} v_1 (A^{(1)})^2 + 2 v_0 A^{(1)} A^{(2)} + v_1 A^{(2)} - \kappa A^{(1)} - \Phi\left(2 c_0 A^{(2)} + c_1 A^{(1)}\right) + q = 0$$

$$A_t^{(2)} + \frac{1}{2} v_2 (A^{(1)})^2 + 2 v_0 (A^{(2)})^2 + 2 v_1 A^{(1)} A^{(2)} + v_2 A^{(2)} - 2\kappa A^{(2)} - \Phi\left(2 c_1 A^{(2)} + c_2 A^{(1)}\right) + \frac{1}{2} q =$$

where

$$q = \sigma^2\left(\Phi^2 + \Phi\right), \; v_0 = \varepsilon^2 + \beta^2\sigma^2, \; v_1 = 2\beta^2\sigma^2, \; v_2 = \beta^2\sigma^2, \; c_0 = \beta\sigma^2, \; c_1 = 2\beta\sigma^2, \; c_2 = \beta\sigma^2$$

This is system is solved by means of Runge-Kutta methods

It is straightforward to incorporate time-dependent model parameters (but not space-dependent local volatility $\sigma_{LSV}(t, X)$)

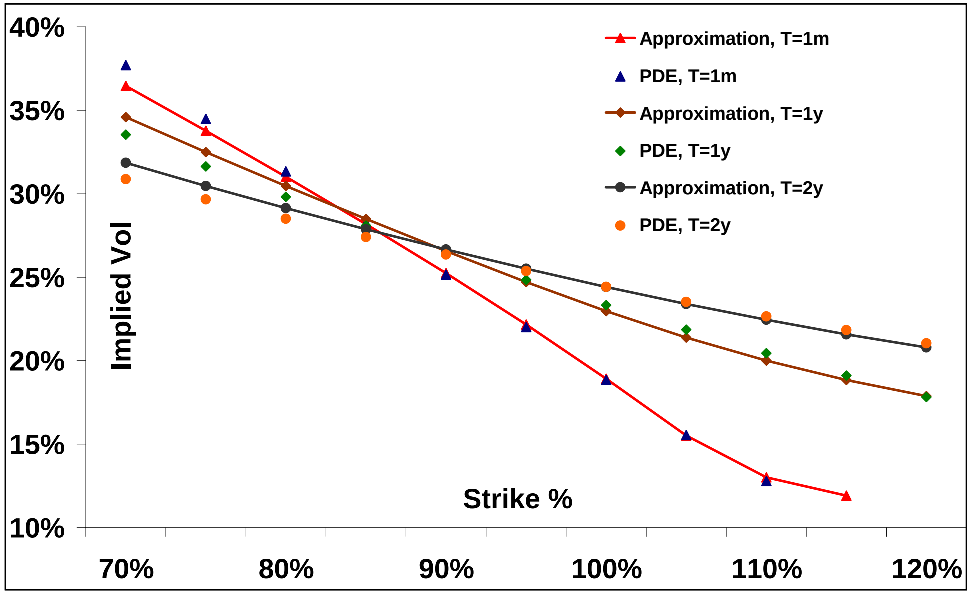
Beta SV model. Approximation for call price II
As a result, for pricing vanilla options, we can apply the standard methods based on the Fourier inversion

The value of the call option with strike K is computed by applying Lipton-Lewis formula:

$$C(t,T,S,Y) = e^{-\int_t^T r(t')dt'}\left(e^{\int_t^T \mu(t')dt'}S - \frac{K}{\pi}\int_0^\infty \Re\left[\frac{G(t,T,x,Y;ik-1/2)}{k^2+1/4}\right]dk\right)$$

where $x = \ln(S/K) + \int_t^T \mu(t')dt'$

Beta SV model. Approximation for call price III



Implied model volatilities computed by approximation formula vs numerical PDE using $\beta = -7.63$, $\varepsilon = 0.35$, $\kappa = 4.32$ and constant volatilities: $\sigma_{CV}(1m) = 19.14\%$, $\sigma_{CV}(1y) = 23.96\%$, $\sigma_{CV}(2y) = 24.21\%$

Properties of the volatility process, assuming constant vol σ_{CV}
Instantaneous variance of $Y(t)$ is given by:

$$dY(t)dY(t) = \left(\beta^2\sigma_{CV}^2(1+Y(t))^2 + \varepsilon^2\right) dt$$

which has systemic part proportional to $Y(t)$ and idiosyncratic part ε

In a stress regime, for large values of $Y(t)$, the variance is dominated by $\beta^2\sigma_{CV}^2 Y^2(t)$ (close to a log-normal model for volatility process)

The volatility process has **steady-state variance** (so that the volatility approaches stationary distribution in the long run):

$$\mathbb{E}\left[Y^2(t)\,|\,Y(0)=0\right] = \frac{\varepsilon^2+\beta^2\sigma_{CV}^2}{2\kappa-\beta^2\sigma_{CV}^2}\left(1-e^{-(2\kappa-\beta^2\sigma_{CV}^2)t}\right)$$

Effective mean-reversion for the volatility of variance is:

$$2\kappa-\beta^2\sigma_{CV}^2$$

Steady state variance of volatility is

$$\frac{\varepsilon^2+\beta^2\sigma_{CV}^2}{2\kappa-\beta^2\sigma_{CV}^2}$$

Instantaneous correlation between $dY(t)$ and $dX(t)$:

$$\rho(dX(t)dY(t)) = \frac{\beta\sigma_{CV}^2(1+Y(t))^2}{\sqrt{\left(\varepsilon^2+\beta^2\sigma_{CV}^2(1+Y(t))^2\right)}\sqrt{\sigma_{CV}^2(1+Y(t))^2}}$$

With high volatility $Y(t)$ is large so letting $Y(t)\to\infty$ we obtain that

$$\rho(dX(t)dY(t))|_{Y(t)\approx\infty} = -1$$

In a normal regime, $Y(t)\approx 0$, so that obtain:

$$\rho(dX(t)dY(t))|_{Y(t)\approx 0} = -\frac{1}{\sqrt{\left(\frac{\varepsilon^2}{\beta^2\sigma_{CV}^2}+1\right)}}$$

The beta SV model introduces **state-dependent spot-volatility correlation**, with high volatility leading to absolute negative correlation

In contrast, Heston and Ornstein-Uhlenbeck based SV models always assume constant instantaneous correlation

The steady state density

Steady state density function $G(Y)$ of volatility factor $Y(t)$ in dynamics (7) solves the following equation:

$$\frac{1}{2}\left[\left(\varepsilon^2+\beta^2\sigma_{CV}^2\left(1+2Y+Y^2\right)\right)G\right]_{YY}+\left[\kappa Y G\right]_Y=0$$

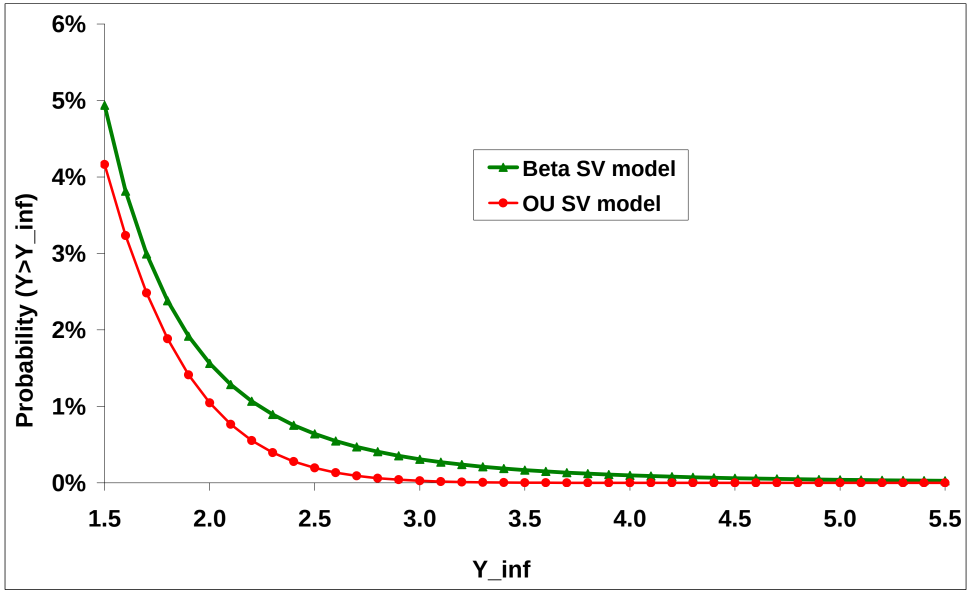
We can show that $G(Y)$ exhibits the **power-like behavior** for large values of Y:

$$\lim_{Y\to+\infty} G(Y)=Y^{-\alpha}\ ,\ \ \alpha=2\left(1+\frac{\kappa}{(\beta\sigma_{CV})^2}\right)$$

This power-like behavior contrasts with Heston and exponential volatility models which imply exponential tails for the steady-state density of the volatility

Thus, the beta SV model predicts higher probabilities of large values of instantaneous volatility

The steady state density. Tails



Tails in the beta SV model (green) vs Ornstein-Uhlenbeck (OU) exponential SV model (red) with $\alpha = 5$ in beta SV model and equivalent vol-of-vol in OU SV model $\epsilon_{OU} = 0.61$

Case Study I: Correlation skew
Apply experiment:
1) Compute implied volatilities from options on the index (say, S&P500)
2) Using implied volatilities (probability density function) of stocks in this index, and stock-stock Gaussian correlations, compute option prices on the index, compute the implied volatility from these prices

Empirical observation (**correlation skew**):
The index skew computed in 1) is steeper than that computed in 2)
Explanation:
Stocks become strongly correlated during big sell-offs
Index skew reflects premium for buying puts on a basket of stocks

Modelling approach:
Correlation skew cannot be replicated using Gaussian correlation
Stochastic and/or local correlations can be applied
But only **SV model with jumps can produce realistic dynamics and reproduce the correlation skew**

Next we augment the beta SV model with jumps and apply it to reproduce the correlation skew

SPY and select sector ETF-s

Consider:

SPY - the ETF tracking the S&P500 index

Select sector ETF-s - ETF-s tracking 9 sectors of the S&P500 index

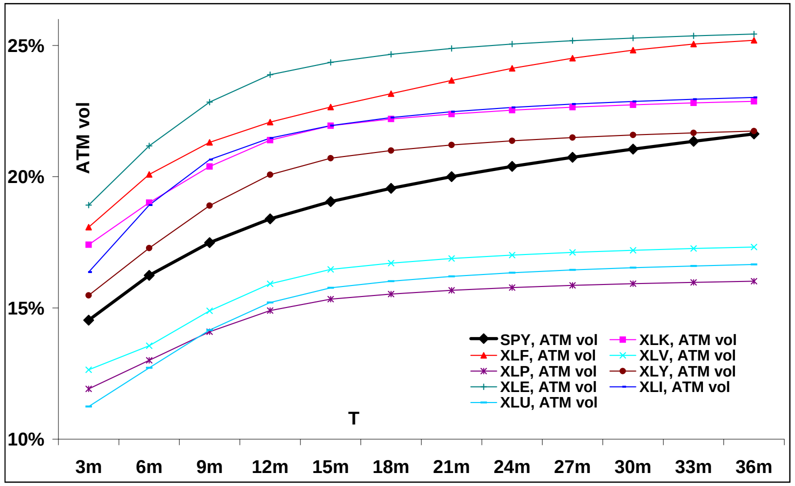
	ETF	SPY weight	Sector
1	XLK	20.46%	INFORMATION TECHNOLOGY
2	XLF	14.88%	FINANCIALS
3	XLV	12.38%	HEALTH CARE
4	XLP	11.66%	CONSUMER STAPLES
5	XLY	11.31%	CONSUMER DISCRETIONARY
6	XLE	11.15%	ENERGY
7	XLI	10.81%	INDUSTRIALS
8	XLU	3.84%	UTILITIES
9	XLB	3.51%	MATERIALS

