TN 2 - Basic Calculus with Financial Applications

Notation

$$y = f(x)$$

$$y' = f'(x) = \frac{d}{dx} f(x)$$

$$f(x) = \int f'(x)\, dx$$

Foreword

Calculus can be considered as the mathematics of motion and change. It is a BIG topic with applications spanning the natural sciences and also some social sciences such as economics and finance. In this TN we can only review a few basic concepts that are most likely to be useful for some finance-oriented modules of Master courses. The discussion will be conducted with exclusive reference to real-valued **univariate calculus** (calculus of one variable) to benefit from its analytical simplicity and ease of visualization.

§1 Functions and Limits

The first use of the word function is credited to Leibniz (1646-1716). Until the mid-1800s the concept of function was that of a relatively straightforward mathematical formula expressing the relationship between the values of a dependent variable (y) and those of one or more independent variables (**univariate** and **multivariate calculus**). In the 19th century, the concepts of function and limit were generalized and made a lot more rigorous, thereby providing a solid foundation for the further development of calculus.

A real-valued mathematical expression, such as the quadratic function in exhibit 1.1, has no defined numerical value until you assign a value to (x). Thus, we say that (y) is a function of (x). Functions are also called **transformations** because they transform the value of (x) into a value of (y).

A very important element in the definition of a function is the requirement that for every given point on its **domain** (*x*-axis) there is one and only one function value. In other words, if you draw a straight line parallel to the y-axis it must cross only once the function's graph.

Thus, if there is more than one value of (y) corresponding to one value of (x) we are dealing with two or more functions instead of one (see exhibit 1.2).

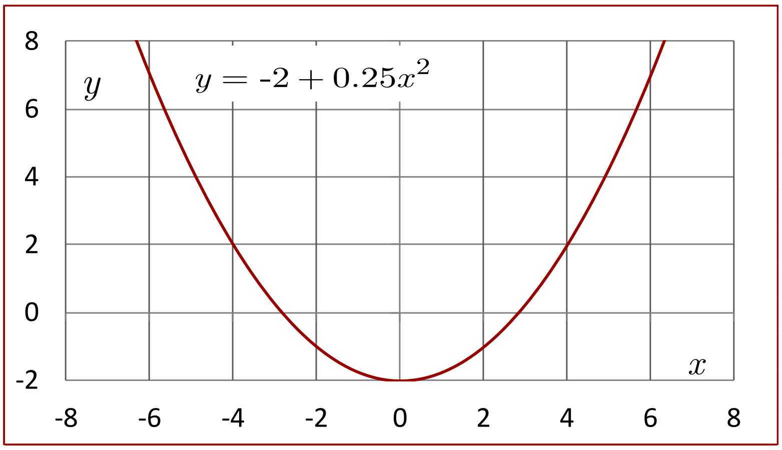


Exhibit 1.1 - Graph of the quadratic function

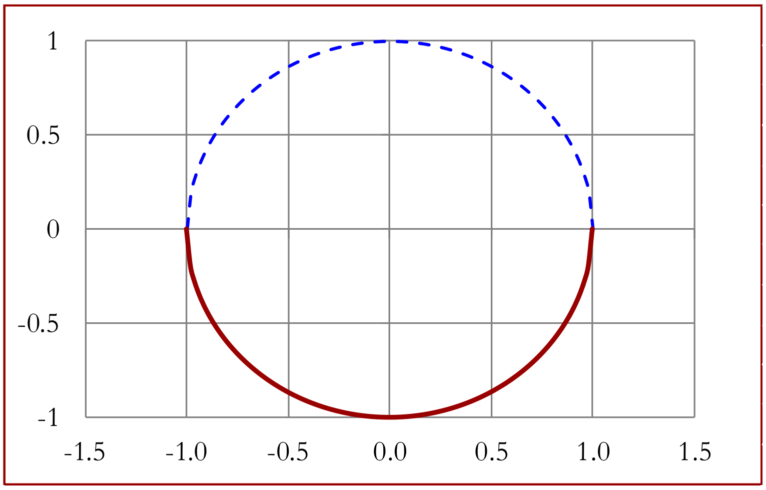


Exhibit 1.2 - Graph of unit circle (radius = 1). This is composed by two functions: one for the positive values of (y) and one for the negative.

Limits

The concept of limit is now all-pervasive in calculus and its applications. The rigorous definition of the limit of a function was worked out in the mid-1800s. There are several types of limits. However, we shall concentrate only on two, chosen for their relevance for our studies.

FIRST, the limit of a function that tends to zero for $(x \to 0)$, where the symbol (→) stand for "approaches". This limit is the cornerstone for the definition and calculation of the derivative of a function and will be discussed in §2.

$$\lim_{x \to 0} f(x) = 0$$

SECOND, the limit of a function that tends to zero for $(x \to \infty)$.

$$\lim_{x \to \infty} f(x) = 0$$

The above limit is true if, given an arbitrarily small number (ε), there is a number (δ) such that:

$$x > \delta \Rightarrow \left| f(x) - 0 \right| < \varepsilon$$

In other words, there is always a value (δ) large enough to obtain the desired result. This is also known as the (δ, ε) approach.

