Modeling and Pricing Longevity Derivatives with Stochastic Mortality Using the Esscher Transform

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Lee-Carter Model

Lee and Carter (1992) model the mortality rate as a function of age group x and time t (in years).

$$m_{x,t} = \exp(a_x + b_x k_t + \epsilon_{x,t})$$

x: age group

t: time

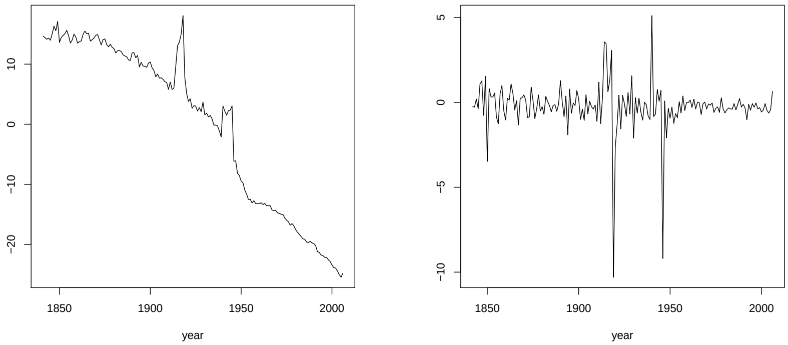
a_x: the general shape of the mortality curve for age group x

k_t: the mortality rate time (year) index

b_x: each age group's response to the mortality rate index

$\epsilon_{x,t}$: error term

Example of Lee-Carter Time Trend k_t



(a) k_t (b) $\Delta k_t = k_t - k_{t-1}$

Data: Male of England and Wales from 1841 to 2006

Modified Lee-Carter Model

Mitchell et al (2011) modify Lee-Carter model by taking difference of the log mortality rate for the first step.

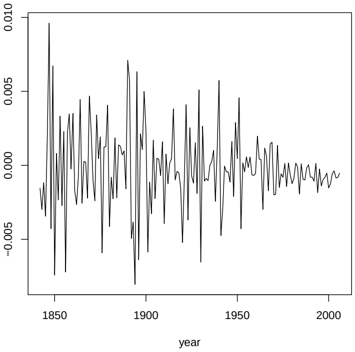
$$m_{x,t} = m_{x,t-1} \exp(a_x + b_x k_t + \epsilon_{x,t})$$
$$\rightarrow log(m_{x,t}) - log(m_{x,t-1}) = a_x + b_x k_t + \epsilon_{x,t}.$$

This modified model performs better than

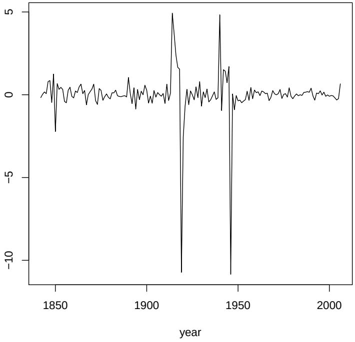
- Lee-Carter model with or without cohort effects
- Logit parametric models
- Several other modification of Lee-Carter models

The paper uses data from 11 countries, including U.S.A, U.K., Canada, and several other countries. This modified model performs better than other models whatever the data are used.

Example of Modified Lee-Carter Time Trend k_t



(c) $log(m_{[65,69],t}) - log(m_{[65,69],t-1})$



(d) k_t

Data: Male of England and Wales from 1841 to 2006

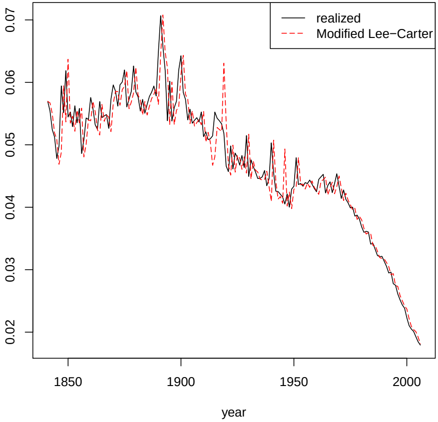
Normal Inverse Gaussian Distribution

To forecast future mortality rate, Michell et al (2011) suggest fitting k_t in the modified Lee-Carter model with a Normal Inverse Gaussian (NIG) process. The NIG distribution has a density

$$f_{NIG}(y;\alpha,\beta,\mu,\delta) = \frac{\alpha}{\pi}\exp(\delta\sqrt{\alpha^2-\beta^2}+\beta(y-\mu))\frac{K_1\left(\alpha\delta\sqrt{1+\left(\frac{y-\mu}{\delta}\right)^2}\right)}{\sqrt{1+\left(\frac{y-\mu}{\delta}\right)^2}}$$