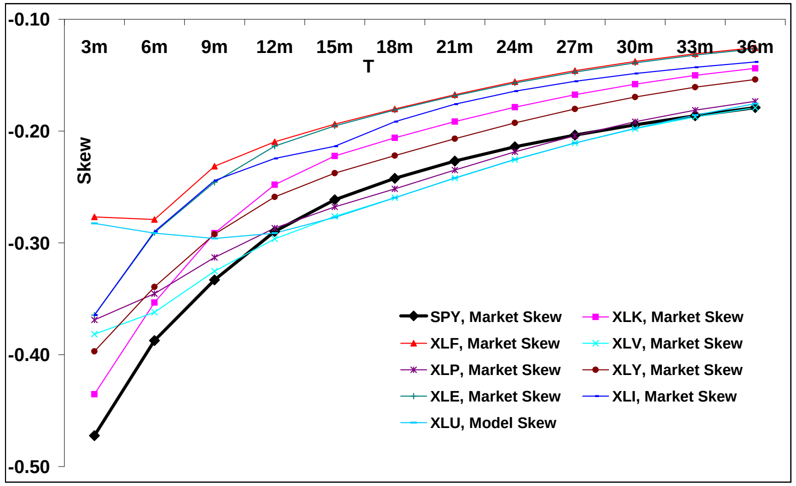
Index and sector vols and skews

Term structure of ATM volatilities (left) and 105%-95% skews (right)

SPY ATM vol (black line) **can be viewed as a weighted average of sector ATM vols**

SPY skew (black line) **is steeper than weighted average skews of sectors**





Calibration of the beta SV model I
First, calibrate the beta SV model with constant SV parameters
Calibration is based on intuition and experience with the model

Volatility beta, β, is set by

$$\beta = \frac{\sigma_{IMP}(6m, 5\%) - \sigma_{IMP}(6m, -5\%)}{0.05\sigma_{IMP}(6m, 0\%)} = \frac{2\text{Skew}_{5\%}(6m)}{\sigma_{ATM}(6m)}$$

where $\sigma_{IMP}(6m, k\%)$ is 6m implied vol for forward-based log-strike k
Idiosyncratic volatility ε is set according to:

$$\varepsilon^2 = \sigma^2_{IMP}(6m, 0\%)\beta^2 \frac{1 - (\rho^*)^2}{(\rho^*)^2}$$

ρ^* is spot-vol correlation for 6m vol implied by SV model with Orstein-Uhlenbeck process for SV driver

Reversion speed κ is adjusted to fit term structure of 1y-3y $105\% -$ 95% skew

Term structure of model level vols $\sigma_{DV}(t)$ are calibrated by construction (by root search) so that the ATM implied vol is fitted exactly

Calibration of the beta SV model II

Calibrated parameters of the beta SV model to SPY and sector ETF-s

	SPY	XLK	XLF	XLV	XLP	XLY	XLE	XLI	XLU	XLB
β	-4.77	-3.72	-2.78	-5.34	-5.31	-3.93	-2.75	-3.06	-5.55	-2.89
ε	0.40	0.39	0.67	0.35	0.33	0.42	0.55	0.41	0.38	0.39
κ	1.45	1.60	1.30	1.25	1.15	1.40	1.40	1.30	1.25	1.45
ρ^*	-0.81	-0.78	-0.60	-0.80	-0.80	-0.76	-0.66	-0.72	-0.79	-0.73

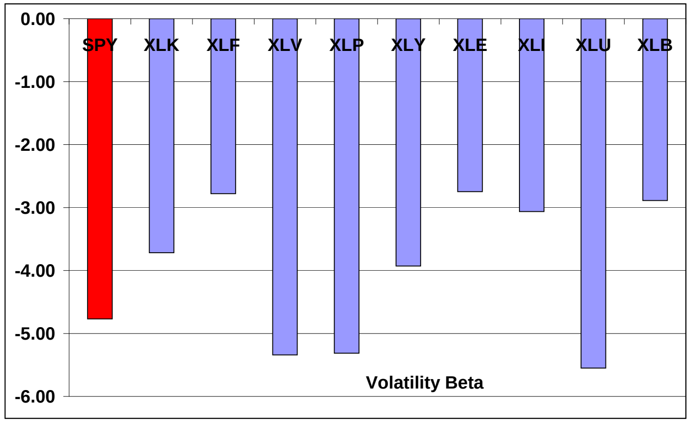
Next we illustrate plots of the term structure of market and model implied $105\% - 95\%$ skew and 1y implied vols accross range of strikes

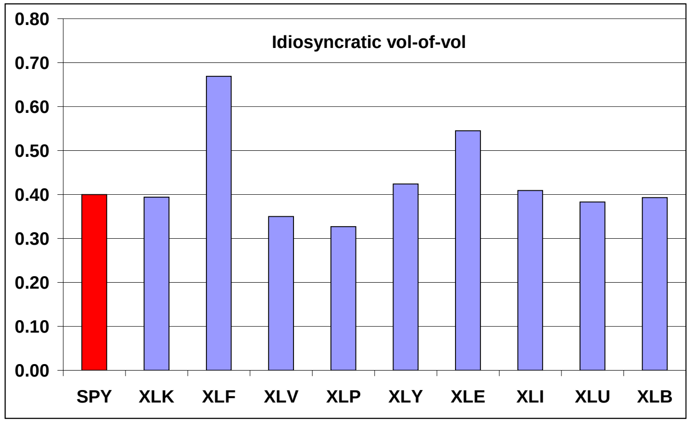
Typically, if the beta SV model is fits $105\% - 95\%$ skew, then it will fit the skew accross different strikes

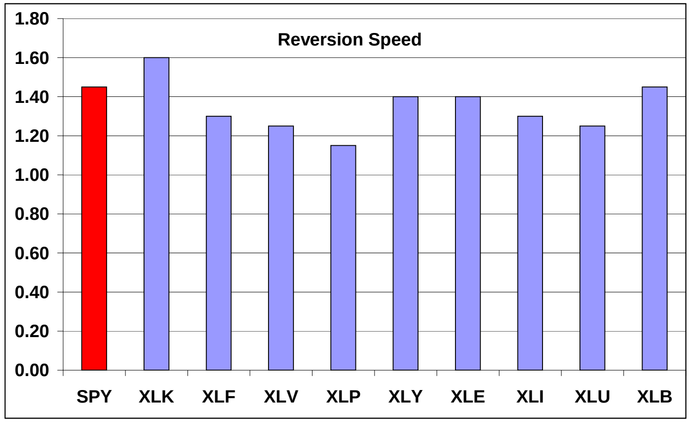
Beta SV model is similar to one-factor SV models - the model fits well longer-term skews (above one-year) while it is unable to fit short-term skews (up to one year) unless beta parameter β is large

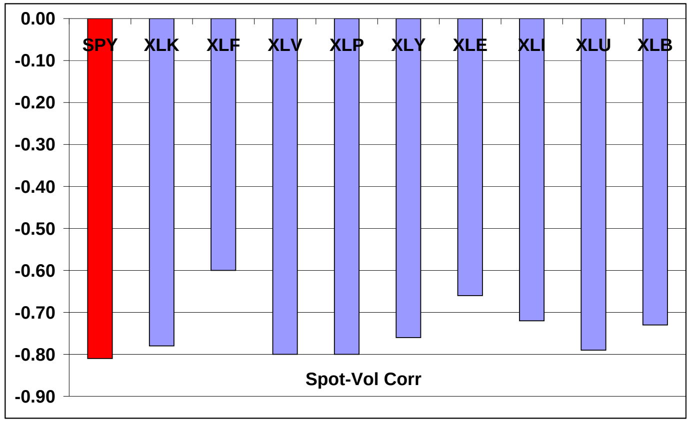
By actual pricing, small discrepancies in implied vols are eliminated by local vol part