Zero Coupon Bond Example

Consider the price (Z) of a zero coupon bond due in (T) years, given a constant compounded yield rate $(Y = 6\%)$. The time-to-maturity is also known as **tenor** in finance-speak:

1.1 Limit of the Price of a Zero Coupon Bond

$$Z = K(1+Y)^{-T} = \$100 \times 1.06^{-T}$$
$$\lim_{T \to \infty} Z = 0$$

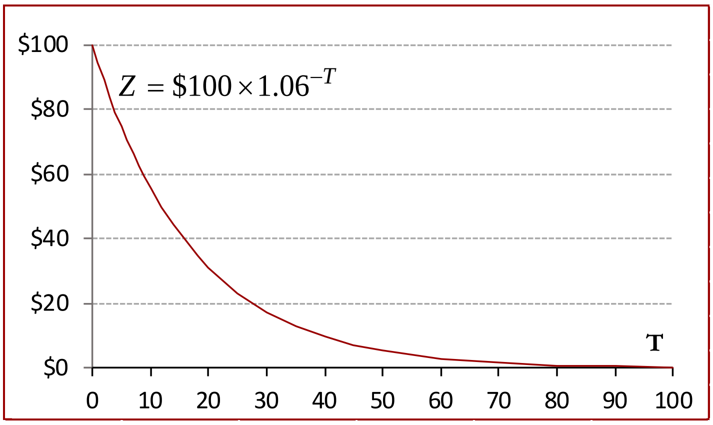


Exhibit 1.3 – Limit of the zero coupon bond price as a function of tenor.

§2 – Derivatives

The twin problems of calculating the tangent to a curve and the area delimited by a curve were solved in the late 1600s. It came as a surprise that the tangent and the area problems are interconnected. The tangent is calculated with the derivative and the area with the integral.

The central idea of differential calculus is the notion of derivative. The derivative of a real-valued function $y = f(x)$ in correspondence of a given value of (x) is a number that measures the slope of the function at that point. Thus, a straight line through this point, with a slope equal to the derivative, will be a tangent to the function $y = f(x)$.

More generally, the derivative of a function (often indicated as the **primitive**) is another function that gives the slope of $f(x)$ for each value of (x) on the domain of the function. Derivatives can be denoted in many different ways.

2.1 Notation for Derivatives

$$\frac{dy}{dx} = \frac{d}{dx} f(x) = f'(x) = y'$$

Exhibit 2.1 visualizes the derivative and tangent of the price of a zero coupon bond as a function of yield (for K = \$100, T = 30, and Y = 4%). The derivative of bond prices as a function of yield is widely used in fixed-income and is the foundation of **duration** and **convexity** analysis. We have chosen a long-maturity zero coupon bond (30 years) to make convexity clearly visible.

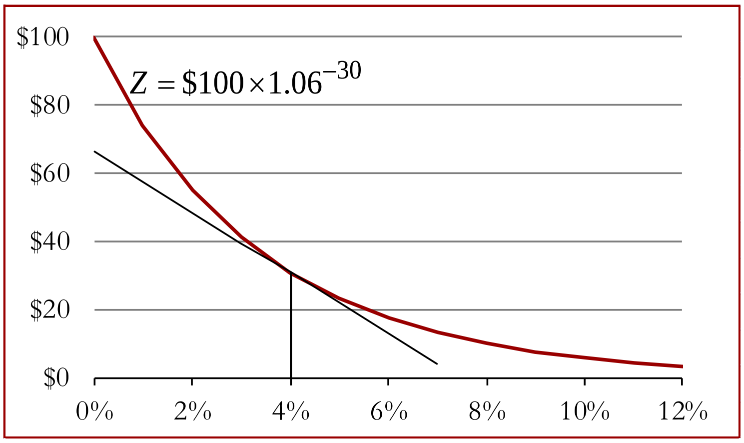


Exhibit 2.1 – Derivative and tangent to the price of a zero coupon bond as a function of spot yield (at $Y = 4\%$)

Derivative and Limit

The formula for calculating a derivative relies on the concept of limit and is both rigorous and intuitively obvious.

2.2 Derivative and Limit

$$y' = \frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

There are two properties to keep in mind because they play a role in understanding a number of economic and financial applications:

- Adding a constant (a) to a function will just shift the function upwards or downwards, while leaving its slope (derivative) unaltered
- Multiplying the independent variable by a constant (b) multiplies its derivative by (b)

We can show how this works by using the quadratic function:

2.3 Derivative of a Quadratic Function

$$\begin{aligned}\frac{d}{dx}\left(bx^2\right) &= \lim_{h \to 0} \frac{b(x+h)^2 - bx^2}{h} \\ &= \lim_{h \to 0} \frac{bx^2 + b2xh + bh^2 - bx^2}{h} \\ &= \lim_{h \to 0} \frac{b2xh + bh}{h} = \lim_{h \to 0} b2x + bh = 2bx\end{aligned}$$

Differentiable Functions

We should add that, to be differentiable for a given value of (x), a function must be both continuous and "smooth" (well-behaved) at that point. Two intuitive examples are provided by exhibits 2.2 and 2.3, which show that a differentiable function may well be non-differentiable in one point.

