Lifetime Dependence Modelling using a Truncated Multivariate Gamma Distribution

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Plan

- Introduction
- Dependent Lifetimes and a Multivariate Gamma Distribution
- Parameter Estimation and Truncation
- Applications and Fitting Population Data
- Conclusion

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Introduction

- Motivation: Assess impact of dependent lifetimes on annuity valuation and risk management.
  - Underlying assumption is that systematic mortality improvements induce dependence.
- Investigate a multivariate gamma distribution for two reasons.
  - The gamma distribution has been applied to single lifetimes.
  - Induces dependence exactly in the manner we envision, namely, it categorizes mortality into systematic and idiosyncratic components.
- Theoretical contribution: resolve parameter estimation in the presence of truncation.
- Practical contribution: provide evidence that dependence plays a significant role in pricing and risk management of bulk annuities.
- Future research: improve fit to data and generalize model.

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Modelling Dependent Lifetimes

- We have M pools of N lives, the lives within a pool are dependent.
- Let $T_{i,j}$ be the life time of individual $i \in \{1, \ldots, N\}$ in pool $j \in \{1, \ldots, M\}$.

We suppose the following model for lifetimes:

$$T_{i,j} = \frac{\alpha_0}{\alpha_j} Y_{0,j} + Y_{i,j},$$

$Y_{i,j} \sim G(\gamma_j, \alpha_j)$ i.i.d. $(i \neq 0)$, $Y_{0,j} \sim G(\gamma_0, \alpha_0 \equiv 1)$ independent of all $Y_{i,j}$.

- The intuition for the construction is that there is a common component $\frac{\alpha_0}{\alpha_j} Y_{0,j}$, representative of systematic mortality.
  - The value of $Y_{0,j}$ impacts each life in pool j.
- The individual component $Y_{i,j}$ is representative of idiosyncratic mortality.
  - The parameters α_j, γ_j that govern this component describe the general risk characteristics of the pool.

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Parameter Estimation for one Pool

We estimate the parameters α_j, γ_j and predict $Y_{0,j}$ using the method of moments.

- Within pool j, the central moments of $T_{i,j}$ are equal to the central moments of the idiosyncratic component $Y_{i,j}$.

$$
\begin{aligned}
\widetilde{m}_2(\mathbf{T}_j) &= \frac{1}{N-1}\sum_{i=1}^{N}(T_{i,j} - a_1(\mathbf{T}_j))^2 \\
&= \frac{1}{N-1}\sum_{i=1}^{N}\left(\frac{\alpha_0}{\alpha_j}Y_{0,j} + Y_{i,j} - \frac{\alpha_0}{\alpha_j}Y_{0,j} - a_1(\mathbf{Y}_j)\right)^2 \\
&= \frac{1}{N-1}\sum_{i=1}^{N}(Y_{i,j} - a_1(\mathbf{Y}_j))^2 = \widetilde{m}_2(\mathbf{Y}_j),
\end{aligned}
$$

where $a_1()$ is the first raw moment, and $\widetilde{m}_2()$ the unbiased second central moment of samples $\mathbf{T}_j$ and $\mathbf{Y}_j$.

Method of Moments

We obtain the following unbiased estimates of α_j, γ_j:

$$\begin{aligned}\widehat{\alpha}_j &= 2\frac{\widetilde{m}_2(\mathbf{T}_j)}{\widetilde{m}_3(\mathbf{T}_j)},\\ \widehat{\gamma}_j &= 4\frac{\widetilde{m}_2^3(\mathbf{T}_j)}{\widetilde{m}_3^2(\mathbf{T}_j)},\end{aligned}$$

In order to predict $Y_{0,j}$, we use the first raw moment,

$$a_1(\mathbf{T}_j) = \frac{1}{N}\sum_{i=1}^{N} T_{i,j} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\alpha_j}Y_{0,j} + \frac{1}{N}\sum_{i=1}^{N} Y_{i,j} = \frac{1}{\alpha_j}Y_{0,j} + a_1(\mathbf{Y}_j),$$

and the fact that for $N \to \infty$,

$$a_1(\mathbf{T}_j) \xrightarrow{P} \frac{1}{\alpha_j}Y_{0,j} + \frac{\gamma_j}{\alpha_j},$$

to produce predictor:

$$\widehat{Y}_{0,j} = a_1(\mathbf{T}_j)\widehat{\alpha}_j - \widehat{\gamma}_j.$$

Parameter Estimation in the Presence of Truncation

Our interest is on the effect of dependence on annuitants, where the focus is typically on retirement planning (e.g. ages 60+).

