Modeling Trades in the Life Market as Nash Bargaining Problems

Rui Zhou Johnny Li Ken Seng Tan

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Outline

Existing pricing methods for mortality-linked securities

No arbitrage approaches

- Cairns et al. (2006), Chen and Cox (2009), Li and Ng (2011)
- require market prices of other products
- need a criterion to select a unique risk-neutral measure

Other approaches

- Zhou et al. (2010, 2011)
  - a gradual calibration of supply and demand
  - assume a competitive market
- Bonnen et al. (2011)
  - model risk redistribution between life insurers and pension funds by a bargaining game
  - assume that trade is fully customized

Our objectives

Model a trade by a Nash bargaining game

- Model a trade between a mortality/longevity risk hedger and an investor
- Participants negotiate price and quantity of mortality-linked securities
- Apply two-player Nash bargaining solution to the trade

Features

- Avoid the difficulties of no-arbitrage approaches
- No requirement for competitiveness
- Fixed structure of the hedging instrument
- Easy to implement

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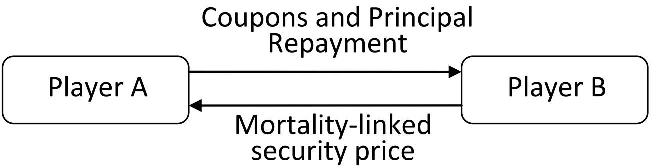
A Multi-Period Mortality-Linked Security



- A: hedger with life contingent liabilities
- B: investor

Notations

- Payments occur at $t = 1, \ldots, T$, where T is maturity time
- f_t: Life contingent liabilities at time t
- g_t: Payouts from each unit of the mortality-linked security at time t
- θ: trading quantity
- P: trading price
- ω^A and ω^B: the initial wealths of Players A and B
- r: continuously compounding risk-free interest rate
- U^A and U^B: utility functions for Players A and B

Assumptions

- Trading is only permitted at time 0
- Two investment choices
  - the mortality-linked security
  - lend/borrow at the same risk-free interest rate r
- Homogeneous believes on the future mortality dynamics

Wealth Process

$$\text{Player A} \quad \begin{aligned} W_0^A &= \omega^A + \theta P \\ W_1^A &= W_0^A e^r - \theta g_1 - f_1 \\ W_t^A &= W_{t-1}^A e^r - \theta g_t - f_t \end{aligned}$$

$$\text{Player B} \quad \begin{aligned} W_0^B &= \omega^B - \theta P \\ W_1^B &= W_0^B e^r + \theta g_1 \\ W_t^B &= W_{t-1}^B e^r + \theta g_t \end{aligned}$$

Terminal Wealth

$$\begin{aligned} W_T^A &= (\omega^A + P\theta)e^{rT} - \theta \sum_{t=1}^{T} g_t e^{r(T-t)} - \sum_{t=1}^{T} f_t e^{r(T-t)} \\ &= (\omega^A + P\theta)e^{rT} - \theta G - F \end{aligned}$$

$$\begin{aligned} W_T^B &= (\omega^B - P\theta)e^{rT} + \theta \sum_{t=1}^{T} g_t e^{r(T-t)} \\ &= (\omega^B - P\theta)e^{rT} + \theta G \end{aligned}$$

- Terminal utility payoffs: $U^A(W_T^A)$ and $U^B(W_T^B)$

Two-player Nash bargaining (Nash, 1950)

- Model a two-player bargaining game by a pair (S, d)
- S: the set of feasible expected utility payoffs to the players
- s: a typical element in S, and $s = (s_1, s_2)$
- $d = (d_1, d_2)$: the disagreement payoff, and $d \in S$
- Results of bargaining:
  - agree on a point $y = (y_1, y_2)$ in S: resulting utility payoffs to the two players are y_1 and y_2
  - no agreement: the players receive d_1 and d_2, respectively

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Nash bargaining solution

A solution intended to model bargaining among rational players should possess the following properties:

1. Pareto optimality
2. Symmetry
3. Independence of irrelevant alternatives
4. Independence of equivalent utility representatives

Nash bargaining solution

- Assume that S contains at least one point s such that $s > d$.
- There exists a unique solution which possesses Properties 1-4.
- This solution is the same with that solves the problem

$$\max_{(y_1,y_2)} \quad (y_1 - d_1)(y_2 - d_2)$$
$$\text{subject to } (y_1, y_2) \in S \text{ and } (y_1, y_2) \geq (d_1, d_2)$$

