Skew Index Option Pricing

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Outline

- The Skew Index.
- Remarks on Option Pricing.
- Hedging with Options on Related Assets.
- Marginal Distributions of Returns to Option Maturity.
- Joint Distributions for Hedged and Hedging Returns.
- Distribution of Hedged Return Conditional of Hedging Asset Return.
- Pricing of Residual Risk.
- Function to be Hedged in the Related Asset Market.
- Pricing at Cost of Hedge.
- Skew Index Details.
- Examples JPM via XLF.
- VIX via SPY.
- Skew via SPY.
- VIX and Skew of Skew.

The VIX Index

- The Index construction follows the principles of the VIX.
- The VIX is the forward price quote for receiving the sum of squared returns for a month.
- For horizon h this value is the following risk neutral expectation

$$VIX_t = \sqrt{\frac{1}{h} E^Q \left[-2 \ln \left(\frac{S(t+h)}{S(t)} \right) \right]}.$$

- A portfolio of stock options hedges twice the negative of the log contract to deliver the VIX.
- There is a substantial market in options on the VIX index.

The Skew Index

- The Skewness of returns is the centered third moment divided by the variance to the power 1.5.
- For return to horizon h,

$$R = \ln\left(\frac{S(t+h)}{S(t)}\right),$$

- The skewness is

$$s = \frac{E^Q\left[\left(R - E^Q(R)\right)^3\right]}{E^Q\left[\left(R - E^Q(R)\right)^2\right]^{1.5}}.$$

- All powered return expectations required may be hedged to determine the skewness s.
- It is typically negative for equity assets and the Skewness Index is

$$S = 100 - 10s$$

- where s is the skewness of the one month return on the S&P 500 index.

Skew Index Options

- The VIX shot up past 80 in COVID time with an increase of 500% before coming down.
- The Skew Index however, has steadily risen to 148.27 in late June from a mid March low of 113.54.
- Currently there are no options trading on the Skew Index.
- We shall price such options here.

Remarks on Option Pricing

- The Black and Scholes (1973), and Merton (1973) theory taught us how to price options at the cost of a replicating hedge.
- It solved a complicated and difficult problem of defining option values that converged to the kinked payoff at maturity.
- Under strong assumptions replication was delivered by dynamic trading in the stock.
- The result was a formula for pricing options valid under assumptions that turned out to be false.
- The formula was soon abandoned for pricing options and is used mainly to quote prices and manage risks.

Option Pricing Models

- The intervening years saw the development many parametric option pricing models.
- They were at best technologies for dimensional reduction inferring the prices of now thousands of options from a handful.
- In particular, there isn't an associated hedge and/or the cost of such a hedge.

Pricing at Cost of Hedge

- Back, prior to 1973, options did not trade as extensively as they do now.
- Hedges were sought in the liquid market for the stock.
- Today one may seek to hedge the risk of say an option on the Skew or VIX index by positions in options for the stock.
- But replication is not to be expected under realistic assumptions.
- Strong assumptions delivering replication may suffer the same fate of being false.
- In the absence of replication, residual risk needs to be directly priced.
- One has to price the risk that still remains after the hedge is put in place.

The Agenda Before Us

- Describing the joint risk of the hedged and hedging returns.
- Describing the residual or post hedge risk.
- Pricing the residual risk.
- Constructing the hedge.
- Pricing at the cost of the hedge.
- Applying the methodology and evaluating the results.

Univariate Return Distributions

- In appreciation of Gauss attention is restricted to limit laws.
- In recognition of the principle of likelihoods entropy is maximized.
- All the limit laws are the self decomposable laws and the selected distribution is in this class.
- Entropy maximization delivers the Gaussian density for a known variance.
- Entropy maximization for a random variance delivers the variance gamma model.
- The variance gamma is a difference of two independent gamma processes with the same variance rate or speed.
- Recognizing that up moves are both more frequent and smaller than down moves leads to the bilateral gamma (BG) model.
- The BG is a four parameter limit law calibrating freely the drift, volatility, skewness and kurtosis.
- Four parameters are a minimal requirement given that there are separate drifts and volatilities for the up and down moves.