It is interesting to note that there are functions which are both everywhere continuous and nowhere differentiable. One of these function is the Brownian motion, which plays a central role in the theory of option pricing in finance. However, this is an advanced topic in stochastic calculus that cannot be covered in these induction lectures.

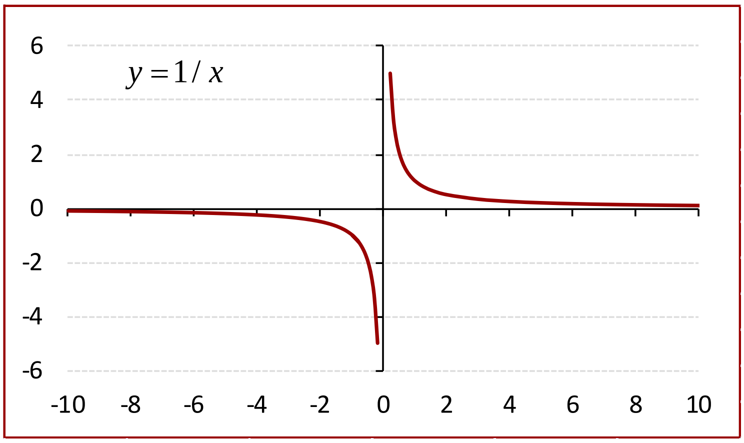


Exhibit 2.2 – This is a rectangular hyperbola. It is everywhere continuous and differentiable, apart from a discontinuity (singularity) for x = 0.

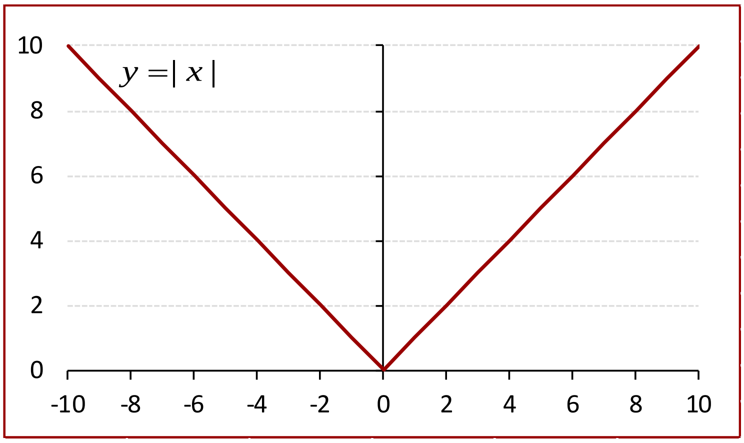


Exhibit 2.3 – This function is everywhere continuous. However it is not differentiable for x = 0 where the slopes on the left and on the right are not the same. In fact the derivative jumps from -1 to +1.

Higher Order Derivatives

If we take the derivative of a derivative we get what is called the second order derivative, which is usually written as:

$$f''(y) = \frac{d^2y}{dx^2}$$

The second derivative plays a relevant role in a number of applications, such as:

- Taylor series approximation
- Measuring convexity
- Determining maxima and minima of a function

§3 – Taylor Series

If we examine again exhibit 2.1 we can't fail noticing that, over a small interval, the tangent is a good approximation to the original function. Therefore, in a small interval around a value

(x_0) the function can be expressed with the following equation which is a one-term Taylor series. (This underlies the duration metric in bond mathematics).

3.1 First Order Taylor Expansion

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

In bond mathematics we also use a two-term Taylor expansion, which uses the first and the second derivative of the function.

3.2 Second Order Taylor Expansion

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$

§4 – Maxima and Minima of Functions

The problem of finding the maximum or the minimum value of a function is one of the most pervasive in the sciences and in economics. In many advanced problems, we must deal with multivariate functions and with inequality conditions. These advanced applications are solved numerically with sophisticated software (linear and non-linear optimization, simulation). We should note here that Excel solver (see TN 4) is not an industrial-strength application, but is a useful piece of software that allows us to familiarize with the structure of these optimization problems.

Some simpler problems can be solved using derivatives. In this case we have the advantage of obtaining an analytical solution. An example of this approach is the determination of Ordinary Least Squares (OLS) regression line coefficients.

If we examine exhibit 4.1 we realize intuitively that in correspondence of a maximum or a minimum of $y = f(x)$ the tangent to the function must have slope zero, thus implying that the derivative must be zero.

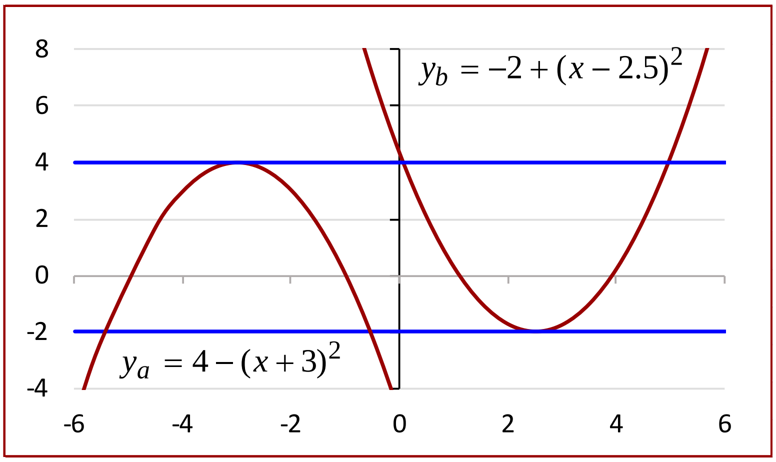


Exhibit 4.1 – Tangents to the maximum and the minimum of quadratic functions

We must now determine if setting the first derivative to zero identified a maximum or a minimum. This can be done using the second derivative if the function. Let us consider the case of a maximum for the function $(y = -x^2)$. The 1st derivative will be positive on the left of the maximum and negative of the right. It follows that the 2nd derivative will be negative (this is visualized in exhibit 4.2)

