The Impact of Long Memory in Mortality Differentials on Index-based Longevity Hedges

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Overview

Mortality differentials and population basis risk

- Population basis risk arises from the difference in mortality improvements (mortality differentials) between populations.
- In an index-based longevity hedge, population basis risk exists between the hedger's own population and the population to which the hedging instrument is linked.
- The effectiveness of an index-based hedge highly depends on the level of population basis risk involved.
- How to accurately capture population basis risk in index-based longevity hedging remains an open research question.

The augmented common factor (ACF) model

Let $m_{x,t}^{(i)}$ be the i-th population's central death rate at age x in year t. The ACF model specifies

$$\ln(m_{x,t}^{(i)}) = a_x^{(i)} + B_x K_t + b_x^{(i)} k_t^{(i)} + \epsilon_{x,t}^{(i)}, \quad i = 1, \ldots, P,$$

where

- $a_x^{(i)}$, B_x and $b_x^{(i)}$ are age-specific parameters,
- K_t is the common time-varying index shared by all populations, and
- $k_t^{(i)}$ is a population-specific time-varying index, representing the mortality differentials from the common trend.

Capturing population basis risk

To capture the evolution of $k_t^{(i)}$ over time, a short memory stochastic process is often used.

- An AR(1) process:

$$k_t^{(i)} = \phi_0 + \phi_1 k_{t-1}^{(i)} + \epsilon_t^{(i)},$$

- An ARMA(P, Q) process:

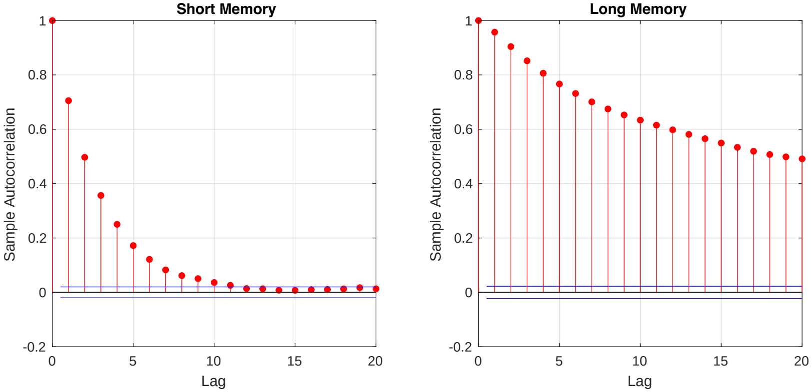
$$k_t^{(i)} = \phi_0 + \sum_{s=1}^{P} \phi_s k_{t-s}^{(i)} + \sum_{s=1}^{Q} \theta_s \epsilon_{t-s}^{(i)} + \epsilon_t^{(i)}$$

Question: Is a short memory process adequate for capturing the dynamics of $k_t^{(i)}$ over time?

The objectives

- Show the existence of long memory in mortality differentials and the necessity of using a long memory process.
- Study the modeling implications of long memory:
  - The rate of mean reversion
  - The level of forecast uncertainty
- Study the hedging implications of long memory:
  - The impact of population basis risk
  - The robustness of hedging strategies

What is long memory?



The autoregressive fractionally integrated moving average (ARFIMA) process

The ARFIMA process:

$$\Phi(B)(1-B)^{2d}k_t^{(i)} = \Theta(B)\epsilon_t^{(i)}, \quad \epsilon_t^{(i)} \sim \mathrm{N}(0,\sigma^2),$$

where B is the backshift operator, $\Phi(B)$ is the autoregressive operator, $\Theta(B)$ is the moving-average operator, and d is the fractional difference parameter.

- If $d = 0$, the ARFIMA process reduces to an ARMA process.
- If $d = 0.5$, the ARFIMA process becomes a first-order ARIMA process.
- If $d \in (0, 0.5)$, the ARFIMA process is said to have long memory.