Bilateral Gamma Model

- Let $\gamma_p(t)$, $\gamma_n(t)$ be two independent standard gamma processes with unit mean and variance rates.
- The BG process $X(t)$ parameters b_p, c_p, b_n, and c_n is defined as

$$X(t) = b_p\gamma_p\left(c_p t\right) - b_n\gamma_n(c_n t),$$

- where b_p, b_n are the scale parameters and c_p, c_n are the speed parameters for the positive and negative moves.

Bilateral Gamma Characteristic Function

- The characteristic function of the Bilateral Gamma at time t is given by

$$\phi_{BG}(u,t) = \left(\frac{1}{1 - iub_p}\right)^{c_p t} \left(\frac{1}{1 + iub_n}\right)^{c_n t}.$$

Bilateral Gamma Density

- The Bilateral Gamma density at unit time, for $x > 0$, is given by

$$
\begin{aligned}
f_{BG}(x) \quad = \quad & \left(\frac{1}{b_p}\right)^{c_p} \left(\frac{1}{b_n}\right)^{c_n} \left(\frac{1}{(1/b_p + 1/b_n)^{(c_p+c_n)/2}\,\Gamma(c_p)}\right) \times \\
& x^{(c_p+c_n)/2-1} \exp\left(-x/2\left(1/b_p + 1/b_n\right)\right) \times \\
& whitta\ker W\left(\frac{c_p - c_n}{2}, \frac{c_p + c_n - 1}{2}, \left(\frac{1}{b_p} + \frac{1}{b_n}\right), x\right).
\end{aligned}
$$

- For $x < 0$ the roles of b_p, c_p and b_n, c_n are reversed with an evaluation at $|x|$.

Bilateral Gamma Lévy Density

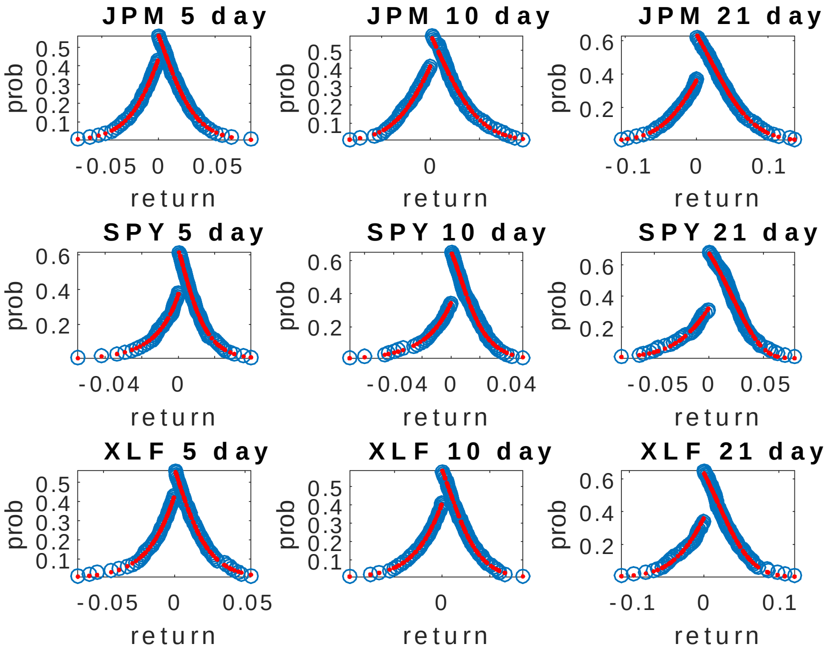
- The Bilateral Gamma Lévy density is given by

$$k_{BG}(x) = c_p \frac{\exp\left(-\frac{x}{b_p}\right)}{x} \mathbf{1}_{x>0} + c_n \frac{\exp\left(-\frac{|x|}{b_p}\right)}{|x|} \mathbf{1}_{x>0}$$

- As $|x| k_{BG}(x)$ is decreasing in $|x|$ the unit time law is self decomposable and thus is a limit law.