We can

- translate observations, or
- truncate observations.

Given the nature of the gamma distribution (behaviour of density at zero), the latter solution is much better suited to produce meaningful results.

The presence of truncation complicates matters, but we can still apply the same principle as before in order to obtain a non-linear system of equations.

- We assume a uniform truncation point across pools, $\tau_j \equiv \tau$.

Truncation Adjustment Coefficient

In order to apply the method of moments, we require a means of adjusting moments to accommodate the effects of truncation.

Lemma (The Truncation Adjustment Coefficient)

Consider $Y \sim G(\gamma, \alpha)$ *with probability density and survival function denoted* $g(y, \gamma, \alpha)$ *and* $\overline{Ga}(y, \gamma, \alpha)$*, respectively. Define associated truncated random variable* ${}_{\tau}Y = Y | Y > \tau$*, where* $\tau \geq 0$*. The* k^{th} *raw moment,* $k \in \mathbb{Z}^{+}$*, of* ${}_{\tau}Y$ *is given by*

$$\alpha_k({}_{\tau}Y) = \alpha_k(Y) K_k(\tau, \gamma, \alpha),$$

where

$$K_k(\tau, \gamma, \alpha) = \frac{\overline{Ga}(\tau, \gamma + k, \alpha)}{\overline{Ga}(\tau, \gamma, \alpha)}.$$

Truncation: The Simplified Case

If we proceed as before, the resulting system of non-linear equations is unstable and cannot be solved using either iterative or numerical methods.

We have to simplify the assumptions.

- Assume all pools share the same risk characteristics.
  - $\alpha_j \equiv \alpha$ and $\gamma_j \equiv \gamma$.

Note that this does not imply that all pools have the same dependence.

Step 1: Work with the global sample, ${}_\tau\mathbf{T}$, to estimate α.

$$\begin{aligned} E[a_1({}_\tau\mathbf{T})] &= \alpha_1({}_\tau T_{1,1}) = \frac{\tilde{\gamma}}{\alpha} K_1(\tau, \tilde{\gamma}, \alpha), \\ E[a_2({}_\tau\mathbf{T})] &= \alpha_2({}_\tau T_{1,1}) = \frac{\tilde{\gamma}(\tilde{\gamma}+1)}{\alpha^2} K_2(\tau, \tilde{\gamma}, \alpha), \end{aligned}$$

where $\tilde{\gamma} = \gamma_0 + \gamma$. Using an iterative algorithm, we obtain $\widehat{\alpha}$.

Truncation: The Simplified Case

Step 2: Armed with $\widehat{\alpha}$, we consider individual pool j to obtain an estimate of γ and to predict $Y_{0,j}$.

$$
\begin{aligned}
E[a_1({}_{\tau}\mathbf{T}_j)|Y_{0,j}] &\approx \frac{1}{\widehat{\alpha}}Y_{0,j} + \frac{\gamma}{\widehat{\alpha}}K_1(\tau', \gamma, \widehat{\alpha}),\\
E[\widetilde{m}_2({}_{\tau}\mathbf{T}_j)|Y_{0,j}] &\approx \frac{\gamma(\gamma+1)}{\widehat{\alpha}^2}K_2(\tau', \gamma, \widehat{\alpha}) - \frac{\gamma^2}{\widehat{\alpha}^2}K_1(\tau', \gamma, \widehat{\alpha})^2.
\end{aligned}
$$

We are (again) presented with a non-trivial system of equations. This time, both an iterative and a numerical solution can be obtained.

- $\widehat{\gamma}^{(j)}$ and $\widehat{Y}_{0,j}$ from an iterative algorithm.
- $\widehat{\gamma}^{(j,BB)}$ and $\widehat{Y}_{0,j}^{(BB)}$ from the Barzilai-Borwein numerical procedure.

Step 3: We average the estimates from each pool to obtain $\widehat{\gamma}$, and $\widehat{\gamma}_0$.

Parameter Estimation: Calibration Results

In the absence of truncation, parameter estimation is identical to calibrating a translated gamma distribution.