Existing studies and results

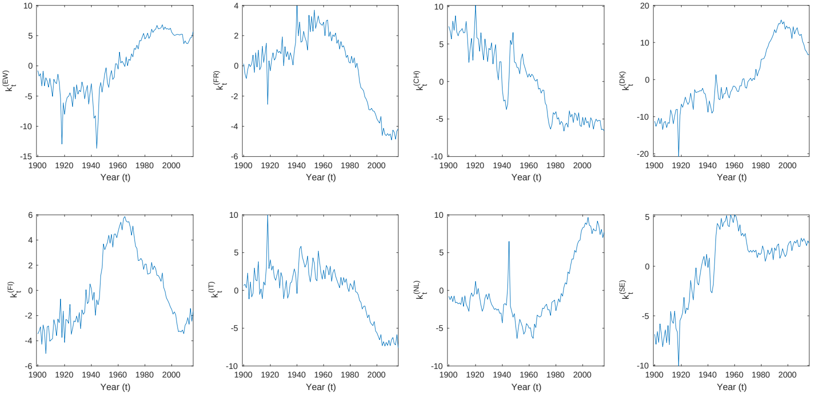
- On continuous-time mortality modeling
  - Wang et al. (2021) proposed the Volterra mortality model with long-range dependence for actuarial valuation and risk management.
  - Wang and Wong (2021) developed a time-consistent mean-variance longevity hedge based on the Volterra model.
- On discrete-time mortality modeling
  - Yan et al. (2021) empirically showed the existence of long memory in age-specific mortality rates from 16 countries, and developed a long memory mortality model based on GLM.
  - Yan et al. (2020) extended their long memory model to multivariate mortality modeling with cohort effects.

The above studies all focused on long memory in mortality rate or intensity, but not on mortality differentials.

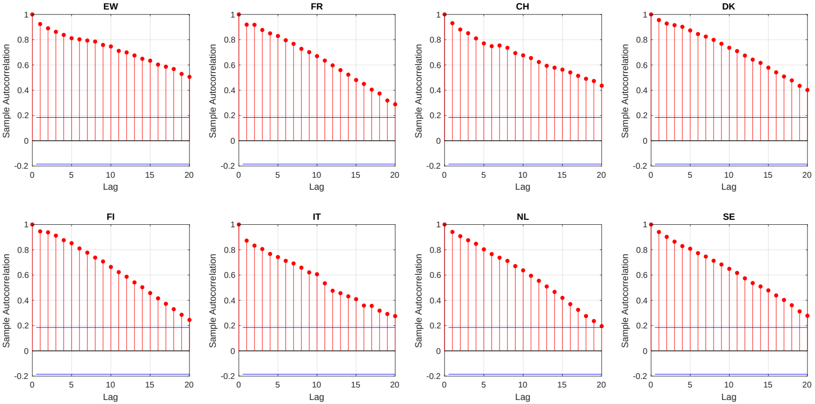
Data

Country	Calibration window	Age range	Gender
EW	1900-2016	40-89	Female
France	1900-2016	40-89	Female
Switzerland	1900-2016	40-89	Female
Denmark	1900-2016	40-89	Female
Finland	1900-2016	40-89	Female
Italy	1900-2016	40-89	Female
Netherlands	1900-2016	40-89	Female
Sweden	1900-2016	40-89	Female