Calibrated parameters

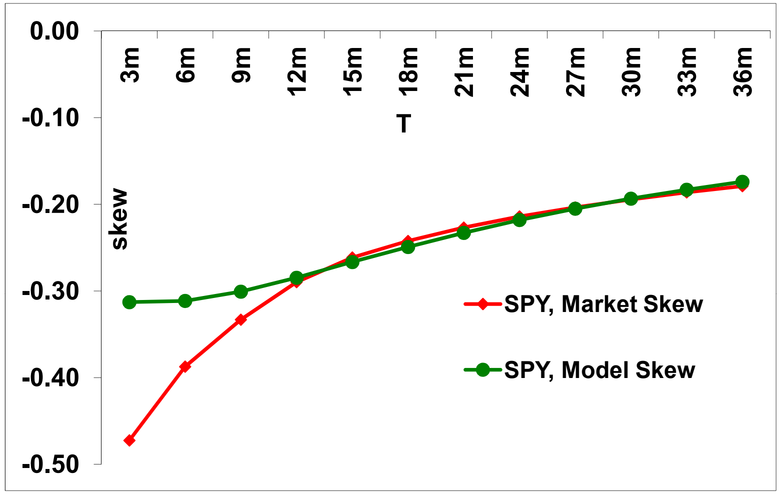


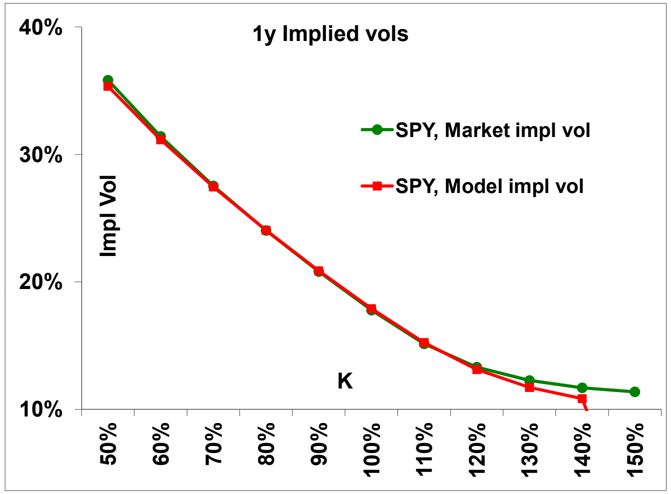




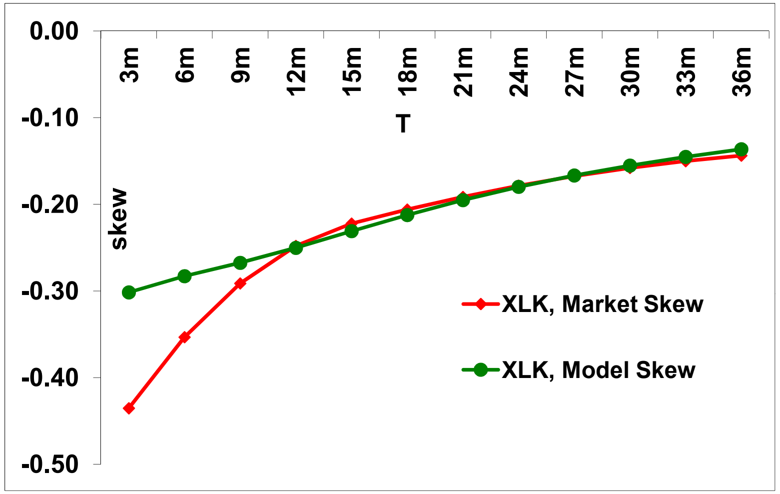


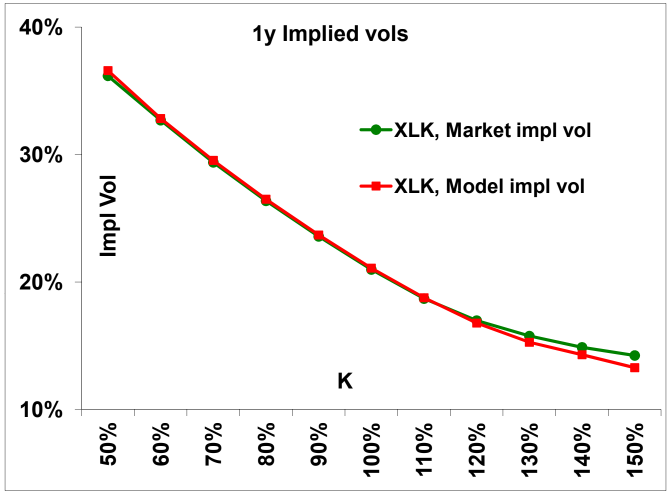
SPY


0.00
-0.10
-0.20
-0.30
-0.40
-0.50
3m
6m
9m
12m
15m
18m
21m
24m
27m
30m
33m
36m
T
skew
SPY, Market Skew
SPY, Model Skew

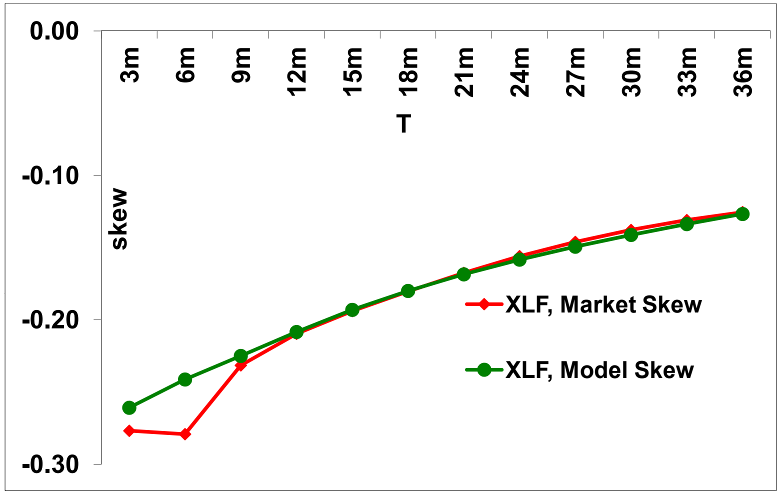

1y Implied vols
40%
30%
20%
10%
Impl Vol
SPY, Market impl vol
SPY, Model impl vol
K
50%
60%
70%
80%
90%
100%
110%
120%
130%
140%
150%