In the presence of truncation, we obtain the following:

Simulation	1	2	3	4
N	1,000	100,000	10,000	1,000
M	1	1	50	1,000
τ	60	60	60	60
α	0.500	0.500	0.500	0.500
$\widehat{\alpha}$	0.522	0.703	0.524	0.497
γ	30.000	30.000	30.000	30.000
$\widehat{\gamma}$	26.366	59.966	32.407	28.692
$\widehat{\gamma}^{(BB)}$	32.991	59.966	33.415	29.875
Y_0 / γ_0	2.191	12.680	5.000	5.000
$\widehat{Y}_0$ / $\widehat{\gamma}_0$	7.741	0.000	4.421	6.415
$\widehat{Y}_0$ / $\widehat{\gamma}_0{}^{(BB)}$	-0.116	-0.001	3.179	4.949

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The Actuarial Present Value of Annuities

- Let ${}_\tau A_j$ denote the value of a bulk annuity sold to members (aged τ) in pool j at time $t = 0$.
- The annuity pays \$1 at the *end* of each year to the surviving members.

$$ {}_\tau A_j = \sum_{t=1}^{\infty} {}_\tau S_{t,j} v^t, $$

where $v = e^{-\delta}$, the discount factor with constant force of interest δ, and ${}_\tau S_{t,j}$ the distribution of joint survival.

Annuity Valuation: Numerical Results

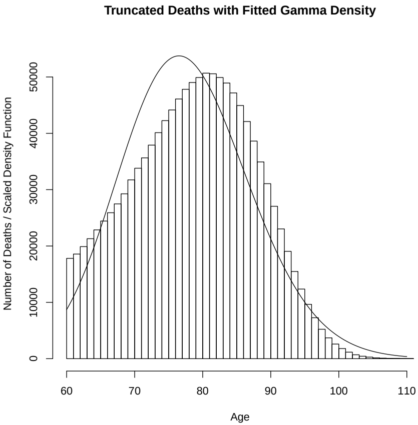
We assume the following parameter values:

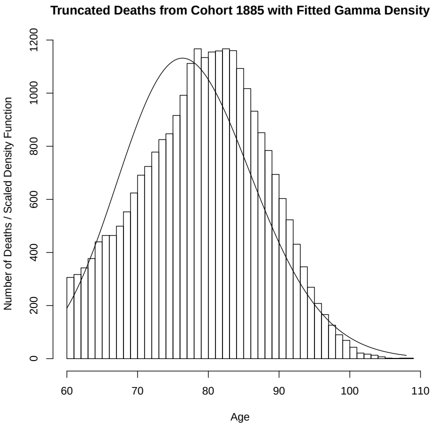
- $\alpha = 0.5$, $\gamma = 30$, $\gamma_0 = 10$
- $\tau = 60$, $\delta = 2\%$.

N		1	10	100
Theoretical results				
$E[_{\tau}A_j]$	MVG	15.73	157.24	*1,482.70*
	Ind.	15.73	157.32	1,573.20
Simulation results				
M (000's)		10	10	10
Mean	MVG	15.75	157.40	1,572.08
	Ind.	15.75	158.95	1,589.29
Standard	MVG	7.51	41.03	356.22
Deviation	Ind.	7.51	23.54	75.00

Fitting Norwegian Population Data

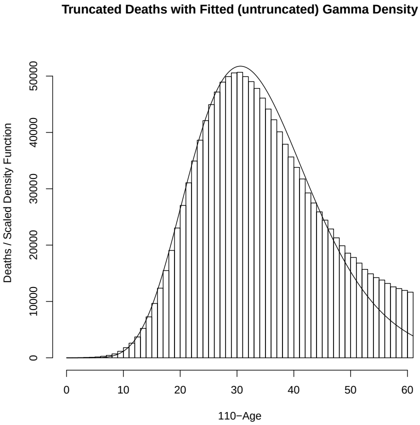
- Human Mortality Database:
  - Use cohort data from birth years 1846-1898 (53 pools).
  - We transform rates into *crude* lifetimes.
  - With a truncation point of 60, we have 1,234,957 deaths.

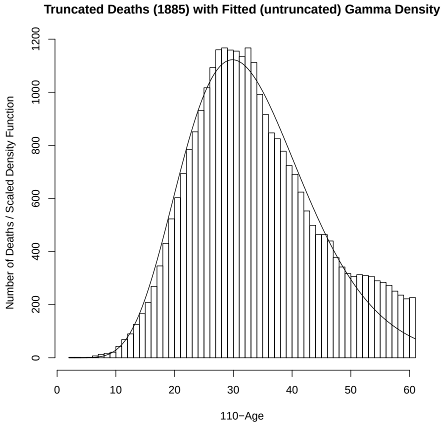




Fitting Norwegian Data: An Adjustment

- Consider the following adjustment to data $t_{i,j}$: $t'_{i,j} = \omega - t_{i,j}$.
  - Maximum attainable age is ω.
  - Data is now right truncated.
  - Extend the model to allow for translation and right truncation.





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Thank you!