The estimated values of $k_t^{(i)}$



The sample ACF of $k_t^{(i)}$



The modified R/S test statistics of $k_t^{(i)}$

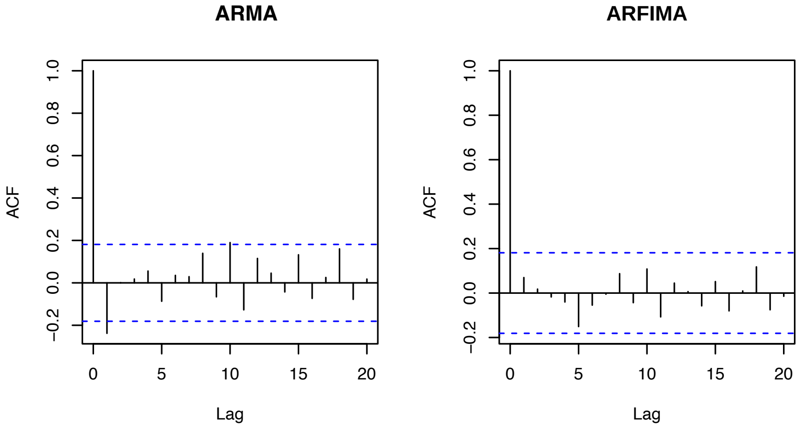
# of Lags	0	1	3	5
UK	4.677(***)	3.361(***)	2.422(***)	2.009(***)
France	4.165(***)	3.031(***)	2.172(***)	1.780(**)
Switzerland	4.604(***)	3.343(***)	2.412(***)	1.986(***)
Denmark	4.538(***)	3.246(***)	2.323(***)	1.911(***)
Finland	4.731(***)	3.383(***)	2.417(***)	1.998(***)
Italy	4.074(***)	2.997(***)	2.176(***)	1.799(**)
Netherlands	4.174(***)	2.980(***)	2.142(***)	1.782(**)
Sweden	4.406(***)	3.138(***)	2.259(***)	1.886(***)

Table: The critical values at 2.5%, 5% and 10% significance levels are 1.862, 1.747 and 1.620, respectively (Lo, 1991).

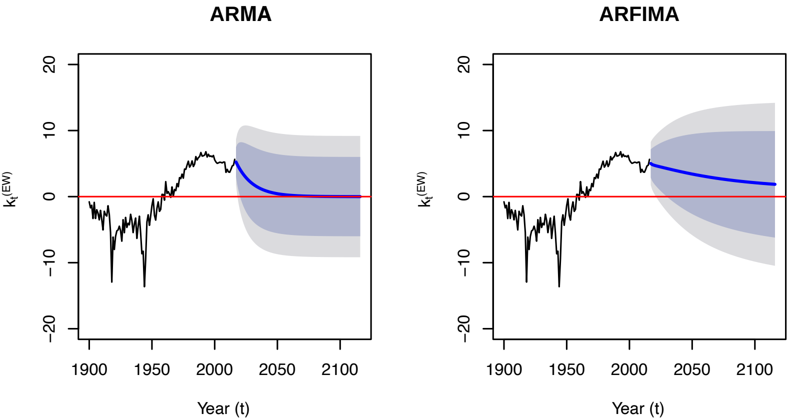
Parameter estimates for $k_t^{(EW)}$

Process	Para.	Estimate	Std. Error	p-Value	BIC
ARMA	ϕ_1	0.9277	0.0325	<2.2e-16	473.28
	σ^2	3.0570	N/A	N/A	
ARFIMA	d	0.4045	0.0053	<2.2e-16	471.83
	ϕ_1	0.9672	0.0694	<2.2e-16	
	θ_1	0.8113	0.0287	<2.2e-16	
	σ^2	2.7694	N/A	N/A	

The sample ACF of residuals



Forecast results



Hedge assumptions

- The liability being hedged is a 20-year deferred 30-year term annuity issued to an individual from population $i =$ EW or Denmark.
- The hedging instrument is a S-forward linked to the EW population and freshly launched at time 0.
- The hedge is established at time 0, and calibrated as either a delta hedge or a variance hedge.
- The hedger assumes that K_t follows a random walk with drift, and $k_t^{(i)}$ follows either an ARMA process or an ARFIMA process.
- The hedge effectiveness is measured by the reduction in cash flow variance resulted from a hedge.