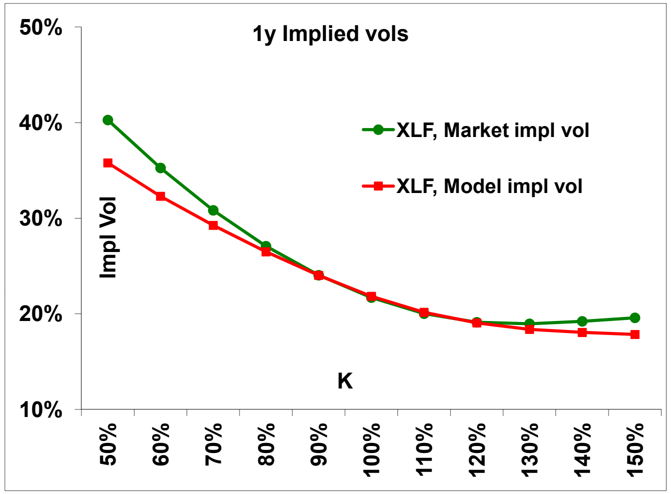
XLK - information technology



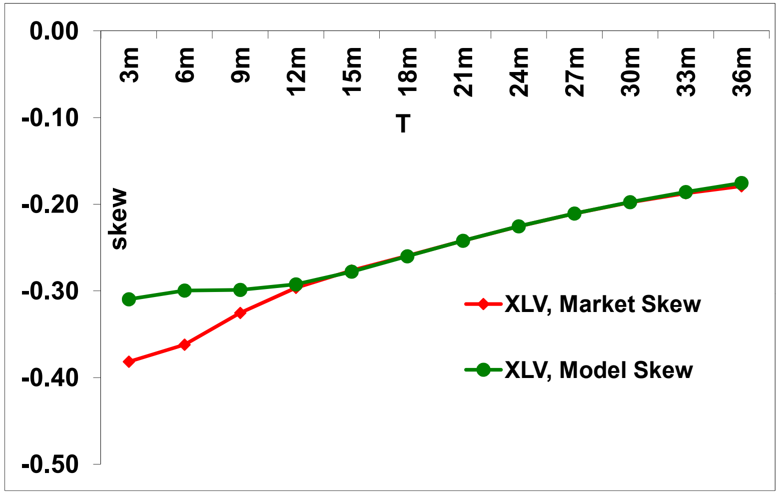


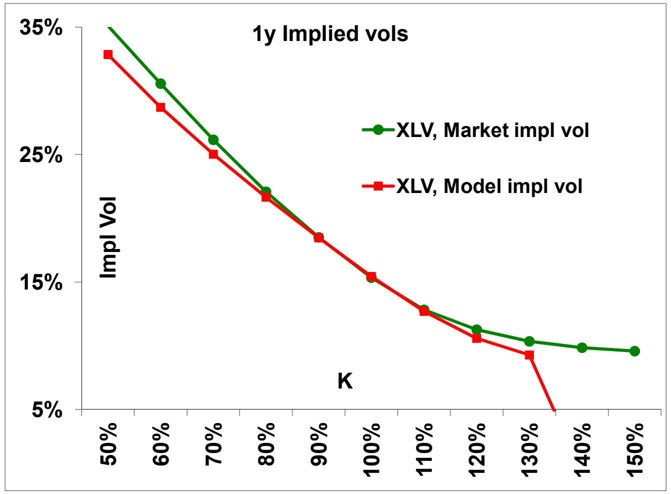
XLF - financials



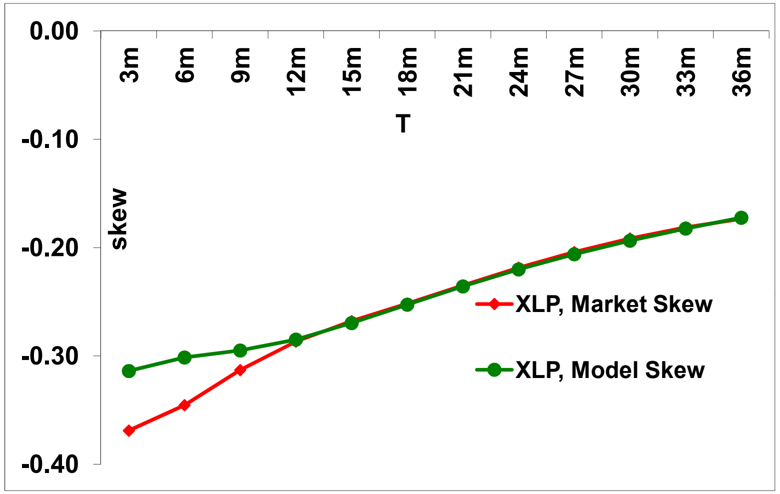


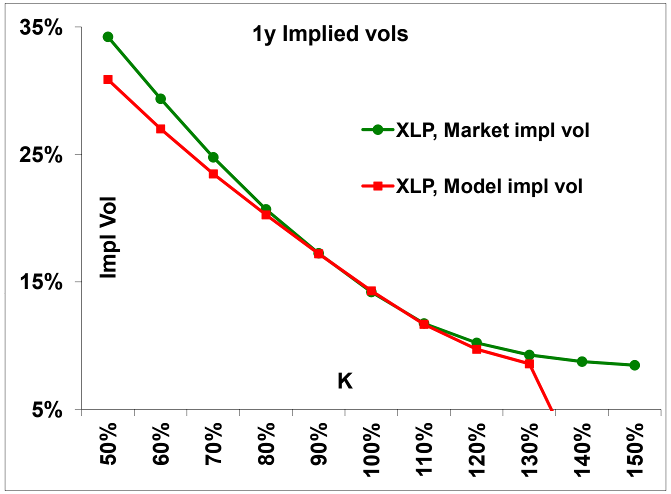
XLV - health care



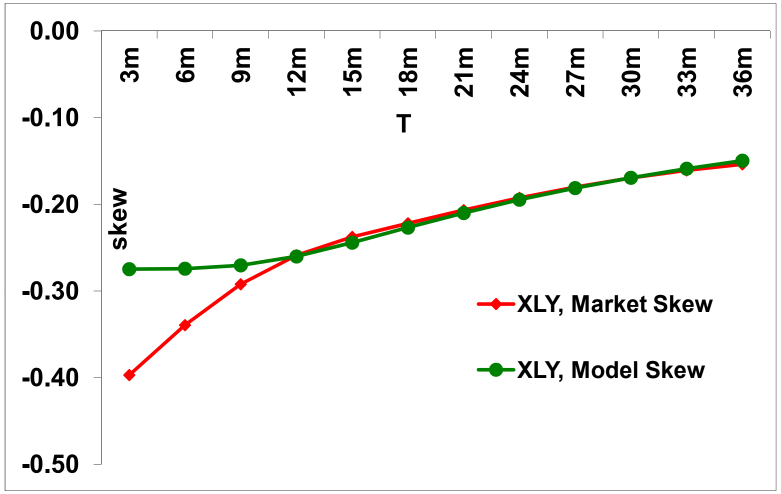


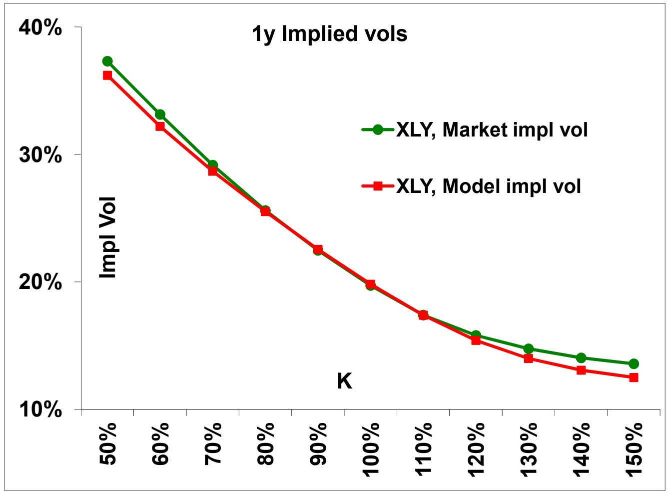
XLP - consumer staples



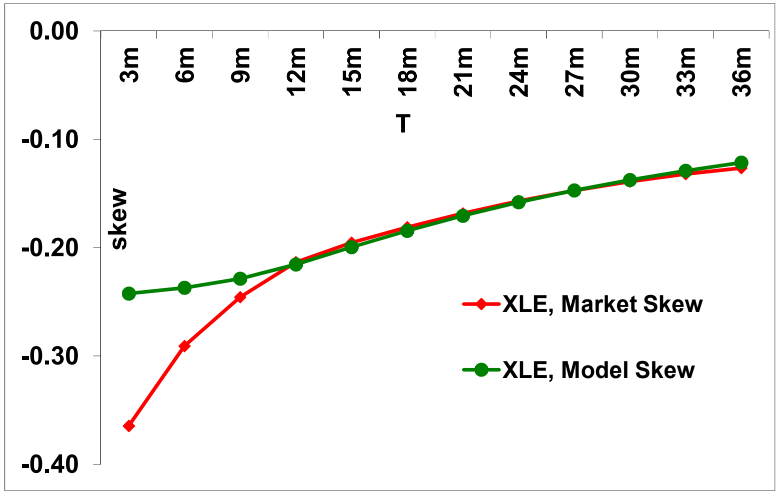


XLY - consumer discretionary





XLE - energy



0.00
-0.10
-0.20
-0.30
-0.40
skew
3m
6m
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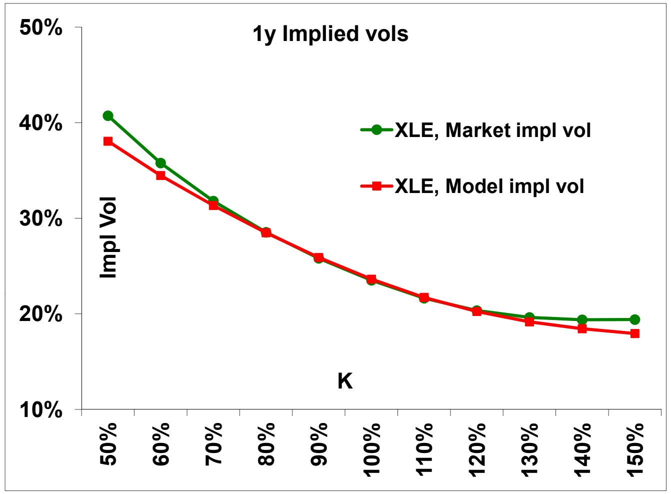
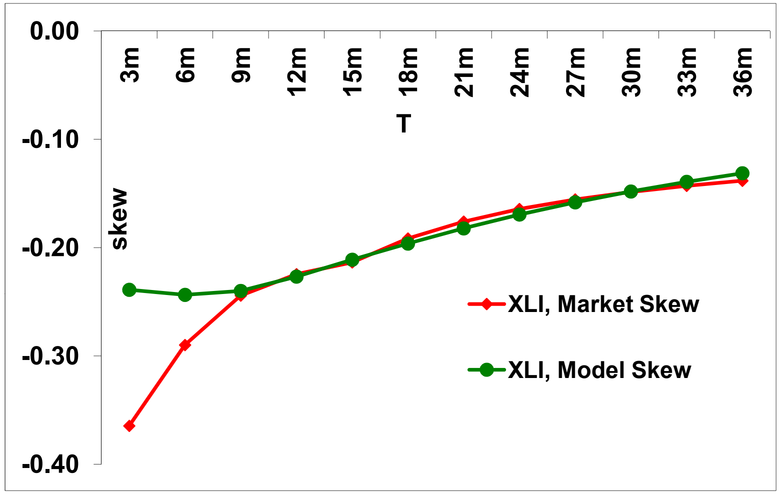
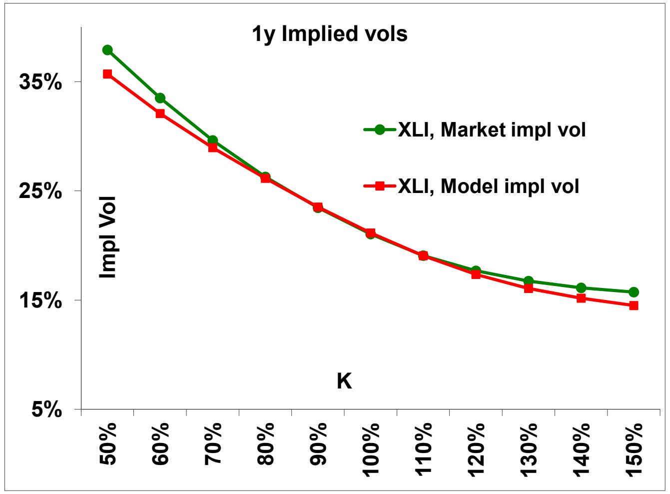
33m
36m
T
XLE, Market Skew
XLE, Model Skew



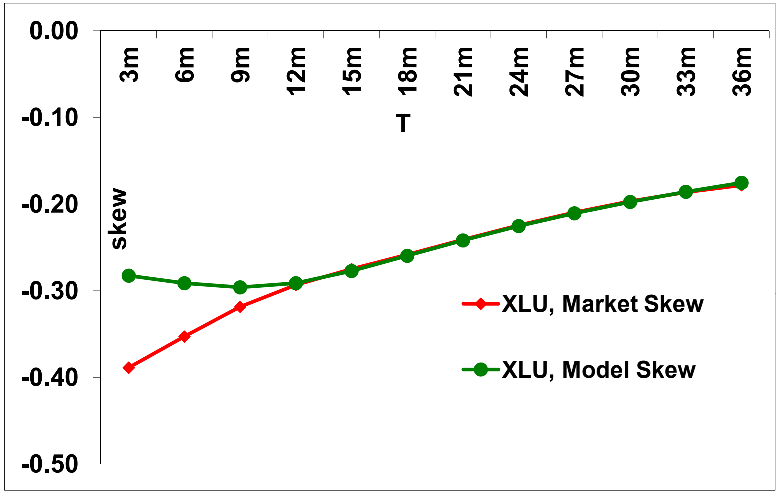
1y Implied vols
50%
40%
30%
20%
10%
Impl Vol
XLE, Market impl vol
XLE, Model impl vol
K
50%
60%
70%
80%
90%
100%
110%
120%
130%
140%
150%

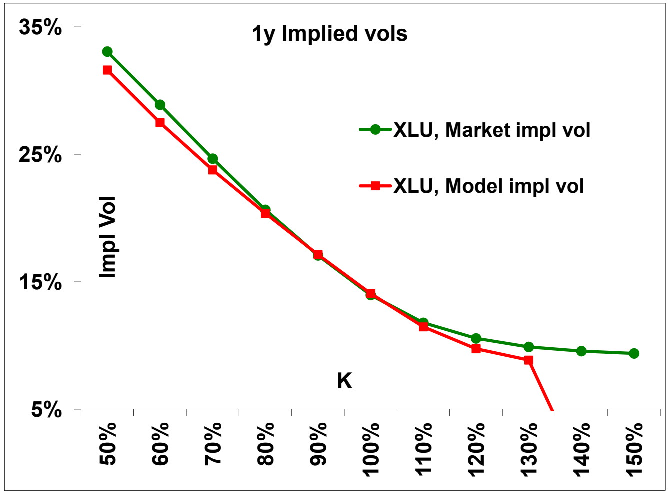
XLI - industrials



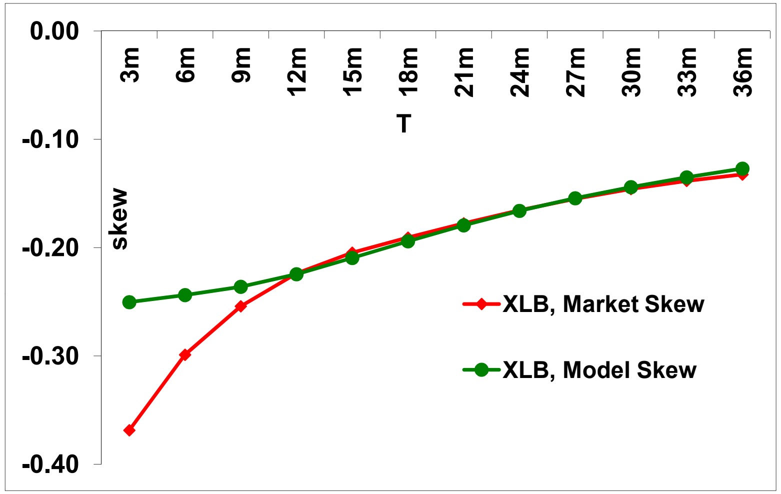


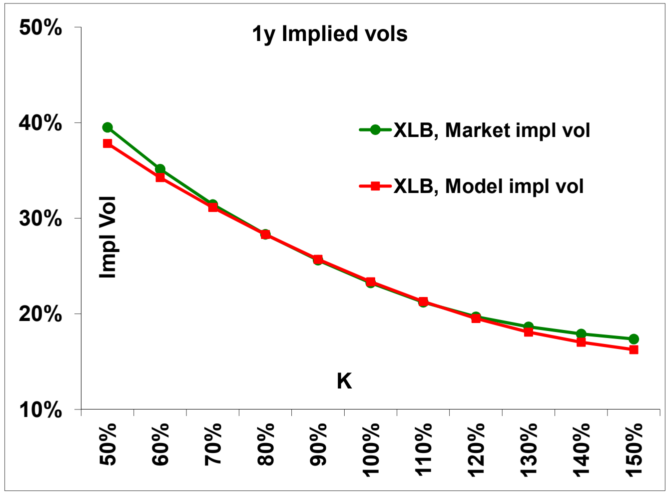
XLU - utilities





XLB - materials





Multi-asset beta SV model
The SV model without jumps cannot reproduce the correlation skew

We present multi-asset beta SV model with simultaneous jumps in assets and their volatilities:

$$\frac{dS_n(t)}{S_n(t)} = \mu_n(t)dt + (1 + Y_n(t))\sigma_n dW_n(t) + \left(e^{\nu_n} - 1\right)\left(dN(t) - \lambda dt\right)$$

$$dY_n(t) = -\kappa_n Y_n(t)dt + \beta_n(1 + Y_n(t))\sigma_n dW_n(t) + \varepsilon_n dW^{(1)}(t) + \eta_n dN(t)$$

where $n = 1, ..., N$
$dW_n(t)$ are Brownians for asset prices with specified correlation matrix
$W^{(1)}(t)$ is the joint driver for idiosyncratic volatilities
$N(t)$ is the joint Poisson process with intensity λ for simultaneous shocks in prices and volatilities
ν_n, $\nu_n < 0$, are constant jump amplitudes in log-price
η_n, $\eta_n > 0$, are constant jump amplitudes in volatilities