Bilateral Gamma for Bootstrapped Returns

- The bilateral gamma model fits have been reported for daily returns in time series data and at option maturities risk neutrally.
- The interest here is for the physical return distribution at the option maturity.
- Empirically the returns are constructed by bootstrapping data on immediately prior price histories.
- We present a graph of the model fit to five, ten and 21 day bootstrapped returns.


JPM 5 day
JPM 10 day
JPM 21 day
SPY 5 day
SPY 10 day
SPY 21 day
XLF 5 day
XLF 10 day
XLF 21 day
prob
return

Bilateral Gamma fit Statistics

- We present a Table of the root mean square error percentiles across 195 underlying equity assets.
-

TABLE 1

Root Mean Square Error of observed and model tail probabilities in basis points

Percentile	5 day return	10 day return	21 day return
1	50.27	51.05	57.91
5	53.55	58.45	63.33
10	57.27	62.67	67.51
25	66.02	72.51	83.90
50	78.89	85.30	102.19
75	91.26	102.77	130.42
90	104.12	126.93	154.88
95	114.52	142.86	172.32
99	159.87	160.04	203.83

- The fit is better at the shorter maturities.

		TABLE 2		
Representative Bilateral Gamma Parameter Values				
		5 Days		
bp	cp	bn	cn	proportion
15.9027	123.3173	58.3982	63.1796	0.0256
9.5064	76.4077	57.7419	12.0104	0.0256
59.6724	11.5148	106.2390	6.2204	0.0923
5.6208	248.9640	58.0989	21.8050	0.0205
139.4795	3.5603	175.1647	2.5233	0.2462
20.8004	30.0944	74.6795	8.1088	0.0205
168.2422	2.0285	229.9694	1.3094	0.5538
		10 days		
bp	cp	bn	cn	proportion
32.6195	98.8760	32.4550	98.8760	0.0205
6.5111	260.3097	93.0103	18.6578	0.0462
27.0037	37.9756	119.3985	8.0190	0.0667
3.8680	339.6846	95.5937	13.9218	0.0564
81.9528	52.5407	55.9518	67.0157	0.0205
12.7121	167.2849	94.4198	25.8470	0.0513
222.1890	3.8762	283.5553	2.7577	0.7179
		21 days		
bp	cp	bn	cn	proportion
1.8854	553.0553	191.3070	5.0389	0.0410
67.7515	93.1767	41.3999	133.2399	0.1026
7.2949	303.4949	153.8004	13.9832	0.1333
58.2010	46.8212	127.9390	19.8097	0.0872
3.8009	424.7692	210.6399	7.8749	0.0821
13.0678	215.7955	122.7883	26.3819	0.1179
360.5090	5.3657	419.0594	3.6803	0.3846

Multivariate Bilateral Gamma

- Just as the multivariate normal distribution is a joint distribution with Gaussian marginals there is multivariate law consistent with prespecified bilateral gamma marginals.
- The multivariate law is that of multivariate variance gamma plus independent bilateral gamma laws.
- Multivariate variance gamma is multivariate Brownian motion with drift θ and covariance matrix Σ time changed by a gamma process with unit mean and variance rate ν.
- The dependency parameters are just ν and the correlation matrix C of the Brownian motions.
- The drifts and variances are given by the marginal laws.

Multivariate Bilateral Gamma Parameters

- Let the marginal parameter vectors be b_p, c_p, b_n, and c_n.
- For dependency parameters ν, C define

$$\begin{aligned} \theta_i &= \frac{b_{pi} - b_{ni}}{\nu} \\ \sigma_i^2 &= \frac{2b_{pi}b_{ni}}{\nu} \end{aligned}$$

- and set

$$\Sigma = \Delta(\sigma)C\Delta(\sigma).$$

- Let Y be a vector of independent bilateral gamma variates with parameters b_p, $c_p - 1/\nu$, b_n, $c_n - 1/\nu$.
- The multivariate bilateral gamma (MBG) variate X has the distribution of

$$X = \theta g + \Delta(\sigma)\sqrt{g}Z + Y$$

- where g is has a gamma distribution with unit mean and variance ν.

