Stochastic Lifestyling with Uncertain Mortality Rates
How Longevity Risk can Impact the Replacement Ratio

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Outline

Introduction

- Our objective is to highlight the importance of accounting for longevity risk in the context of DC pension plan design.
- Why *stochastic lifestyling*?
  - The stochastic lifestyling model, derived by Cairns, Blake and Dowd (2006), is an optimal dynamic asset allocation model for DC pension plans.
  - The model assumes that DC plan members derive utility from the replacement ratio as opposed to a cash lump sum.
  - This type of assumption underpins most contemporary DC plan design proposals.

Introduction (contd.)

- Where does longevity risk fit in?
  - Longevity risk was not considered in the original derivation of the stochastic lifestyling model.
  - The annuity rate used in the definition of the replacement ratio was assumed to depend solely on the interest rate.
  - We introduce a force of mortality state variable into the model framework, adopting the stochastic mortality model of Dahl and Møller.
  - We gauge the impact of uncertain mortality rates through the demand for, and utility gained from, a synthetic mortality linked security.

Problems with the traditional ‘design’

- DC plans are not explicitly geared towards delivering a standard of living in retirement.
- In the absence of this mandate they are essentially reduced to merely (tax favored) savings accounts, or mutual funds.
- It is largely left to the individual plan members to determine whether their pensions are appropriately funded and what action to take in the event that they are not.
- This involves a willingness to actively participate, and a level of financial understanding, that few pension plan members possess.

A plan member oriented design

- A potential solution is perhaps obvious
  - Embed the ‘standard of living’ assumption in the design of the plan i.e. The plan should be designed “from desired outputs to required inputs” (Blake et al., 2009).
- The principal here is that the plan is designed with an intrinsic optimal dynamic asset allocation model which is sensitive to the ‘desired output’.
  - Once the desired standard of living is determined the asset allocation model determines the optimal investment strategy.
- In order for this model to be considered truly robust it must take into account all uncertainty associated with the investment return, the fund contributions, and the desired standard of living (in however it is quantified).

The stochastic lifestyling model

- The stochastic lifestyling model presents a framework for a dynamic asset allocation model driven by a standard of living in retirement.
- The model utilizes Merton's dynamic programming portfolio optimization approach.
- The objective function is set as a function of the replacement ratio (as opposed to the nominal fund wealth).
- The replacement ratio is measured in terms of the number of annuities paying the plan member's salary immediately prior to retirement.
- The resulting asset allocation strategy resembles the traditional lifestyling strategy, in that the general trend involves a gradual (stochastic) switch from 'risky' assets to 'safe' assets.

Systematic longevity risk and the annuity decision

- The stochastic lifestyling model implicitly assumes that the plan member wishes to hedge all exposure to unsystematic longevity risk through the purchase of annuity.
- The model does not take into account systematic longevity risk.
  - The price of the annuity at retirement is uncertain prior to retirement due, in part, to systematic longevity risk.
- The restriction of full annuitization may seem stringent in the traditional economic context of maximizing utility over the life-cycle.
- In the context of pension plan design, however, we are merely catering to those who value this decision.

Hedging longevity risk over the life-cycle

- The majority of studies in this area are concerned with hedging unsystematic longevity risk
  - e.g. Milevsky and Young (2007), Horneff et al. (2008), Koijen et al. (2011).
- The lack of a developed market in mortality-linked securities presents a barrier to the consideration of systematic mortality risk in this respect.
- Menoncin (2008), and Cocco and Gomes (2012), introduce synthetic mortality-linked securities, namely 'longevity bonds', into the asset mix; an approach we too adopt.
- As such, our work can be viewed as an emphasis on the importance of the development of a strong market in similar types of securities.

Discrete Vs Continuous time

- Discrete-time stochastic mortality models are considerably more popular than continuous-time models
  - e.g. Lee and Carter, Renshaw and Haberman, Cairns, Blake and Dowd.
- This is due to both their flexibility, and the ease to which they can be adapted for numerical simulation.
- In order to work within the stochastic lifestyling framework we require a continuous-time stochastic mortality model.
- We focus on the class of affine mortality models due to their analytical tractability; in particular, we focus on the model of Dahl & Møller (2006).