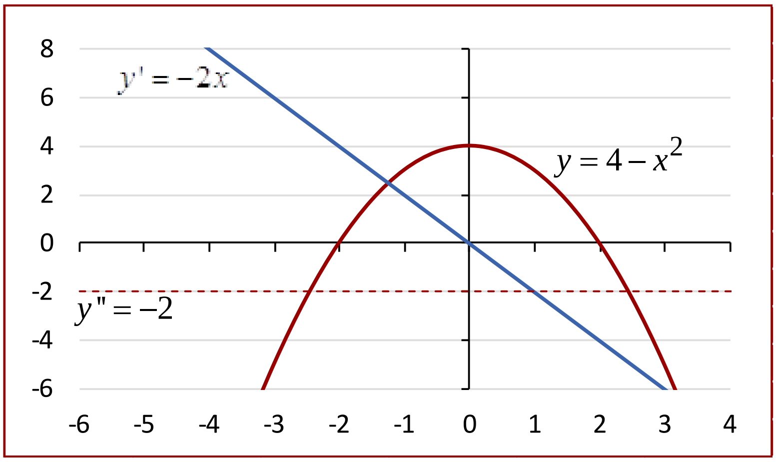


Exhibit 4.2 – the second order derivative indicates that this is a maximum

§5 Power, Exponential, and Log Functions

Power Function

The power function (PF) can be written as follows, denoting the independent variable with (x) and the fixed exponent with (a) :

$$y = x^a$$

The quadratic and cubic functions that we discussed in TN 1 are clearly power functions. Numerical values can be calculated with the Excel POWER spreadsheet function or the $(y = x \hat{} a)$ syntax. The power function has a number of applications in finance. Suffice to note that discount factors and value relatives are power functions.

Derivative of the Power Function

The derivative of the power function is quite straightforward:

5.1 Derivative of the power function

$$\frac{d}{dx} x^a = a\, x^{a-1}$$

The derivative with respect to (Y) of the above equation lies at the foundation of the duration approach to measuring interest rate risk for debt securities. In fact this derivative is knows in finance as dollar duration (D$).

5.2 Dollar Duration (D$)

$$\text{D\$} = \frac{d}{dY}(1+Y)^{-T} = -T(1+Y)^{-T-1}$$

Chain Rule (Derivative of a Function of a Function)

The chain rule is often used in both simple and advanced financial applications. Hence it is necessary to understand how it works. Denoting two functions with (f) and (g) we have a function of a function if:

$$y(x) = f[g(x)]$$

The derivative of (y) relative to (x) is obtained by applying the chain rule in the following way:

5.3 – Chain Rule and Power Function

$$\frac{dy}{dx} = \frac{dy}{d[g(x)]} \times \frac{d[g(x)]}{dx}$$

The Natural Exponential Function

The exponential and the logarithmic functions are a cornerstone of calculus, widely used in economic and financial applications. Log yields, also known as continuously compounded yields, are based on the natural exponential function.

Exponential functions are the standard tool when modeling proportional growth (either positive or negative). It has been proven that proportional positive or negative growth can be modeled with, and only with, the exponential function.

Exponential functions (as well as logarithms) can have different basis. However the most useful one is the irrational number (e = 2.718282 ...) which is the base of the natural exponential and logarithmic functions. The equation is:

$$y = e^x = \exp(x)$$

The reason for choosing the natural exponential function is that its derivative has the useful property of being equal to the primitive function, and this simplifies considerably mathematical manipulations without any loss of generality. If the exponent of (e) is a function of (x) we must apply the chain rule as shown in the following equations.

5.4 Derivatives of the Natural Exponential Function

$$\frac{d}{dx}e^x = e^x, \quad \frac{d}{dx}e^{f(x)} = e^{f(x)}e^x$$

Due to its widespread use, the exponential function can be calculated not only in Excel, with the spreadsheet function **EXP**, but also with most handheld calculators. The natural exponential function turns out to be necessary to define continuously compounded yield (exponential yield). The $\exp(x)$ function can model both exponential growth when $(x > 0)$ and exponential decline when $(x < 0)$.