Jump calibration
Based on my presentation for Global Derivatives in Paris, 2011
The idea is based on linear impact of jumps on the short-term implied skew:

$$\sigma_{\text{imp}}(K) \approx \sigma - \frac{\lambda\nu}{\sigma} \ln\left(S/K\right)$$

Specify w_{jd} - the percentage of the skew attributed to jumps
Set jump intensity as follows:

$$\lambda = \frac{\left(\text{Skew}_{5\%}(1y)\right)^2}{w_{jd}}$$

The jump size is implied as follows:

$$\nu = -\frac{\sqrt{w_{jd}}\sigma_{ATM}(1y)}{\sqrt{\lambda}} = \frac{w_{jd}\sigma_{ATM}(1y)}{\text{Skew}_{5\%}(1y)}$$

Jump size in volatility, η, can be calibrated to options on the VIX skew or options on the realized variance
Empirically, $\eta \approx 2$

Jump calibration for ETF. I
Set $w_{jd} = 50\%$ and imply **jump intensity** λ_{SPY} from 1y 5% skew for SPY ETF
Individual jump sizes are set using λ_{SPY} and sector specific ATM volatility $\sigma_{ATM,n}(1y)$:

$$\nu_n = -\frac{\sqrt{w_{jd}}\sigma_{ATM,n}(1y)}{\sqrt{\lambda_{SPY}}}$$

Jump size in volatility, η, is set uniformly $\eta = 2$ (realized jump in ATM volatility will be proportional to ATM volatility of sector ETF)

Previously specified β, ε and κ are reduced by 25%

Next we illustrate plots of term structure of market and model implied $105\% - 95\%$ skew and 1y implied volatilities accross range of strikes

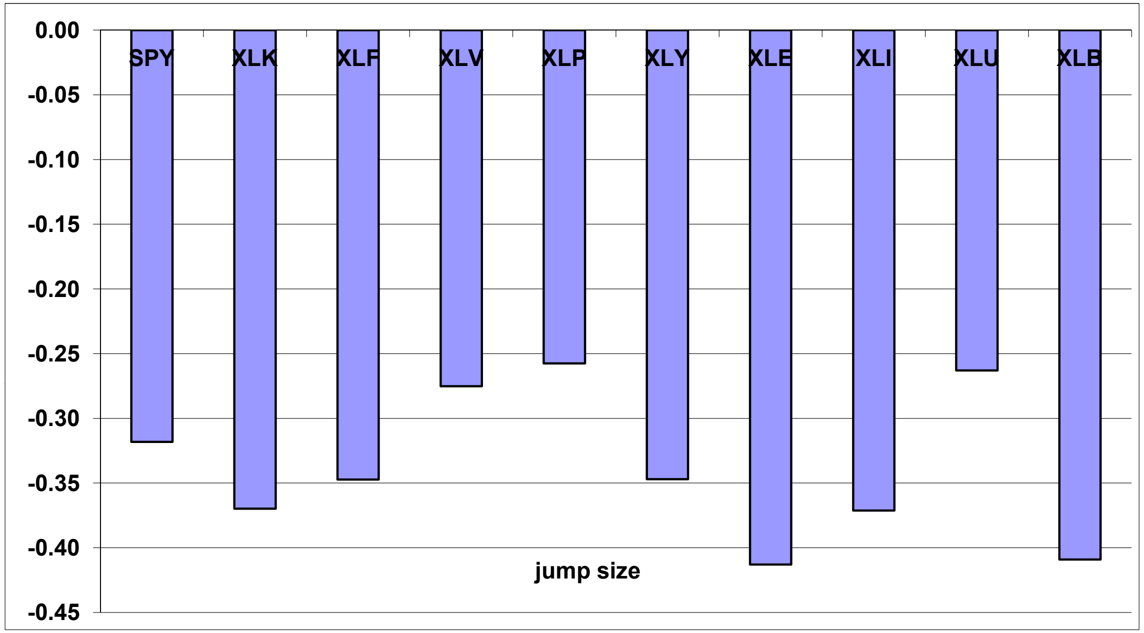
Beta SV model with jumps produces steep forward skews for short maturities and is consistent with term structure of skew

Again, by actual pricing, small discrepancies in implied vols are eliminated by local vol part

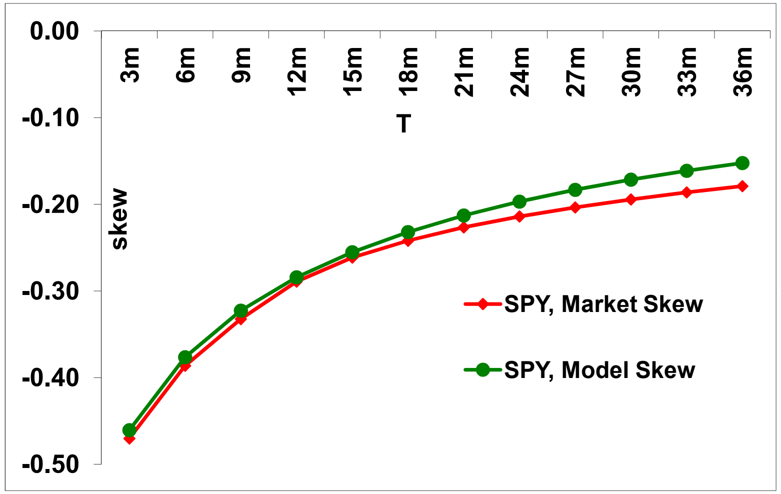
Jump calibration for ETF. II

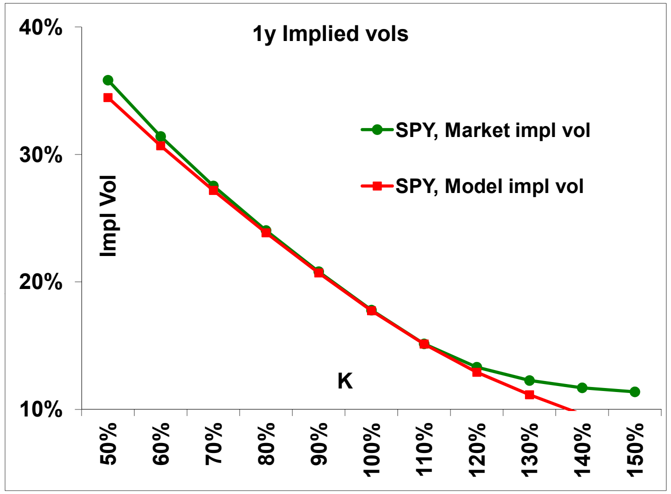
For recent data, $\lambda = 0.17$ and jump sizes are shown in table and figure

SPY	XLK	XLF	XLV	XLP	XLY	XLE	XLI	XLU	XLB
-0.32	-0.37	-0.35	-0.28	-0.26	-0.35	-0.41	-0.37	-0.26	-0.41

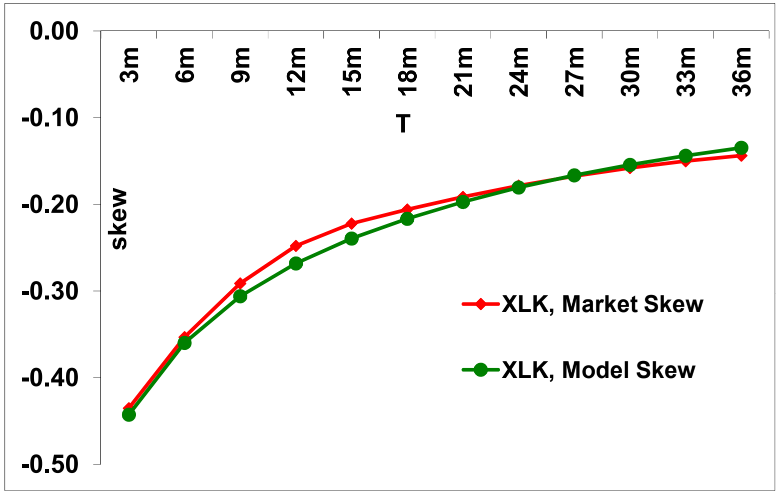


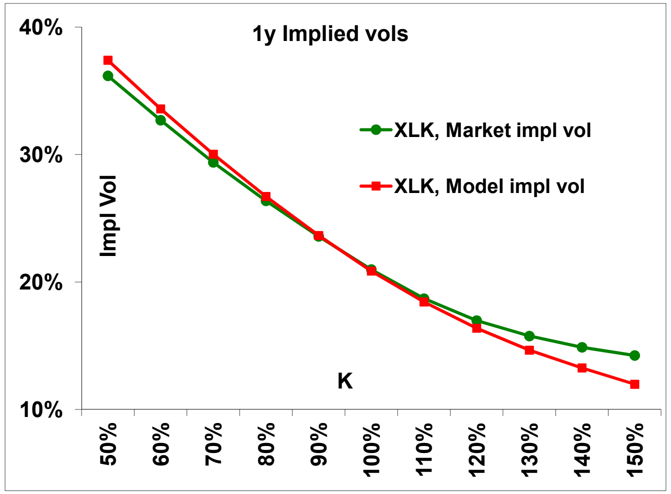
SPY


0.00
-0.10
-0.20
-0.30
-0.40
-0.50
3m
6m
9m
12m
15m
18m
21m
24m
27m
30m
33m
36m
T
skew
SPY, Market Skew
SPY, Model Skew

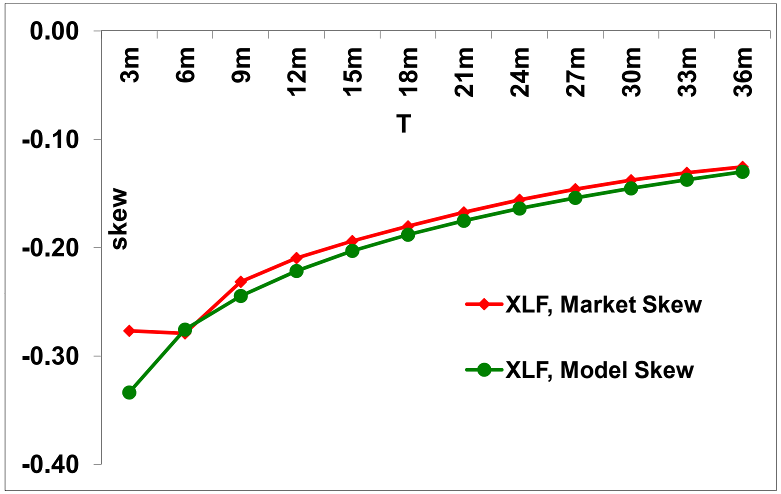

1y Implied vols
40%
30%
20%
10%
Impl Vol
SPY, Market impl vol
SPY, Model impl vol
K
50%
60%
70%
80%
90%
100%
110%
120%
130%
140%
150%