Empirical Characteristic Function Matching

- The dependency parameters may be estimated in higher dimensions by matching the theoretical characteristic function to the empirical characteristic function.
- The *MBG* characteristic function is given by
- $$\begin{aligned}\phi_{MBG}(u) &= E\left[\exp(iu'X)\right] \\ &= \left(\frac{1}{1 - iu'\theta\nu + \frac{\nu}{2}u'\Sigma u}\right)^{\frac{1}{\nu}} \times \\ &\prod_j \left(\frac{1}{1 - iu_j b_{pj}}\right)^{c_{pj} - 1/\nu} \left(\frac{1}{1 + iub_{nj}}\right)^{c_{nj} - 1/\nu}.\end{aligned}$$
- The correlation matrix may be parameterized by the lower triangular part of its matrix logarithm Archakov and Hansen (2018).

MBG Lévy measure

- The multivariate arrival rates or Lévy density $k(x)$ for the multivariate bilateral gamma model may be specified as follows.

$$k(x) = \widetilde{m}(x) + \sum_{j=1}^{n} k_j(x_j) \prod_{\substack{l \neq j \\ l=1}}^{M} \mathbf{1}_{x_l = 0},$$

where

$$\begin{aligned} \widetilde{m}(z) &= \frac{\exp\left(\theta^T \Sigma^{-1} x\right)}{\nu \left(2\pi\right)^{n/2-1} \sqrt{|\Sigma|} \sqrt{x^T \Sigma^{-1} x}} \times \\ & \exp\left(-\sqrt{\left(\theta^T \Sigma^{-1} \theta + \frac{2}{\nu}\right) \left(x^T \Sigma^{-1} x\right)}\right) \\ k_j(x_j) &= \frac{c_{nj-1/\nu}}{|x_j|} \exp\left(-|x_j| / b_{nj}\right) \mathbf{1}_{x_j<0} \\ & + \frac{c_{pj} - 1/\nu}{x_j} \exp\left(-x_j / b_{pj}\right) \mathbf{1}_{x_j>0}. \end{aligned}$$

MBG Joint Density I

- The residual risk of y given x may be simulated from the conditional density of y given x.
- For this we require the joint density of two multivariate bilateral gamma distributed variates.
- The two variates are specified as

$$\begin{aligned} X &= \theta_x g + \sigma_x \sqrt{g} z_1 + y_1 \\ Y &= \theta_y g + \sigma_y \sqrt{g} z_2 + y_2 \end{aligned}$$

- where g is gamma distributed and y_1, y_2 are independent bilateral gamma variates.

MBG Joint Density II

- The joint density of (X, Y) conditional on $g,\ y_1,\ y_2$ is

$$
\begin{aligned}
& f_{MBG}(x_1, x_2; g, y_1, y_2) \\
= \quad & \frac{1}{2\pi\sigma_x\sigma_y g\sqrt{1-\rho^2}} \times \\
& \exp\left(\begin{array}{c} -\frac{(x-\theta_x g-y_1)^2}{2\sigma_x^2 g(1-\rho^2)} - \frac{(y-\theta_y g-y_2)^2}{2\sigma_y^2 g(1-\rho^2)} \\ +\frac{\rho(x-\theta_x g-y_1)(y-\theta_y g-y_2)}{\sigma_x\sigma_y g(1-\rho^2)} \end{array} \right)
\end{aligned}
$$

- The variates g, y_1, y_2 may be integrated out by Monte Carlo.