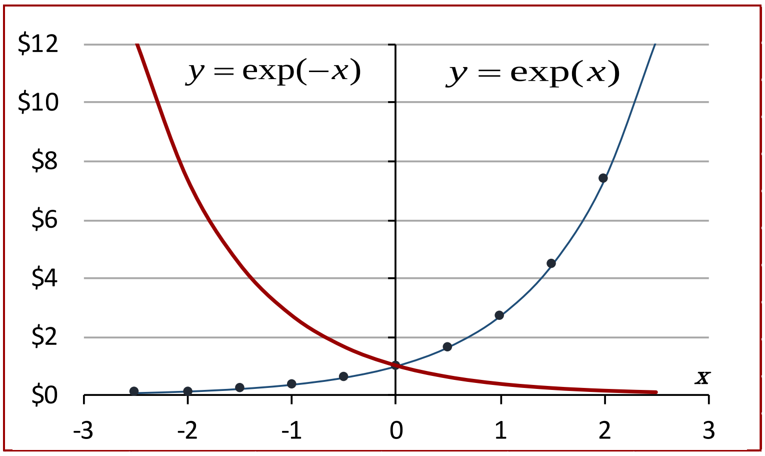


Exhibit 5.1 – Positive and negative growth with the natural exponential function

We are likely to come across the following self-explanatory properties of the exponential function:

$$e^0 = 1, \quad e^1 = e, \quad e^a \cdot e^b = e^{a+b}$$

One last point: $\exp(x)$ will always return a positive number, as visualized in exhibit 6.1. This is a very important property in financial modelling because almost all financial assets have **limited liability**. Their return can well be negative (think of the financial crisis of 2008-09) but their price has zero as lower bound.

The Logarithmic Function

The logarithmic function is the inverse of the exponential function, that is:

5.5 Natural Exponential Function

$$\ln(e^x) = x, \quad e^{\ln(x)} = x$$

This entails that the graphs of the two functions have identical shapes when you exchange the (x) and (y) axis, as shown in exhibit 6.2. We should also note that the logarithmic function is not defined for $(x \le 0)$. Natural logs can be calculated with the Excel function LN or using a handheld calculator. Natural logs are also extensively used in econometric time-series analysis.

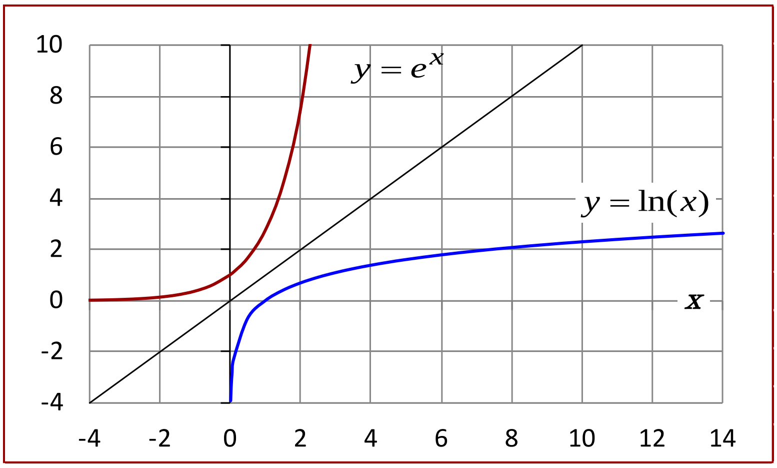


Exhibit 5.2 – The natural log and exponential functions

§6 – Linearity and Convexity

Linear Function of Simple Yields

We have already examined linear functions in TN1. Let us now extend the analysis of what linearity implies when we consider either realized portfolio returns or future expected returns. Consider exhibit 6.1 that shows the realized (end of year) simple returns and value relatives on a 5-asset portfolio over a 1-year time horizon. Clearly, asset weights must add up to one.

Asset	*w* Weights	Simple Returns	Value Relatives
1	10%	-10.0%	0.90
2	20%	0.0%	1.00
3	40%	10.0%	1.10
4	20%	20.0%	1.20
5	10%	30.0%	1.30
Weighted Averages		10.0%	1.10

Exhibit 6.1 – Value relatives of a 5-assets portfolio

The weighted arithmetic mean of value relatives $\mu(v) = 1.10$ equals the value relative calculated with the weighted arithmetic mean of returns $\mu(r) = 10\%$. This can be shown as follows:

6.1 The weighted arithmetic returns with simple rates

$$\begin{aligned}\mu(v) &= w_1(1+r_1) + \cdots + w_n(1+r_n) \quad [\Sigma\,\mathrm{w} = 1] \\ &= 1 + \mu(r)\end{aligned}$$

This result is important because it shows that we must use simple yields when we want to relate portfolio returns to the average return of its components. This can be easily generalized to the stochastic return on some financial asset (stocks, bonds, foreign exchange, etc.) Just substitute the weights with probabilities and we find that the expected value relative $\mathrm{E}[v]$ equals the value relative calculated on the expected yield $\mathrm{E}[r]$.