The general model of Dahl & Møller

- Dahl and Møller define the force of mortality as:

$$\lambda(x,t) = \lambda_o(x+t)\zeta(x,t)$$

  $\lambda_o(x+t)$ is the force of mortality for an individual aged $x+t$ at time t given by the initial force of mortality curve $\lambda_o(\tau)$.
  $\zeta(x,t)$ is the proportional change in the force of mortality for an individual aged $x+t$ over the time interval $(0,t)$.
- $\zeta(x,t)$ is referred to as the *mortality improvement process*, and is allowed to follow a Cox-Ingersoll-Ross process

$$\mathrm{d}\zeta(x,t) = (\theta - \delta\zeta(x,t))\mathrm{d}t + \sigma\sqrt{\zeta(x,t)}\mathrm{d}Z_\lambda(t)$$

- The force of mortality, as a result, follows a time-inhomogeneous CIR process

$$\mathrm{d}\lambda(x,t) = (\theta_\lambda(x,t) - \delta_\lambda(x,t)\lambda(x,t))\mathrm{d}t + \sigma_\lambda(x,t)\sqrt{\lambda(x,t)}\mathrm{d}Z_\lambda(t)$$

The general model of Dahl & Møller (contd.)

- The survival probability, $S(x,t,T)$, defined by

$$S(x,t,T) = \mathrm{E}\left[e^{-\int_t^T \lambda(x,\tau)\mathrm{d}\tau} \mid \mathcal{I}(t)\right]$$

is shown to be of the form

$$S(x,t,T) = e^{A(x,t,T) - B(x,t,T)\lambda(x,t)}$$

where

$$\frac{\partial}{\partial t}B(x,t,T) = \delta_\lambda(x,t)B(x,t,T) + \frac{1}{2}(\sigma_\lambda(x,t))^2 B(x,t,T)^2 - 1$$

$$\frac{\partial}{\partial t}A(x,t,T) = \theta_\lambda(x,t)B(x,t,T)$$

with $\quad B(x,T,T) = 0 \quad$ and $\quad A(x,T,T) = 0$

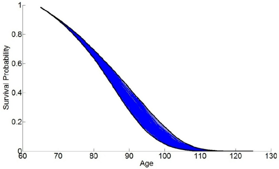
A closed form solution

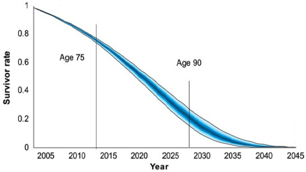
- If we define the initial mortality curve, $\lambda_o(\tau)$, by the Gompertz-Makeham law
$$\lambda_o(\tau) = \frac{1}{b} e^{\frac{1}{b}(\tau - m)}$$
it is possible to solve the coupled ODE in the previous slide explicitly.
- We find, in this case, that

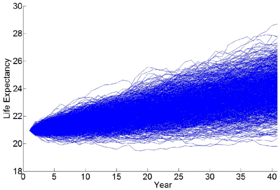
$$B(t,T) = \frac{2b}{z(t)} \left[\frac{I_{-\nu+1}(z(T))I_{\nu-1}(z(t)) - I_{\nu-1}(z(T))I_{-\nu+1}(z(t))}{I_{\nu-1}(z(T))I_{-\nu}(z(t)) - I_{-\nu+1}(z(T))I_{\nu}(z(t))} \right]$$

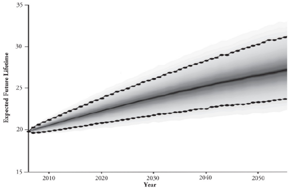
$$A(t,T) = -\frac{\theta\delta}{\sigma^2}(T-t) + \ln \left(\frac{I_{-\nu+1}(z(T))I_{\nu}(z(t)) - I_{\nu-1}(z(T))I_{-\nu}(z(t))}{I_{-\nu+1}(z(T))I_{\nu}(z(T)) - I_{\nu-1}(z(T))I_{-\nu}(z(T))} \right)^{\frac{\gamma\delta}{\sigma^2}}$$

How does the model compare?









The objective function

- The case of power utility

$$\frac{1}{\gamma}\left(\frac{W(T)}{Y(T)a(T,\zeta(T))}\right)^{\gamma}$$

where $(1-\gamma)$ is the coefficient of relative risk aversion.

$W(T) \sim$ Pension fund wealth at time retirement.

$Y(T) \sim$ Salary immediately prior to retirement.

$a(T,\zeta(T)) \sim$ Price of annuity paying one unit of cash annually from retirement until the plan member's death.

$$a(t,\zeta(t)) = e^{-r(T-t)} \sum_{\tau=T}^{\infty} e^{-r(\tau-T)} S(t,\tau)$$

The asset mix

- The 'risk-free' asset $R_0(t)$

$$R_0(t) = R_0(0)\exp(rt)$$

- The 'risky-asset' $R_1(t)$

$$\mathrm{d}R_1(t) = R_1(t)\Big[(r + \xi_1\sigma_1)\mathrm{d}t + \sigma_1\mathrm{d}Z(t)\Big]$$