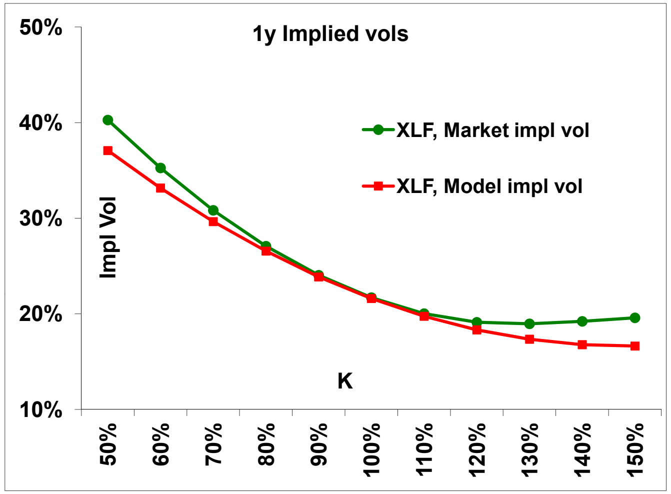
XLK - information technology



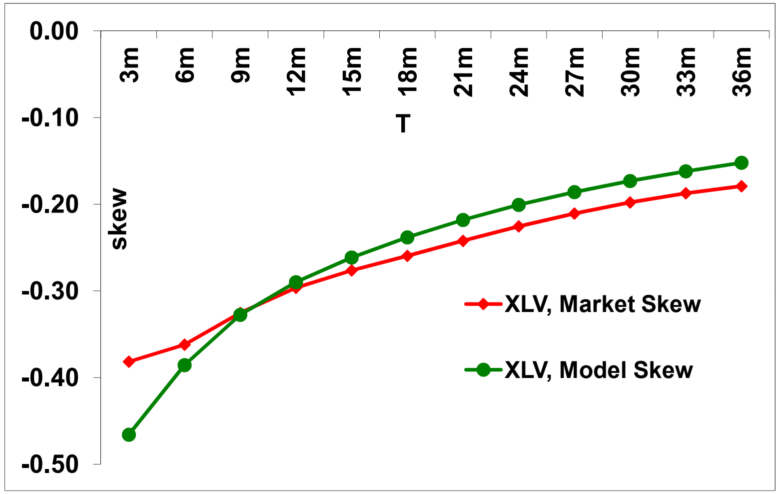


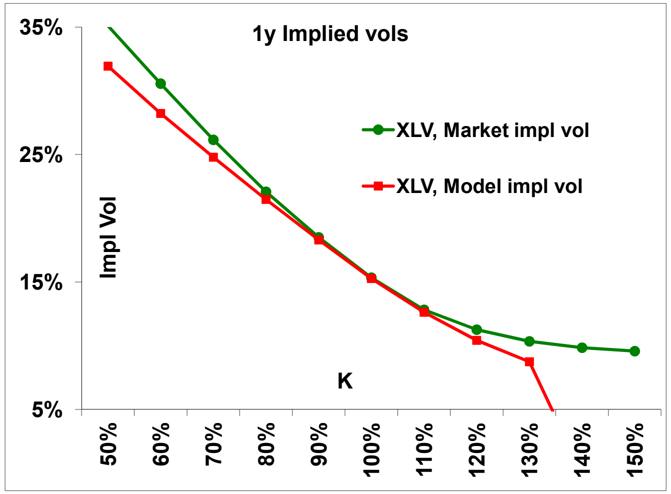
XLF - financials



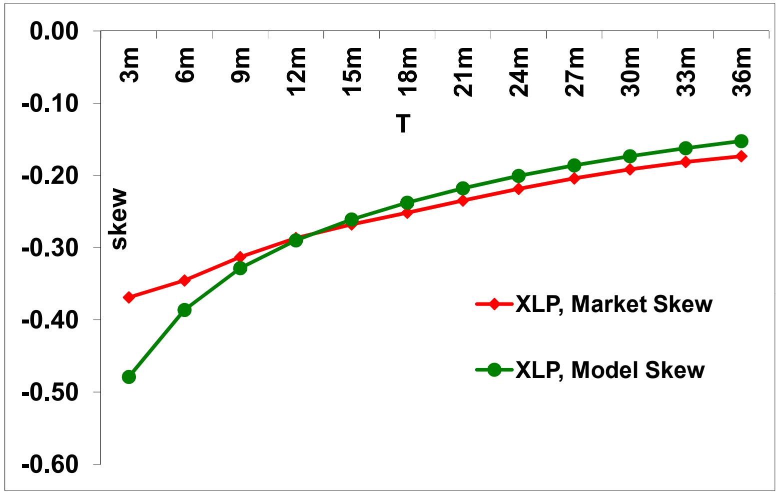


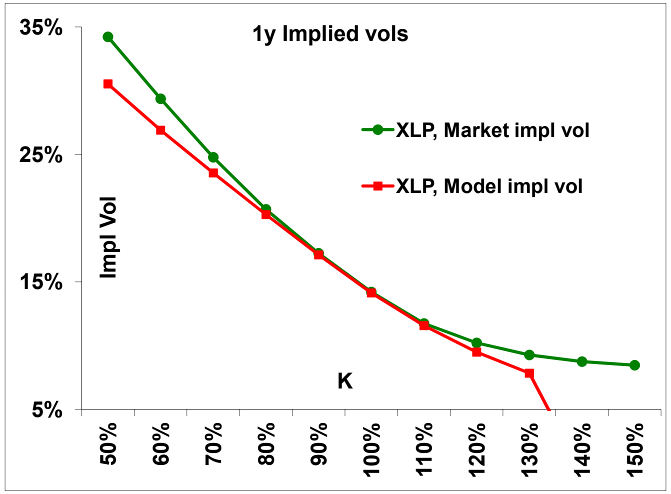
XLV - health care



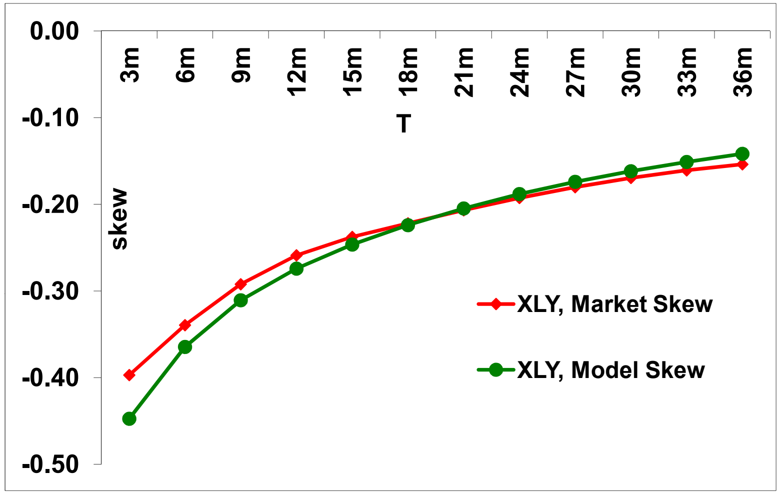


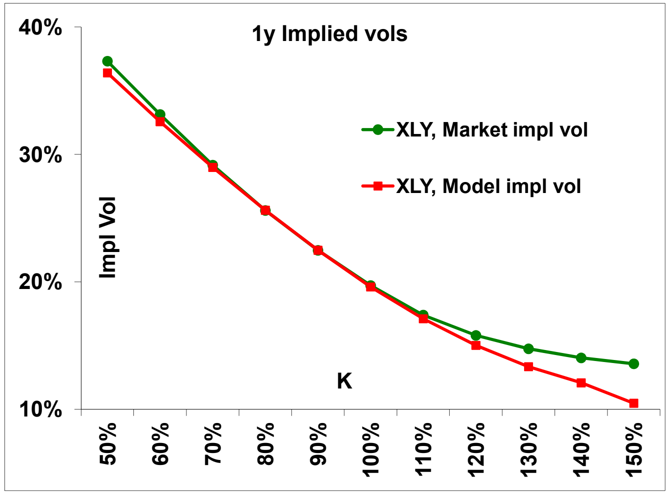
XLP - consumer staples



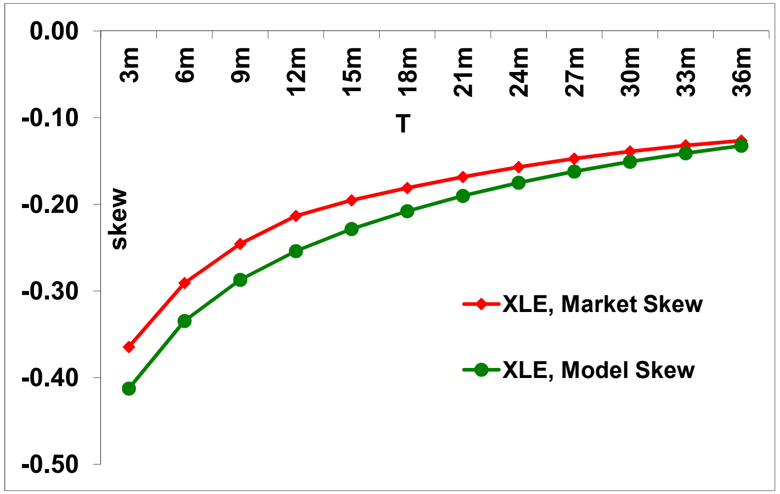


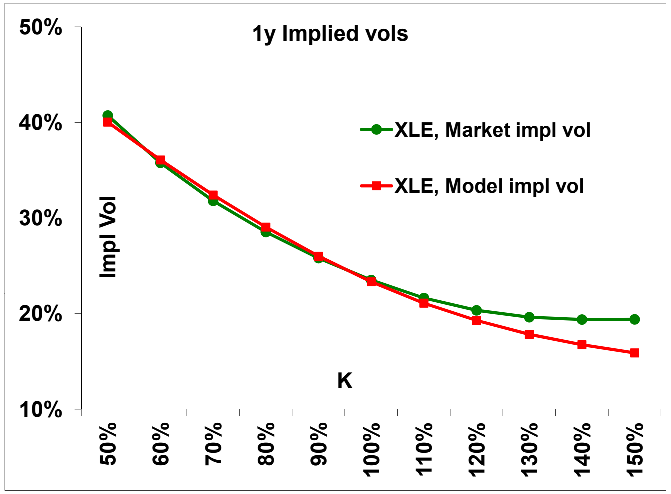
XLY - consumer discretionary



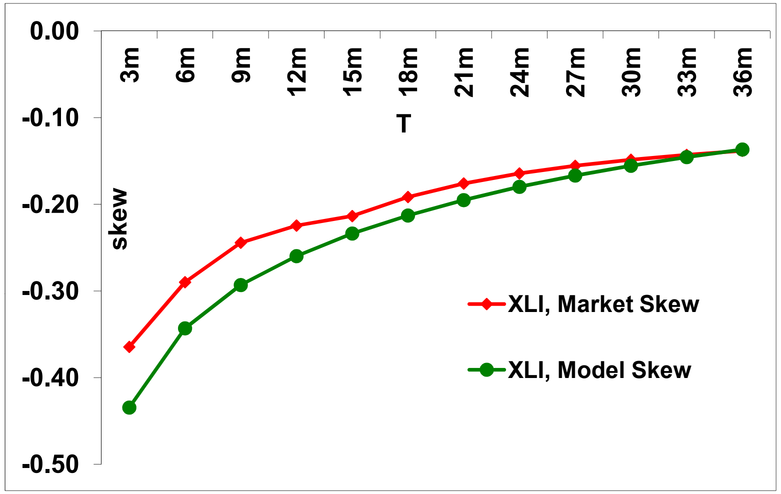


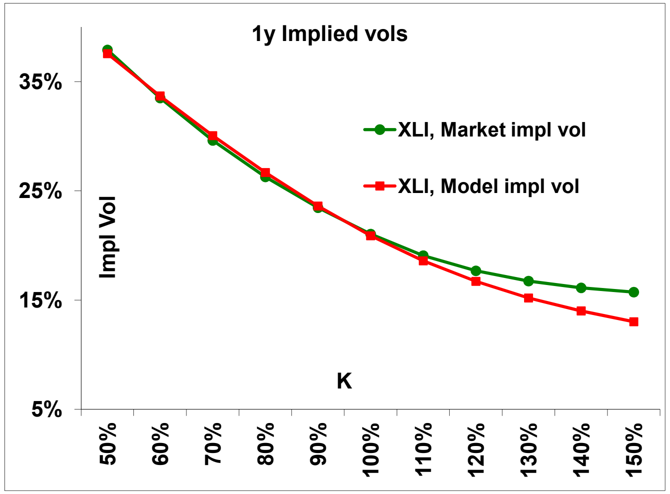
XLE - energy


0.00
-0.10
-0.20
-0.30
-0.40
-0.50
3m
6m
9m
12m
15m
18m
21m
24m
27m
30m
33m
36m
T
skew
XLE, Market Skew
XLE, Model Skew