6.2 Expected return with simple rates

$$\begin{aligned}\mathrm{E}[v] &= P_1(1+r_1) + \cdots + P_n(1+r_n) \quad [\Sigma P = 1] \\ &= 1 + \mathrm{E}[r]\end{aligned}$$

Outcomes	*P* Prob.	Simple Returns	Value Relatives
1	10%	-10.0%	0.90
2	20%	0.0%	1.00
3	40%	10.0%	1.10
4	20%	20.0%	1.20
5	10%	30.0%	1.30
Expected values		10.0%	1.10

Exhibit 6.2 – Expected asset returns and value relatives

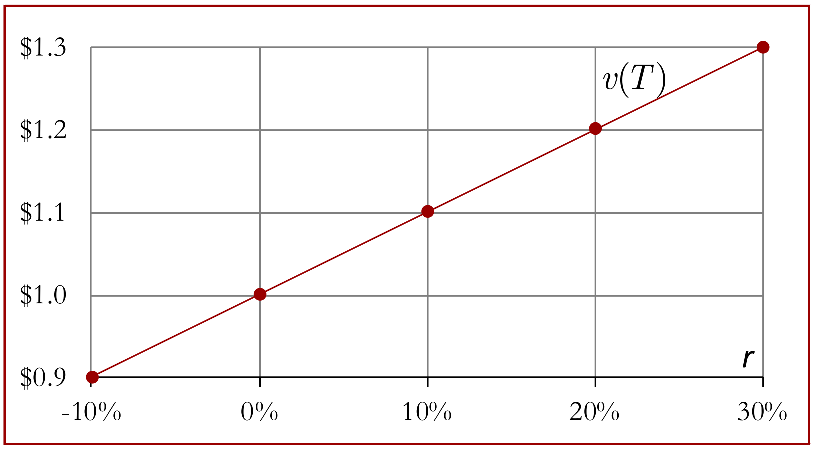


Exhibit 6.3 – Expected asset returns and value relatives

Convexity

A function is **upwards concave** (convex in financial jargon) over some interval of (x) if a straight line (known as **secant**) through any two points of $y = f(x)$ will always lie above the function itself. In finance, the most frequently used convex functions is the exponential function, which is convex over the positive real line.

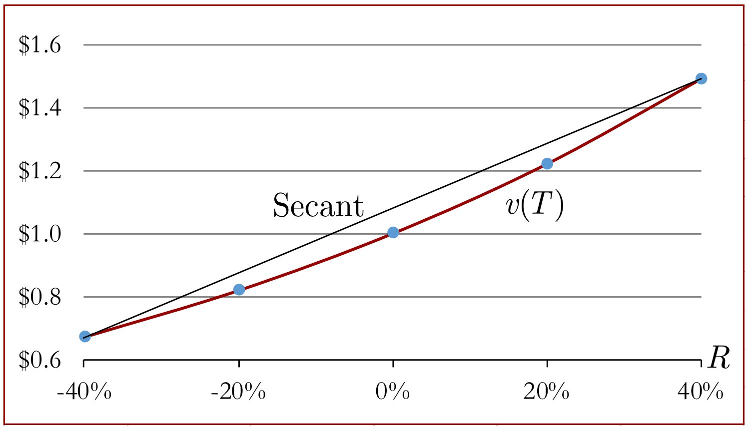


Exhibit 6.4 – Convexity of the function $\exp(R)$

Relevance of Convexity

From the point of view of finance, the most relevant property of convex functions is that the expected value of the function is always higher than the function of the expected value of the independent variable. This property is known as Jensen's inequality (from the Danish mathematician Johan Jensen, who proved it in the early 1900s.)

6.3 Jensen's Inequality

$$\mathrm{E}[v] = \mathrm{E}\Big[(\exp(R)\Big] > \exp\Big(\mathrm{E}[R]\Big)$$

Let us consider a very simple example, related to fixed income securities interest rate risk, to binomial option pricing models, and to the Black-Scholes options pricing model. Consider a 10-year zero coupon bond yielding 5% at time-0. Assume now that market-required yields either decrease to 4% or increase to 6% in a short time-interval (dt), with the same 50% probability. Clearly the expected yield will be 5% but the expected return will not be zero, due to the Jensen's inequality.

time-0		time-dt			
		R	6%		
		Z	54.8812	E[R]	5.00%
R	5%			E[Z]	60.9566
Z	60.65307			ΔZ	0.3035
		R	4%	ΔZ/Z	0.005004
		Z	67.0320		

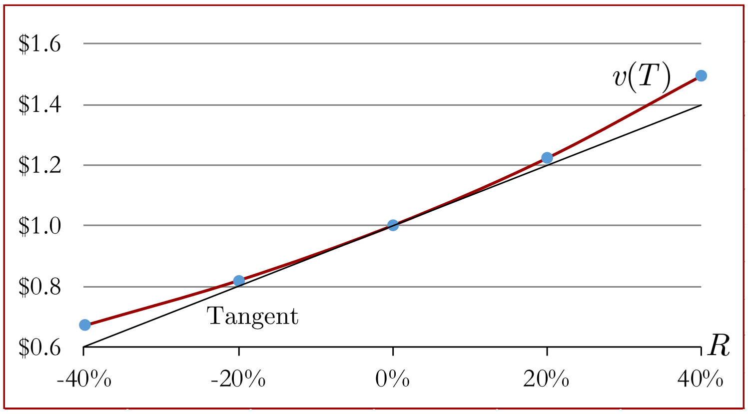
Exhibit 6.5: One binomial step with log yields