Bargaining power

$$\max_{(y_1, y_2)} \quad (y_1 - d_1)^a (y_2 - d_2)^{(1-a)}$$
$$\text{subject to } (y_1, y_2) \in S \text{ and } (y_1, y_2) \geq (d_1, d_2)$$

- Equal bargaining power: $a = 0.5$
- Player 1 has greater bargaining power: $a > 0.5$

Application to mortality-linked security pricing

- A trading contract: (P, θ)
- $W_T^A(P, \theta)$ denotes value of W_T^A given price P and quantity θ
- $d_1 = E\left[U^A(W_T^A(0, 0))\right]$
- $d_2 = E\left[U^B(W_T^B(0, 0))\right]$
- For a trading contract (P, θ),
  - $y_1 = E\left[U^A(W_T^A(P, \theta))\right]$
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The Nash bargaining solution can be expressed as

$$\max_{(P,\theta)} \quad \left\{E\left[U^A(W_T^A(P,\theta))\right] - E\left[U^A(W_T^A(0,0))\right]\right\} \times \left\{E\left[U^B(W_T^B(P,\theta))\right] - E\left[U^B(W_T^B(0,0))\right]\right\}$$

subject to

$$E\left[U^A(W_T^A(P,\theta))\right] - E\left[U^A(W_T^A(0,0))\right] \geq 0$$

$$E\left[U^B(W_T^B(P,\theta))\right] - E\left[U^B(W_T^B(0,0))\right] \geq 0$$

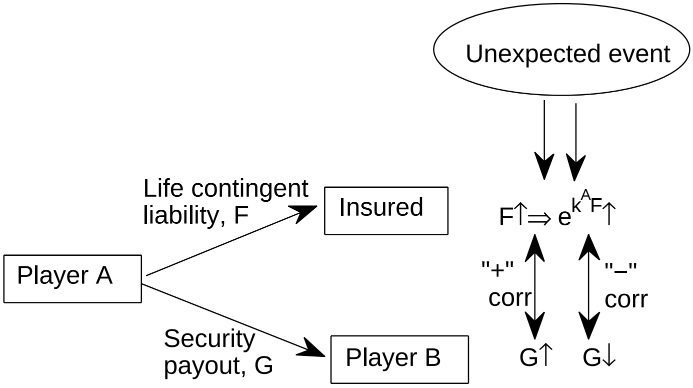
$$\theta \geq 0$$

$$P > 0$$

Findings

Assuming exponential utility functions for both players, there exists s in S such that $s > d$, if and only if $corr(e^{k^A F}, G) < 0$.

- $F = \sum_{t=1}^{T} f_t e^{r(T-t)}$, accumulated value of life contingent liabilities
- $G = \sum_{t=1}^{T} g_t e^{r(T-t)}$, accumulated value of security payouts.
- k^A is the risk aversion parameter for Player A, and $k^A > 0$


Unexpected event
Life contingent liability, F
Insured
Player A
Security payout, G
Player B
F↑⇒ e^{k^A}F↑
"+" corr
"−" corr
G↑
G↓

Pareto Optimality

Suppose both players have exponential utility functions.
When $corr(e^{k^A F}, G) \geq 0$, the trading contract, (P, θ), is pareto optimal if and only if $\theta = 0$.

Findings

- When $corr(e^{k^A F}, G) < 0$, the trading contract, (P, θ), is pareto optimal if and only if

$$\frac{E[e^{k^A \theta G + k^A F} G]}{E[e^{k^A \theta G + k^A F}]} - \frac{E[e^{-k^B \theta G} G]}{E[e^{-k^B \theta G}]} = 0. \qquad (1)$$

- Equation (1) has a unique solution when $corr(e^{k^A F}, G) < 0$.

Conclusion

- Model the trade between a mortality/longevity risk hedger and an investor by a two-player Nash bargaining game
- Provide a unique pair of price and quantity for the trade
- Allowing negotiation fits current market
- Can be used to price standardized hedging instrument

Future Research Plans

Pricing mortality-linked securities

- Multi-player Nash Bargaining game
- Noncooperative bargaining processes and realistic features, such as information asymmetry

Thanks!