- The mortality-linked asset $L(t, \zeta(t))$

$$L(t, \zeta(t)) = \sum_{\tau=T}^{\infty} e^{-r(\tau-t)} S^M(t, \tau)$$

where $S^M(t, \tau)$ is the martingale defined by Dahl & Møller

$$S^M(t, T) = \mathrm{E}\left[e^{-\int_0^T \lambda(\tau)\mathrm{d}\tau}\Big|\mathcal{I}(t)\right] = e^{-\int_0^t \lambda(\tau)\mathrm{d}\tau} S(t, T)$$

Wealth Dynamics

- The case where salary is fully hedgeable

$$\mathrm{d}Y(t) = Y(t)\Big[(r+\mu_Y)\mathrm{d}t + \sigma_{Y_1}\mathrm{d}Z(t)\Big]$$

- The ‘portfolio of longevity bonds’ follows

$$\mathrm{d}L(t,\lambda) = L(t,\zeta)\left[r\mathrm{d}t + \sigma\sqrt{\zeta}\lambda_o(t)\nabla_\lambda \mathrm{d}Z_\lambda(t)\right]$$

  where we have defined $\nabla_\lambda = \dfrac{1}{a(t,\lambda)}\dfrac{\partial}{\partial\lambda}a(t,\lambda)$ as the semielasticity of the annuity price w.r.t the force of mortality.

- The wealth dynamics are governed by

$$\mathrm{d}W(t) = W(t)\Big[\big(r + p(t)\xi_1\sigma_1\big)\mathrm{d}t + p(t)\sigma_1\mathrm{d}Z(t) + q(t)\nabla_\lambda\sigma\lambda_0(t)\sqrt{\zeta}\mathrm{d}Z_\lambda(t)\Big] + \pi Y(t)\mathrm{d}t$$

The optimal strategy

- Upon deriving the *Hamilton-Jacobi-Bellman equation* the optimal hedging strategy can be found by solving the first order condition for $q(t)$.
- The resulting strategy was found to be:

$$q^*(t) = 1 + \frac{V_x}{xV_{xx}} - \frac{1}{\lambda_0 \nabla_\lambda} \frac{V_{x\zeta}}{xV_{xx}}$$

- $V = V(t, x, \zeta)$ in this expression is the value function

$$V(t, x, \zeta) = \sup_{(p,q)\in\mathcal{Q}} \mathrm{E}_P \left[\frac{1}{\gamma} (X(T))^\gamma \middle| \mathcal{G}(t) \right]$$

$$\text{where } X(t) = \frac{W(t)}{Y(t)a(t, \zeta(t))}$$

The value function

- Since the value function is homogeneous in $X(t)$ the HJB equation can be reduced to a Black-Scholes type PDE, the solution of which has a Feynman-Kac representation.
- This, in conjunction with theorem 3.4.1 of Cairns et al. (2006), gives the following:

$$V(t,x,\zeta) = \frac{1}{\gamma}\left(x + \frac{\pi f(t)}{a(t,\zeta)}\right)^{\gamma} \exp\left\{\gamma\left(-\mu_Y + \xi_1\sigma_Y + \frac{1}{2}\frac{1}{(1-\gamma)}(\xi_1 - \sigma_Y)^2\right) \cdot (T-t)\right\} a(t,\zeta)^{\gamma} \mathrm{E}_P\left[a(t,\zeta(T))^{-\frac{\gamma}{1-\gamma}} \middle| \mathcal{G}(t)\right]^{1-\gamma}$$

- $\pi Y(t) f(t)$ is, as defined by Cairns et al., the market price at time t for the premiums payable between t and T.

Generating paths for $q^*(t)$

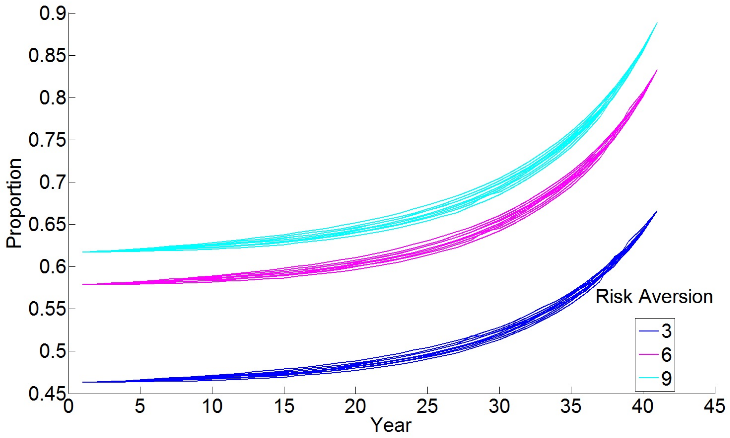
- Evaluating the partial derivatives of the value function as necessary, we can write the optimal hedging strategy as:

$$q^*(t) = \frac{1 + \frac{\pi Y(t) f(t)}{W(t)}}{\lambda_0(t)\nabla_\lambda} \frac{\frac{\partial}{\partial \zeta} \mathrm{E}_P\left[a(t,\zeta(T))^{-\frac{\gamma}{1-\gamma}} \middle| \mathcal{G}(t)\right]}{\mathrm{E}_P\left[a(t,\zeta(T))^{-\frac{\gamma}{1-\gamma}} \middle| \mathcal{G}(t)\right]}$$