Assume that a distribution of log yields has an expected value $\mathrm{E}[R] = 10\%$. We can calculate the tangent to $v = \exp(R)$. If the values of (v) did lie on the tangent we would have:

$$\mathrm{E}[v] = \exp\Big(\mathrm{E}[r]\Big)$$

However, all the values of $\exp(R)$ lie above the tangent, with the only exception of the tangent point $\exp(10\%)$. Therefore, we must have:

$$\mathrm{E}[v] > \exp\Big(\mathrm{E}[R]\Big)$$



§7 Integrals

In §2 of this TN we have seen that the derivative $f(x)$ of a function $F(x)$, known as the **primitive**, measures the slope of $F(x)$ and, therefore, its speed of change. **Integration** reverses this approach to calculate $F(x)$ given its derivative. This turns out to be an extremely relevant development because, both in the natural sciences and in economics and finance, we can often measure rate of change but not the primitive, which must be calculated. The integration symbol is a stylized S, to indicate that an integral $F(x)$ is some form of summation based on $f(x)$. This link between derivatives and integrals lies at the foundations of the two **fundamental theorems of calculus**. In the next sub-sections we shall try to provide an intuitive understanding of the link between integral and derivative.

The Indefinite Integral

In a large number of cases (but not always) we can find the integral equation $F(x)$ of a given function $f(x)$. [This is now made easy by online apps.] In our example we shall refer to the (value relatives/discount factors) equations based on log yields.

7.1 Integral and Derivative of Value Relative

$$F(t \mid R) = \exp(R \times t)$$

$$f(t \mid R) = \frac{d}{dt}\exp(R \times t) = R \times \exp(R \times t)$$

$$F(t \mid R) = \int R \times \exp(R \times t)\, dt$$

From Derivative to Integral

A number of developments on derivatives and integrals are influenced by the tangent and area origins of calculus (see §2). This analysis is perfectly correct and rigorous, but not very intuitive from the point of view of many business students. Therefore, we shall adopt an approach based on the change of value-relative with the passing of time, assuming for simplicity that the log-yield rate remains constant (at 5%). We shall also cheat a little and start with a well-known primitive function, take it derivative and work back to the integral and to its summation meaning.

7.2 Value-relative Equation

$$v(t) = \exp(5\% \times T) \quad [0 \le T \le 30]$$

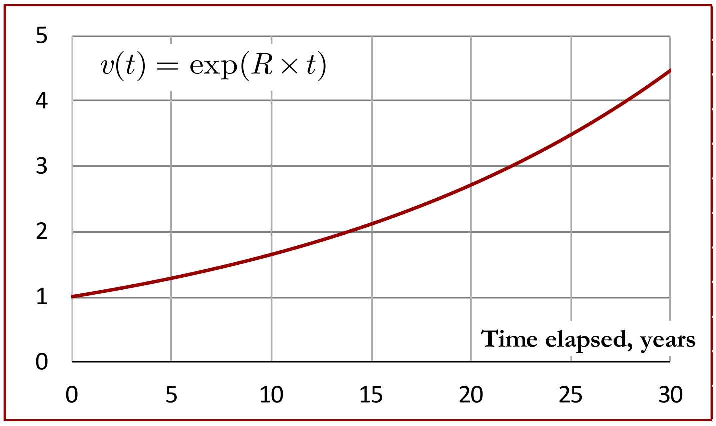


Exhibit 7.1 – Value relative as a function of time elapsed and a 5% constant log yield

Derivative of the value-relative equation

$$\frac{d}{dt}\exp(R \times t) = R \times \exp(R \times t)$$

We can easily see from exhibit 7.2 that the derivative of $v(t)$ is identical to $v(t)$ scaled down by the product with the 5% log yield.

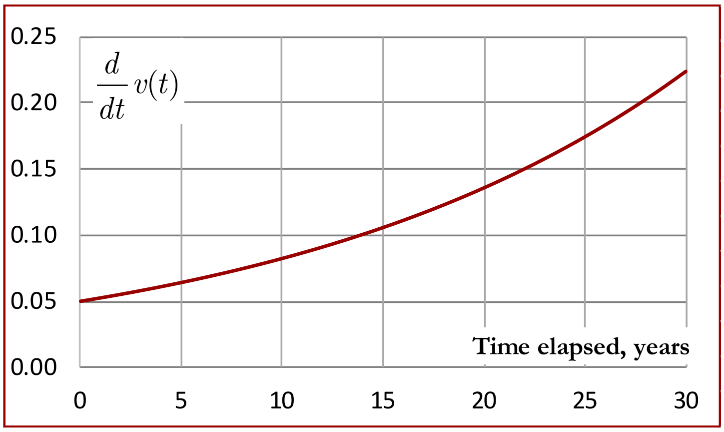


Exhibit 7.2 – Derivative of the value-relative function

At this point we can partition the *t*-axis in a relatively large number of intervals (we have chosen 120, with interval's length of one quarter (four per year for 30 years). As $v(t)$ is very smooth this will already give us an acceptable approximation. Given the value of $v(t \mid n)$ at the beginning of the n-th interval, the value at the beginning of the following in interval can be calculated multiplying the slope of $v(t)$ by the interval's length. Repeating the process we obtain the definite integral. Note that the definite integral is a numerical value

$$\int_{t=0}^{360} \Delta\,\mathrm{t} \times f(t)\,dx$$

$$v(t + \Delta t) = v(t) + \frac{d}{dt}v(t) \times \Delta t$$

Clearly, if we take smaller time-intervals the sum becomes closer to the accurate value of the primitive function. In the calculus approach, the time intervals' length will have zero as a limit We could consider the figure similar to that in Exhibit 7.1 to delimit a surface, as shown in exhibit 7.3. (In this case, both axis represent a distance.)