Hedge overview

- Two populations:
  1. EW (with no population basis risk)
  2. Denmark (with population basis risk)
- Two hedges:
  1. Delta-neutral hedge
  2. Variance-minimizing hedge
- Two processes:
  1. ARMA process (short memory)
  2. ARFIMA process (long memory)

Case I: Under-estimation of population basis risk

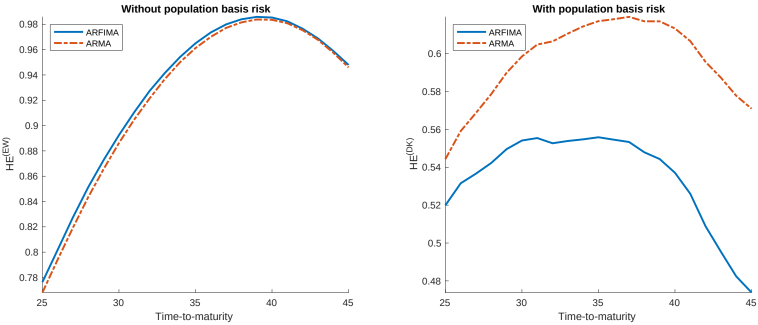


Figure: Hedge effectiveness of variance-minimizing hedge for the EW and Denmark populations under ARMA and ARFIMA processes.

Case I: Under-estimation of population basis risk

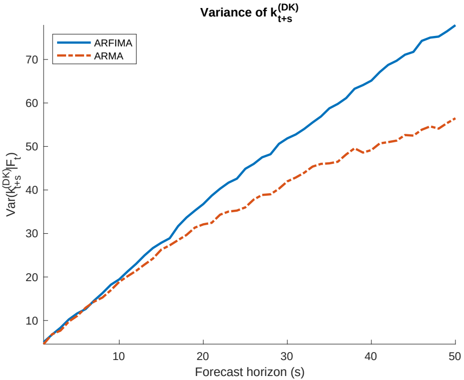


Figure: Variance of the simulated values of $k_t^{(DK)}$ under ARMA and ARFIMA processes.

Case II: Robustness of delta hedge

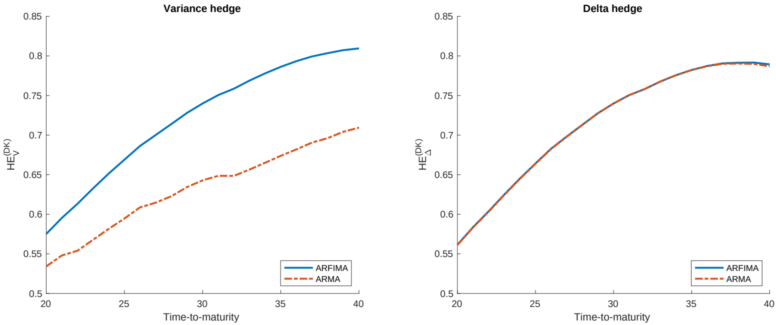


Figure: Hedge effectiveness under variance hedge (left) and delta hedge (right) for the Denmark population.

Case II: Robustness of delta hedge

The robustness of delta hedge can be explained by examining how the hedge ratios are calculated under the two hedging strategies.

- Variance hedge:

$$\frac{\mathrm{Cov}\left(\mathcal{L}^{(DK)}(K_{t+}, k_{t+}^{(DK)}), \mathcal{H}^{(EW)}(K_{t+}, k_{t+}^{(EW)})\right)}{\mathrm{Var}\left(\mathcal{H}^{(EW)}(K_{t+}, k_{t+}^{(EW)})\right)}$$

- Delta hedge:

$$\mathrm{E}\left[\frac{\partial}{\partial K_t}\mathcal{L}^{(DK)}(K_{t+}, k_{t+}^{(DK)})\right] \Bigg/ \mathrm{E}\left[\frac{\partial}{\partial K_t}\mathcal{H}^{(EW)}(K_{t+}, k_{t+}^{(EW)})\right]$$

Conclusion

- Long memory is statistically present in mortality differentials, and needs to be captured by a long memory process.
- Long memory in mortality differentials will lead to
  1. A slower rate of mean reversion, and
  2. A higher level of forecast uncertainty.
- When long memory is present but short memory is assumed, we observe that
  1. Population basis risk will be underestimated, and
  2. Delta hedge is more robust than variance hedge.

Thank You!

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