- 4 parameters, $(\alpha,\beta,\mu,\delta)$, to control shape and location of the distribution.
- It provides a better fit when the distribution has high kurtosis and fat tails.

Fitted Mortality Rate for Age 65 to 69



Data: Male of England and Wales from 1841 to 2006

Model k_t with a NIG Process

$$m_{x,t} = m_{x,t-1} \exp(a_x + b_x k_t + \epsilon_{x,t})$$

The modified Lee-Carter Model

Model k_t with a NIG Process

$$m_{x,t} = m_{x,t-1} \exp(a_x + b_x k_t + \epsilon_{x,t})$$

By iteration, we have

$$= m_{x,0} \exp\left(a_x t + b_x \sum_{i=1}^{t} k_i + \sum_{i=1}^{t} \epsilon_{x,i}\right)$$

Model k_t with a NIG Process

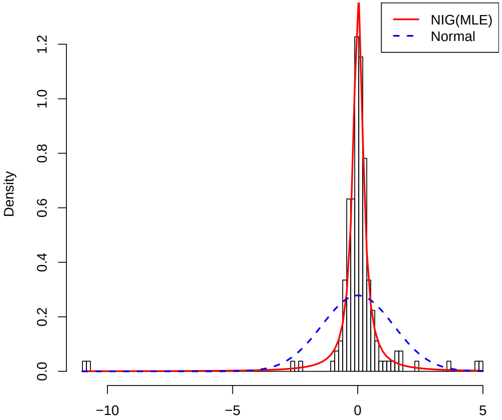
$$m_{x,t} = m_{x,t-1} \exp(a_x + b_x k_t + \epsilon_{x,t})$$
$$= m_{x,0} \exp\left(a_x t + b_x \sum_{i=1}^{t} k_i + \sum_{i=1}^{t} \epsilon_{x,i} \right)$$

Let k_i and $\epsilon_{x,i}$ be i.i.d. random variables, then we have

$$= m_{x,0} \exp(a_x t + b_x \sum_{i=1}^{t} k_1 + \sum_{i=1}^{t} \epsilon_1)$$

where k_1 a NIG random variable.

Fit k_1 with NIG Distribution



Data: Male of England and Wales from 1841 to 2006

Lévy Process

- A $(Y_t)_{t\geq 0}$ is a Lévy process if
  - Independent increments: For $0 \leq t_1 < t_2 \leq t_3 < t_4$, $(Y_{t_2} - Y_{t_1})$ and $(Y_{t_4} - Y_{t_3})$ are independent.
  - Stationary increments: $(Y_{t+h} - Y_t)$ does not depend on t.
  - Right continuous: For all $\epsilon > 0$, $\lim_{h\to 0} \mathbb{P}(|Y_{t+h} - Y_t| > \epsilon) = 0$.
- In financial modeling, the stock price process is usually assumed to be

$$S_t = S_0 \exp(r_f t + Y_t)$$

S_t: stock price at time t
r_f: risk free rate.
Y_t: Lévy process

Mortality Rate with a Stochastic Component

Let the time-varying part be a NIG Lévy process. We have

$$m_{x,t} = m_{x,0} \exp(a_x t + N_{x,t}),$$

where

$$\begin{aligned} N_{x,t} &\sim NIG(\alpha/b_x, \beta/b_x, b_x \mu t, b_x \delta t) \\ &\sim NIG(\alpha', \beta', \mu' t, \delta' t). \end{aligned}$$

Esscher Transform

If a stock process follows a exponential Lévy process, the Esscher transform can be used to find a martingale measure for pricing purpose.
The Esscher transform $\mathbb{P}^{\theta}$ is defined as

$$\frac{d\mathbb{P}^{\theta}}{d\mathbb{P}} = \frac{e^{\theta Y}}{E(e^{\theta Y})},$$

provided that $E(e^{\theta Y})$, the moment generation function, exists.