- Since $\zeta(t)$ follows a CIR process we have an explicit functional representation for the density function of $\zeta(T)$ at any $t < T$

$$f(\zeta(T), T; \zeta(t), t) = \frac{2\delta}{\sigma^2(1 - e^{-\delta(T-t)})} e^{-\frac{2\delta(\zeta(T)+\zeta(t)e^{-\delta(T-t)})}{\sigma^2(1-e^{-\delta(T-t)})}} \cdot \left(\frac{\zeta(T)}{\zeta(t)e^{-\delta(T-t)}}\right)^{\frac{\gamma}{\sigma^2}-\frac{1}{2}} I_{\frac{2\gamma}{\sigma^2}-1}\left(\frac{4\delta}{\sigma^2(1 - e^{-\delta(T-t)})}\sqrt{\zeta(T)\zeta(t)e^{-\delta(T-t)}}\right)$$

Sample paths for the hedging strategy

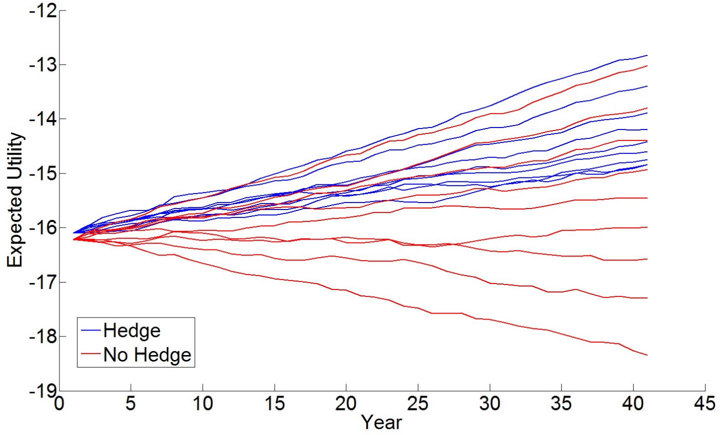


The value function for the non-hedgeable case

- We present the value of the longevity hedge as the percentage change in utility from the non-hedgeable case to the hedgeable case.
- In order to calculate this we need to identify the value function for the non-hedgeable case.
- Following the same procedure with $q(t)$ set to zero for all t, we find that

$$V(t,x,\zeta) = \frac{1}{\gamma}\left(x + \frac{\pi f(t)}{a(t,\zeta)}\right)^{\gamma} \exp\left\{\gamma\left(-\mu_Y + \xi_1\sigma_Y + \frac{1}{2}\frac{1}{(1-\gamma)}(\xi_1 - \sigma_Y)^2\right) \cdot (T-t)\right\} a(t,\zeta)^{\gamma} \mathrm{E}_P\left[a(t,\zeta(T))^{-\gamma} \middle| \mathcal{G}(t)\right]$$

Sample paths for the value functions



Utility gain

Percentile	Life Expect.	Annuity Price	Risk Aversion		
			3	6	9
10th	89.58	13.58	10.76%	30.06%	45.73%
20th	89.02	13.42	4.76%	14.21%	23.05%
30th	88.63	13.31	3.46%	10.48%	17.24%
40th	88.31	13.22	2.54%	7.77%	12.92%
50th	88.00	13.12	1.78%	5.50%	9.21%
60th	87.71	13.04	1.10%	3.40%	5.75%
70th	87.40	12.94	0.44%	1.39%	2.36%
80th	87.05	12.83	-0.21%	-0.67%	-1.14%
90th	86.59	12.68	-0.90%	-2.85%	-4.91%

Summary

- There is a pressing need to re-design the DC scheme so that it can become an adequate replacement for the DB scheme.
- We have focused on the importance of accounting for longevity risk in light of this assertion.
- We introduced a force of mortality state variable into the *stochastic lifestyling* framework.
- The force of mortality was allowed to follow a time-inhomogeneous CIR process permitting a closed form solution for the survival probability.
- We showed that there is significant demand for a longevity risk hedge in this environment and that the resulting utility gains from such a hedge can be considerable.