MBGR and MBGIR

- The *MBG* law may be used on (x, y) but when there is a strong dependence as occurs for the *VIX* and the *SPX* one may first perform a regression

$$y = a + bx + u$$

- If the only dependence is the regression then one models u as a bilateral gamma. This is the model *MBGIR*.
- Alternatively one may model the pair (x, u) as *MBG*. This the model *MBGR*.

Copulas

- The bilateral gamma marginals may be put together into a joint law using a copula.
- In addition we employ a Gaussian and a $'t'$ copula, both of which incorporate a correlation coefficient.
- Four models of dependence are employed, $MBGR$, $MBGIR$, and a Gaussian and a $'t'$ copula.

MBG and Copula Remark

- Copulas concentrate attention on return correlations
- The introduction of common time changes for correlated Brownian motions helps differentiate return correlations from squared return correlations.
- The parameter C drives return correlations while ν controls squared return correlation.

Residual Risk

- Suppose the return on the hedging asset at maturity turns out to be x.
- The option to be priced has a payout C for strike K and w equal to unity for a call and zero otherwise of

$$C = (1-w)\left(K - S_y(0)e^y\right)^+ + w(S_y(0)e^y - K)^+.$$

- Further suppose it is organized that conditional on the hedging return being x we shall receive the funds $L(x)$.
- The residual risk on selling the option is then

$$R = L(x) - C.$$

- If R is positive we have no issue, we payout C and keep the difference.
- The residual risk is then an acceptable risk.
- Our interest then is in the smallest value $L(x)$ such that R is an acceptable risk.

Acceptable Risk

- The theory of acceptable risks developed in Artzner, Delbaen, Eber and Heath (1999) defines acceptable risks more generally as a convex cone of random variables that contains the nonnegative random variables, as these are clearly acceptable.
- Equivalently it is shown that there exists a convex collection $\mathcal{M}$ of test probabilities $Q \in \mathcal{M}$ such that a risk R is acceptable if and only if

$$E^Q[R] \geq 0, \text{ for all } Q \in \mathcal{M}.$$

- It follows that the smallest value $L(x)$ is given by

$$L(x) = \sup_{Q \in \mathcal{M}} E^Q[R]$$

- This is the ask price for selling and option. The bid price for buying it is

$$B(x) = \inf_{Q \in \mathcal{M}} E^Q[R].$$

Distorted Expectations I

- The test probabilities may be defined by a probability altering rule.
- For a concave distribution function $\Psi(u)$, $0 \leq u \leq 1$ define

$$\mathcal{M} = \{Q|\ Q(A) \leq \Psi\left(P(A)\right),\ \text{all } A\}$$

- where P is the original probability measure. Then

$$B(x) = \int_{-\infty}^{\infty} rd\Psi(F_R(r)),$$

- is a distorted expectation and F_R is the distribution function of the risk.
- It may be observed to be below the expectation as it is an expectation under an altered probability that reweights losses upwards and gains downwards.
- The gap between the two depends on the concavity of Ψ.

Distorted Expectations II

- Distorted expectations may be computed from a simulation of cash flows C_n.
- Let $C_{(n)}$ be the sequence of cash flows in ascending order. The distorted expectation

$$\mathcal{D}\left(C\right) \approx \sum_{n=1}^{N} C_{(n)} \left(\Psi\left(\frac{n}{N}\right) - \Psi\left(\frac{n-1}{N}\right)\right).$$