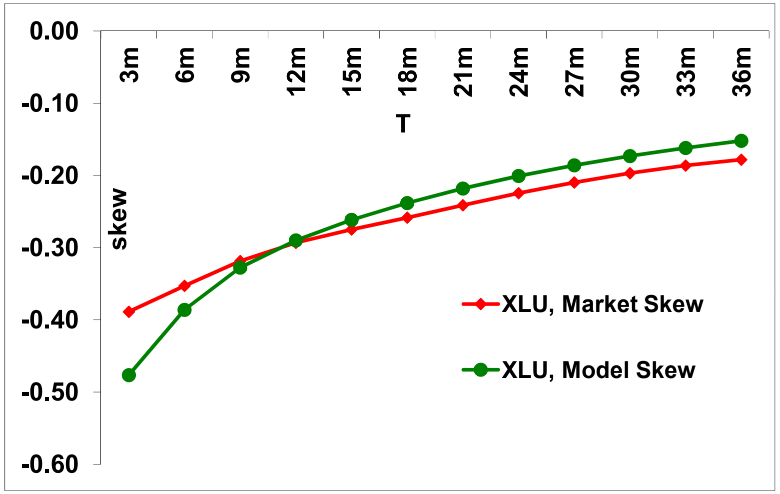

1y Implied vols
50%
40%
30%
20%
10%
Impl Vol
XLE, Market impl vol
XLE, Model impl vol
K
50%
60%
70%
80%
90%
100%
110%
120%
130%
140%
150%

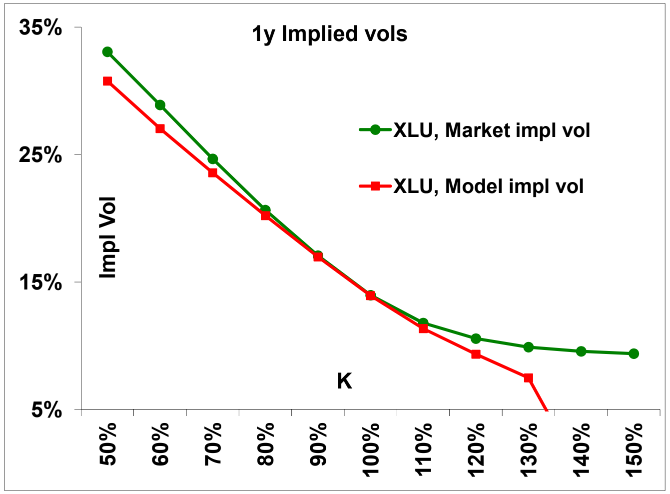
XLI - industrials



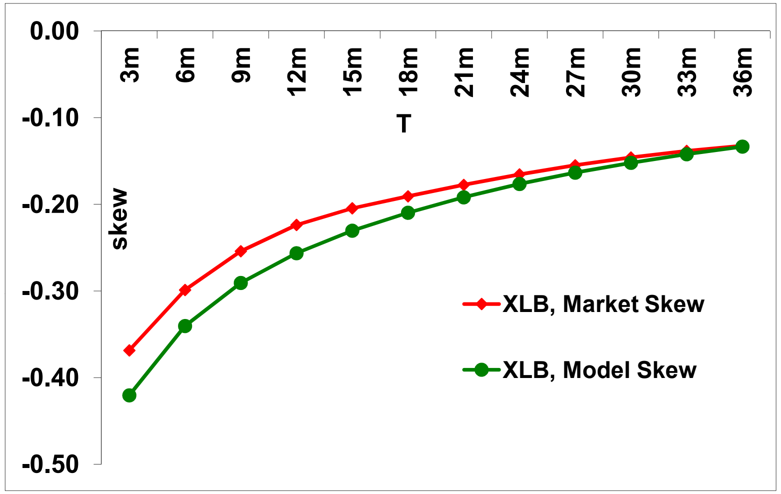


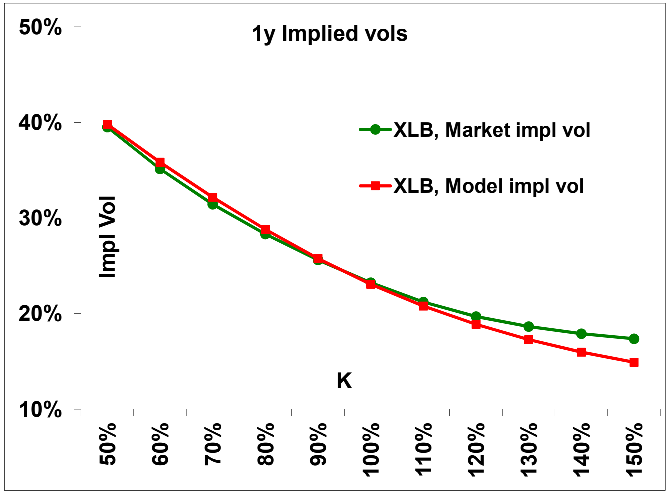
XLU - utilities





XLB - materials





Correlation matrix for sector ETF-s
Sector-wise correlation are estimated using time series

	XLK	XLF	XLV	XLP	XLY	XLE	XLI	XLU	XLB
XLK	100%								
XLF	85%	100%							
XLV	64%	80%	100%						
XLP	69%	77%	77%	100%					
XLY	92%	87%	84%	83%	100%				
XLE	85%	84%	75%	48%	83%	100%			
XLI	90%	89%	86%	81%	94%	88%	100%		
XLU	67%	66%	74%	82%	72%	66%	72%	100%	
XLB	89%	87%	80%	73%	88%	88%	91%	63%	100%

Correlation skew. Illustration

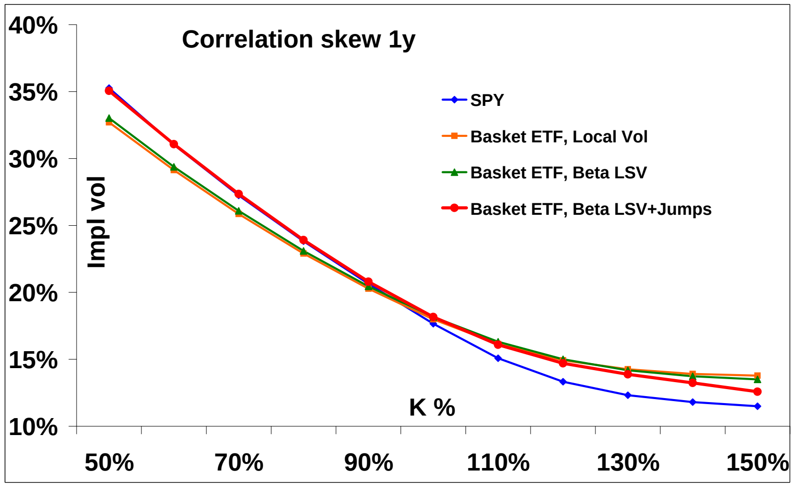
Next we compare the index skew implied from:

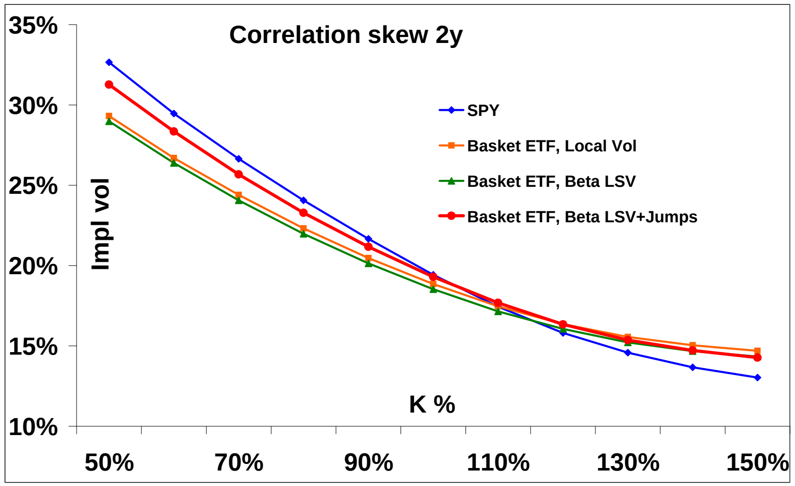
1) SPY options (SPY)

2) basket of ETF priced using local volatility with Gaussian correlations (Local Vol)

3) basket of ETF priced using beta SV model (Beta LSV)

4) basket of ETF priced using beta SV model with jumps (Beta LSV+Jumps)





Correlation skew. Conclusion

Local volatility and stochastic volatility models without jumps cannot reproduce the correlation skew

Only jumps can introduce the correlation skew in a robust way

In my example, I calibrated jumps to 1y skew so the model fits 1y correlation skew, but model correlation skew flattens for 2y (problem with data for long-dated ETF options?)