- The ratio of two measures.
- With Esscher transform, the martingale measure can be calculated easily through the moment generation function.

In financial modeling, the stock price process is assumed to be an exponential Lévy process

$$S_t = S_0 \exp(r_f t + Y_t).$$

- Let $M_Y(\theta)$ be the moment generation function and $\kappa(\theta)$ is the exponential components of $M_Y(\theta)$.
- With the Esscher transform, the θ value needed to obtain the martingale measure can be found by solving

$$\kappa(\theta + 1) - \kappa(\theta) = r_f.$$

See Schoutens (2003).

- In our mortality process model, $\kappa(\theta)$ is known and we solve $\kappa(\theta+1)-\kappa(\theta)=a_x$.
- Example: By using mortality rate data of 65 to 69 years old male in England and Wales from 1841 to 2006, the estimation results are

$$
\begin{aligned}
&(\alpha', \beta', \mu', \delta', a_{[65,69]}, \theta) \\
=&(7.2093, -0.5786, -0.0003, 0.0039, -6.3177, -0.0070).
\end{aligned}
$$

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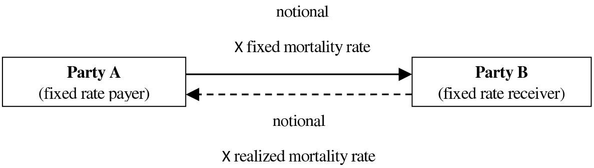
Application: Pricing a Q-forward



- A q-forward exchanges fixed mortality for realized mortality at maturity of the contract.
- The LLMA (Life & Longevity Markets Association)'s structure can be used for pricing. The related documents are available at `http://www.llma.org/publications.html`

LLMA Pricing Structure

Party A is an investment bank. Party B is a pension fund who wants to hedge longevity risk. At maturity, the bank (party A) receives

$$\text{notional amount} \times (m_{\text{realized}} - m_{\text{fixed}}),$$

and the pension fund (party B) receives

$$\text{notional amount} \times (m_{\text{fixed}} - m_{\text{realized}}).$$

To price the product at time 0 before the realized rate is known, the bank will construct a modified mortality rate with an adjustment for risk premium and an adjustment in mortality rate for expected mortality rate movement. Therefore, at maturity, the bank receives

$$\text{notional amount} \times (m_{\text{modified}} - m_{\text{fixed}}).$$

LLMA Pricing Structure

Given discount rate r and contract duration t years, the present value of settlement is

$$\text{notional amount} \times (\mathrm{m}_{\text{modified}} - \mathrm{m}_{\text{fixed}})/(1+r)^t.$$

The modified rate is

$$\mathrm{m}_{\text{modified}} = m_0 \times (1 - (\mathrm{m}_{\text{predicted}} + \xi))^t,$$

where ξ is an adjustment for risk premium. The predicted rate can be obtained by mortality modeling. LLMA suggests an average predicted rate across t years in an age group or desired ages, denoted as $\hat{m}$. Therefore, the adjustment value ξ is the smallest value to solve

$$\mathrm{m}_{\text{fixed}} \leq \mathrm{m}_0(1 - (\hat{m} + \xi))^t.$$

Example: JPMorgan Q-forward

- The notional amount is 50 million GBP.
- 10-year contract (from 12/31/2006 to 12/31/2016).
- The reference group is males in England & Wales who will be 65 to 69 years old in 2016.
- Fixed rate is 1.2%, given by contract.
- Discount rate is 4.37 % (T-bill rate).

Goal: Solve for the adjustment for risk premium ξ in the previous equation.

- Data: death rate for the age group 65-69, 1841-2006
- The predicted mortality rate is calculated from the modified Lee-Carter model with NIG Lévy process and Esscher transform.
- Results:
  - The average predicted mortality rate for age group 65 to 69 is $\hat{m} = 1.70\%$.
  - **The adjustment for risk premium value is ξ=2.08%.**
  - **The present value of settlement is 938,601.38 GBP.**

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