- From the relationship between inf and sup we have that

$$L(x) = -\mathcal{D}\left(-C\right).$$

Minmaxvar Distortion

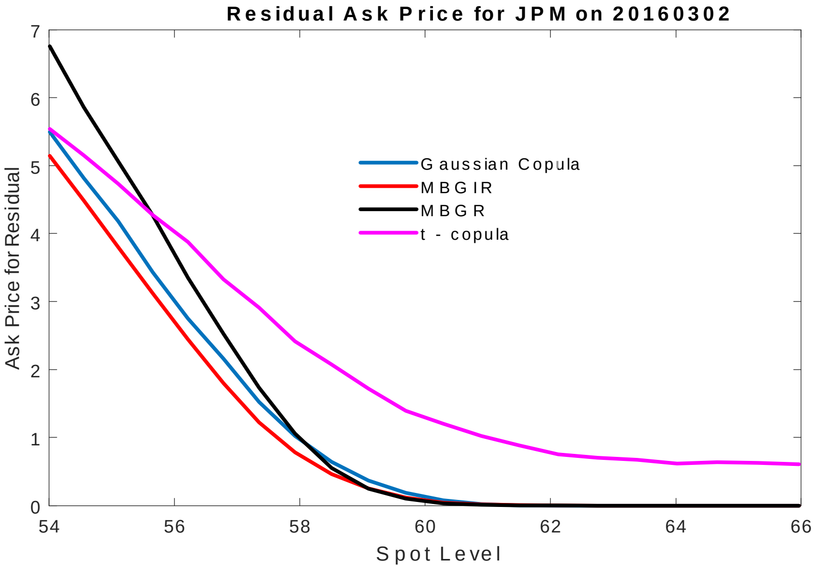
- Cherny and M. (2009) introduced the distortion termed minmaxvar defined by a single stress parameter γ as

$$\Psi^{(\gamma)}(u) = 1 - \left(1 - u^{\frac{1}{1+\gamma}}\right)^{1+\gamma}.$$

- The concavity increases with γ and the reweighting of large losses is infinite with large gains being discounted down to zero.
- $\Psi^{(\gamma)\prime}(u) \to \infty$ as $u \to 0$ and $\Psi^{(\gamma)\prime}(u) \to 0$ as $u \to 1$.

Residual Risk Function

- For a small stress level of $\gamma = 0.05$ we present a graph of the residual risk ask price $L(x)$ as a function of the price of XLF at the maturity of one month for a put option on JPM struck at 57.5 with the spot price at 59.69.
- The four curves are for the four joint laws used in constructing the residual risk from the required conditional distributions.


Residual Ask Price for JPM on 20160302
Ask Price for Residual
Spot Level
Gaussian Copula
MBGIR
MBGR
t - copula
0
1
2
3
4
5
6
7
54
56
58
60
62
64
66

Hedging the Residual Risk Function

- The Residual Risk Function may be hedged by least squares regression of the target cash flows onto to the cash flows delivered by the hedge instruments.
- Let S_x denote the price of the hedging asset at maturity with the target being $L(S_x)$ as evaluated by the residual risk pricing function.
- For hedging assets of a bond, the stock, and options with strikes K_i and w_i unity for calls and zero otherwise the hedge cash flows for positions α, β, η_i are

$$H(S_x) = \alpha + \beta S_x + \sum_i \eta_i \left[(1 - w_i)\left(K_i - S_x\right)^+ + w_i \left(S_x - K_i\right)^+ \right]$$

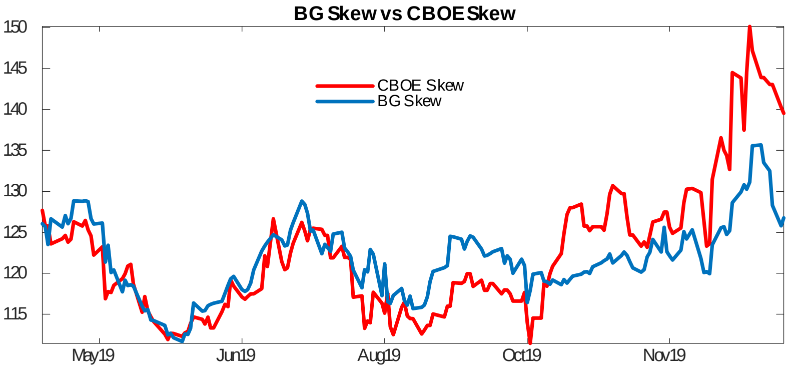
- For a grid of values for S_x one may evaluate the matrix of payouts to each hedge instrument and regress $L(S_x)$ to determine the hedge positions α, β, η_i.
- The cost of the hedge may then be computed.
- For the *JPM* put hedged by *XLF* options the hedge costs were 1.3877, 1.2730, 1.6435 and 2.1357 for the Gaussian Copula, *MBGIR*,