Perhaps more elaborate jump process is necessary (probably though spot- and volatility-dependent intensity process)

Case Study II: Conditional forward skew

We imply forward volatility by computing forward-start option conditional that $S(\tau)$ starts in range $(D-\Delta, D+\Delta)$, typically $\Delta = 5\%$, with the following pay-off:

$$\mathbf{1}_{\{D-\Delta<S(\tau)<D+\Delta\}}\left(\frac{S(T)}{S(\tau)}-K\right)^{+}$$

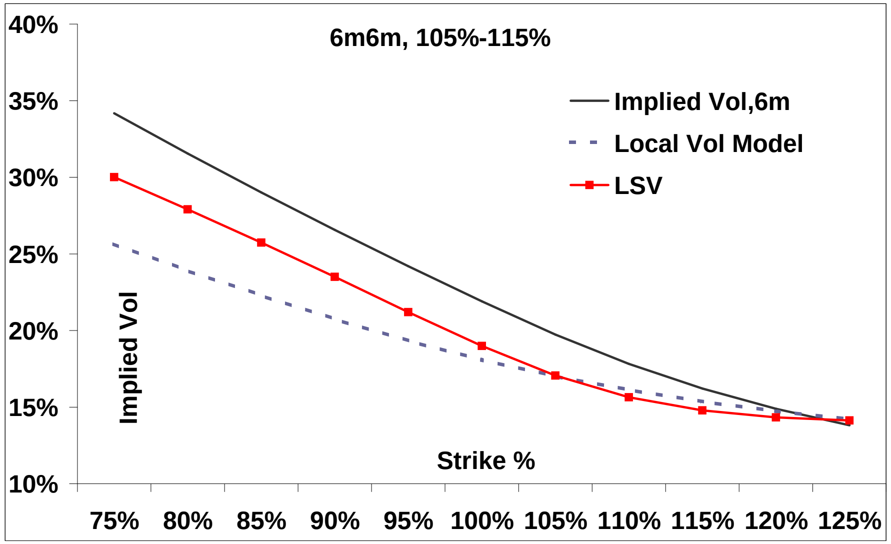
In the BSM model, the variables $S(\tau)$ and $\frac{S(T)}{S(\tau)}$ are independent so that the BSM value of this pay-off is the probability of $S(\tau)$ hitting the range times the value of the forward start call

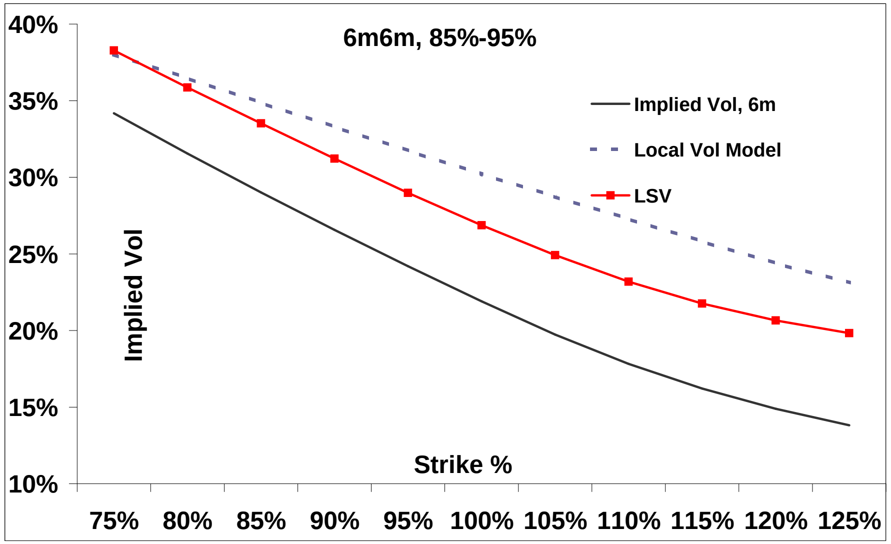
Under alternative models, we compute the above expectation, PV, by means of MC simulations and in addition compute the hitting probability, P, $P = \mathbb{E}\left[\mathbf{1}_{\{D-\Delta<S(\tau)<D+\Delta\}}\right]$

Then we imply the conditional volatility using the BSM inversion for call with strike K, time to maturity $T-\tau$, and value PV/P

We compare two models: local volatility (LV) and beta SV with local vol (LSV)

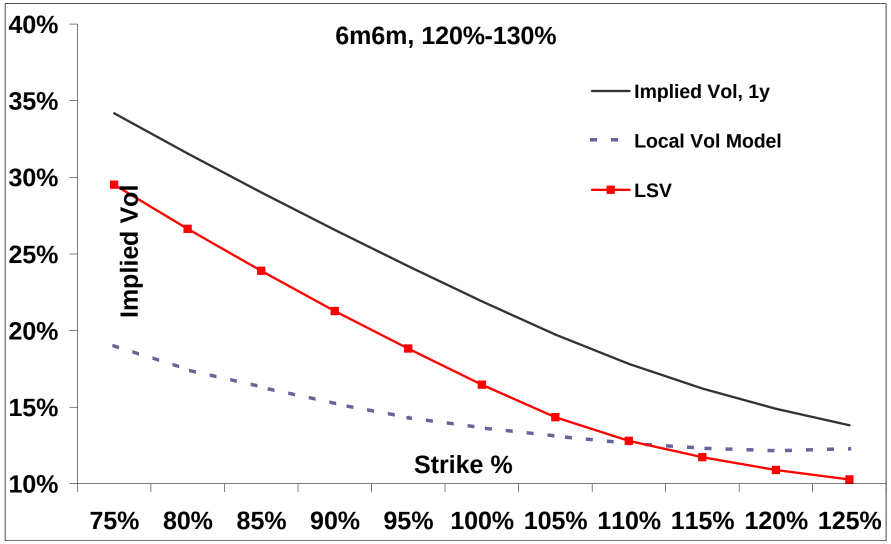
Conditional forward skew II

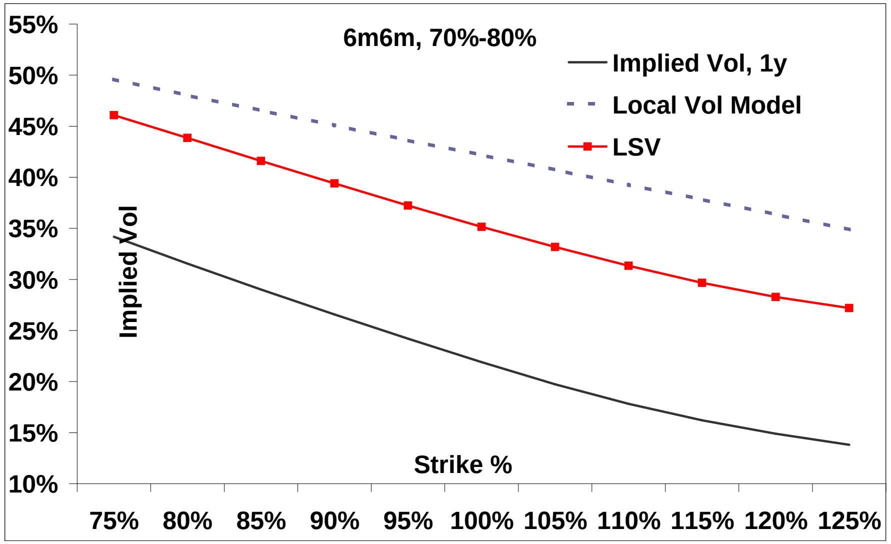




6m6m conditional skew for $D = 110\%$ (left) and $D = 90\%$ (right)

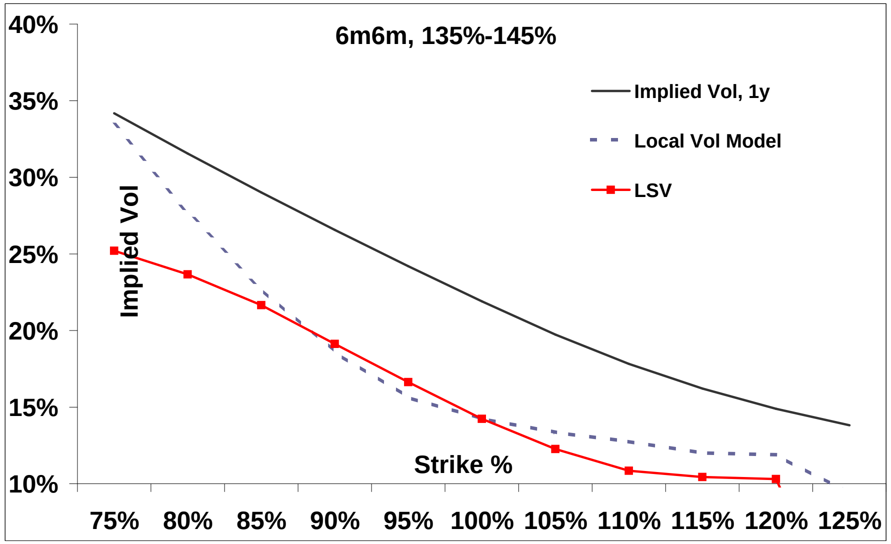
Conditional forward skew III

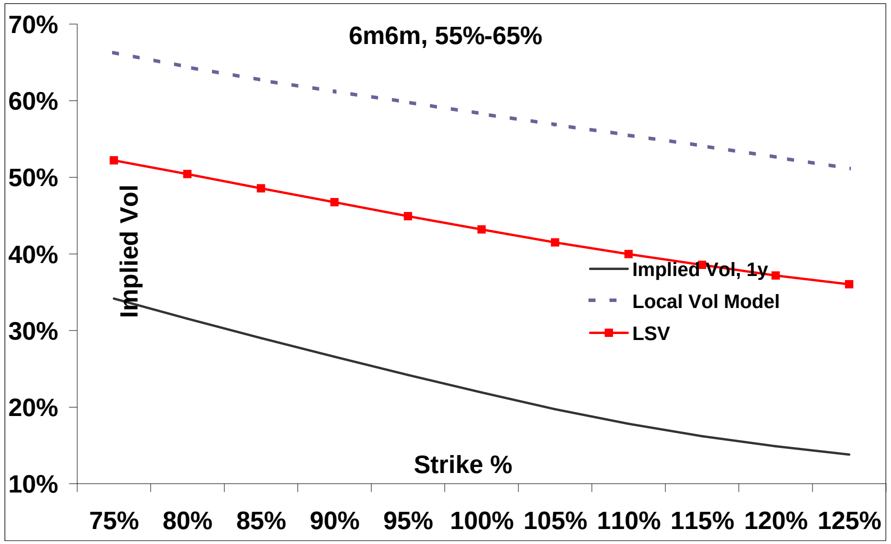




6m6m conditional skew for $D = 125\%$ (left) and $D = 75\%$ (right)

Conditional forward skew IV





6m6m conditional skew for $D = 140\%$ (left) and $D = 60\%$ (right)

Conclusions
I presented the beta stochastic volatility model that

1) Has intuitive parameters (volatility beta) that can be explained using empirical data

2) Calibration of parameters for SV process and jumps is straightforward and intuitive (no non-linear optimization methods are necessary)

3) Allows to mix parameters to reproduce different regimes of volatility and the equity skew

4) Equipped with jumps, allows to reproduce correlation skew for multi-underlyings

5) Produces very steep forward skews

6) The driver for the instantaneous volatility has nice properties: fat tails and level dependent spot-volatility correlations

Open questions
1) Better understanding of relationship between model parameters of market observables (ATM vol, skew, their term structures)

2) Model implied risk incorporating the stickiness ratio

3) Numerical methods (numerical PDE, analytic approximations)

4) Calibration of jumps and correlation skew

5) Illustrate/proof that only SV model with jumps is consistent with observed empirical features:
A) Stickiness ratio is between 1 and 2
B) Steep correlation skew

Models with local volatility and correlation may be consistent with **A)** and **B)** but they are not consistent with observed dynamics thus producing wrong hedges

Thank you for your attention!

Disclaimer

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