We can now take the same first-order Taylor series approach, calculating the derivatives and multiplying them by the x-axis intervals, which we denote with (Δd) to indicate that they are distance-intervals. These products will be (distance * distance), thereby representing areas. This is visualized in exhibit 7.4.

Moreover, a number of functions do not have a closed-form antiderivative. Therefore $F(x)$ *must be calculated with numerical methods. For example, in TN3 we shall see that the normal density function (the well-known bell curve) does not have an equation for its cumulative density. The integral exists, but its values must be calculated.*

We can now use the Taylor series approach (see §2) and multiply each of the 100 slopes by the 6-seconds intervals (Δt).

Indefinite Integral (Antiderivative)

An indefinite integral is written as follows, with no indication of upper and lower bounds. The (dx) is not a multiplier but simply a reminder that the integration takes place with reference to the variable (x). The constant of integration (C) has no defined value and is simply a reminder that a constant (if it exists) drops in taking the derivative and could need to be added back to F(x).

$$F(x) = \int f(x)\,dx + C$$

Clearly, the change in value of F(x) between two values of (x) will be:

$$F = F(b) - F(a) = \int_a^b f(x)dx$$

The above equation is known as the **Second Fundamental Theorem of Calculus**.

§8 Log Yield

With "classic" compounded yield, the yield rate appears in the base $(1+Y)$ of the exponential function. With log yield (often referred to as continuously compounded yield), that we shall denote with (R), the yield rate is at the exponent of the natural exponential function. Because of their mathematical properties, log yields are consistently used in options theory. When using log yields, day count is usually act/365. These are the equations:

Log Yield Equations

$$\begin{aligned} v(T) &= \exp(R \times T) \\ \ln[v(T)] &= R \times T \\ d(T) &= \exp(-R \times T) \\ \ln[d(T)] &= -R \times T \end{aligned}$$

As compounded yields (Y) and log yields are both based on exponential functions they produce the same identical result when we adjust the numerical value of (R) as shown in the log yield equations.

Example 8.1 – Compounded yield (quoted for act/365) is 4.5%. Calculate (R).

$$\exp(R) = 1 + Y$$
$$R = \ln(1.045) = 4.4017$$

Example 8.2 – a 92 days T-bill has a log yield of 2.75%. Its price (B) is:

$$t = 92 \div 365,\ R = 2.75\%,\ K = 100$$
$$B = 100 \times \exp(-0.0275 \times 92 / 365) = 99.3092$$

Example 8.3 – a one-year T-bill is priced at 95.00. Its log yield is computed as follows:

$$t = 1$$
$$R = \ln\left[v(t)\right] \div t = \ln\left(100 \div 95\right) = 5.1293\%$$

Continuously Compounded Yield

Log yields are often denoted as continuously compounded yields because the value-relative that we obtain using the log rate (*R*) is identical to that obtained using (*R*) as a simple annualized rate compounded an infinite number of times. For a simple proof see the mathematical appendix. This can be expressed with the following equation:

$$\exp(R) = \lim_{n \to \infty} \left(1 + R / n\right)^n$$

n	(1 + R/n)^n
1	1.050000
2	1.050625
4	1.050945
12	1.051162
365	1.051267
∞	1.051271

Exhibit 8.1 – Numerical example of the continuous compounding calculation, where (*n*) indicates the number of compounding periods per year

A Common Misconception

We are likely to come across the statement that continuous compounding should be used only when coupons are paid with a very high frequency. For example, let us quote from a well-known fixed income textbook (GNMAs are bonds issued by the Government National Mortgage Association – also known as Ginnie Mae – a U.S. fully owned Government Corporation.)

> *"Eurobonds pay annual coupons, U.S. treasuries' coupons are semiannual, and GNMAs make monthly payments. As the coupons become more frequent, it becomes more accurate to assume exponential continuous compounding."*

Clearly, this does not make sense. Given a value-relative $v(t)$, the log rate (R) is simply a mathematically efficient way to measure the rate of growth (positive or negative), and has nothing to do with the coupon payment frequency. In fact, we use log yields for all sort of securities, including:

- Zero coupon bonds (no coupon payment)
- Stochastic processes for non-dividend-paying securities (in option pricing)

Glossary

Antiderivative	Functions	Primitive
Concavity	Integral (definite)	Multivariate calculus
Continuously comp. yield	Integral (indefinite)	Secant
Convexity	Integration	Simple yields
Derivative	Jensen’s inequality	Taylor expansion
Deivative (first)	Limited liability	Tenor
Deivative (second)	Limits of a function	Transformations (functions)
Differentiable Functions	Linearity	Univariate calculus
Domain	Log yields	Upwards concave
Duration	Maxima of functions	Zero coupon bonds
Exponential function	Minima of functions	