The CBOE Skew Index

- The Skew Index is a linear transformation of the risk neutral skewness.
- For a bilateral gamma density the skewness may be analytically be obtained from the derivatives of the logarithm of the characteristic function.
- Explicitly we have

$$
\begin{aligned}
d &= b_p c_p - b_n c_n \\
v &= \sqrt{b_p^2 c_p + b_n^2 c_n} \\
\widetilde{s} &= 2\frac{b_p^3 c_p - b_n^3 c_n}{v^3} \\
\widetilde{k} &= 3 + 6\frac{b_p^4 c_p + b_n^4 c_n}{v^4}.
\end{aligned}
$$

- for the drift, volatility, skewness and kurtosis.
- We present a graph of the CBOE Skew and the Bilateral Gamma calibrated skewness.


BG Skew vs CBOE Skew
CBOE Skew
BG Skew
150
145
140
135
130
125
120
115
May19
Jun19
Aug19
Oct19
Nov19

VIX via SPY

- For every ten days between January 4, 2016 and December 31 2019 call options on the *VIX* that were between five and fifteen percent out of the money were priced at four hedge costs for the four models.
- There were 56 days and a total of 96 strikes.
- The stress level was 0.025.
- A table presents percentiles of the hedge cost price premia over market.

TABLE 3

Hedge Price Premia Over Market

Percentile	Gaussian	MBGIR	MBGR	t-copula
10	0.4991	0.7263	0.7678	1.4283
25	0.5928	0.8583	0.9012	1.6755
50	0.8413	1.1719	1.2569	2.0340
75	1.3123	1.8572	1.9825	3.1738
90	1.9949	2.8896	3.1072	5.2781

Skew Index Option Prices

- The same methodology was used to price skew index option calls that were similarly out of the money for a monthly maturity.
- The $'t'$ copula was dropped as it priced options a bit too high.
- A Table presents a sample Skew Index Options

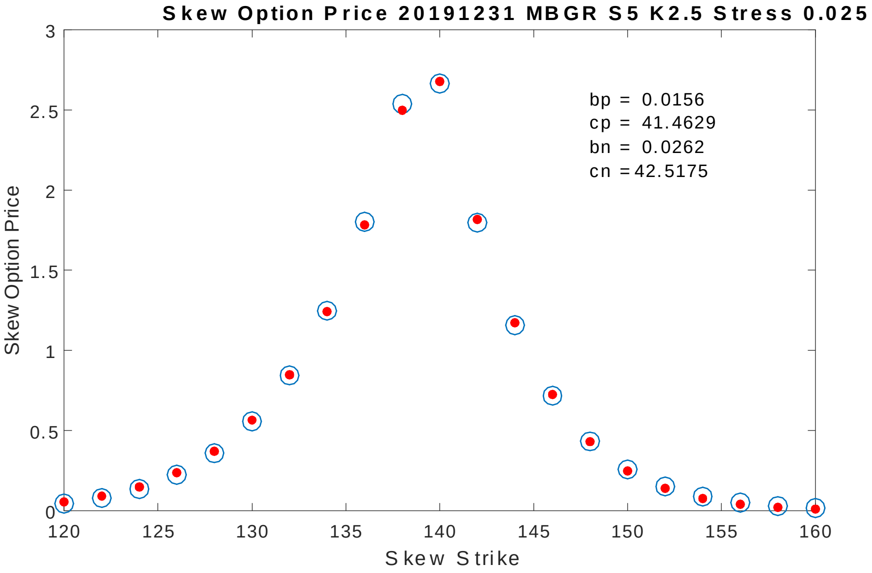
TABLE 7

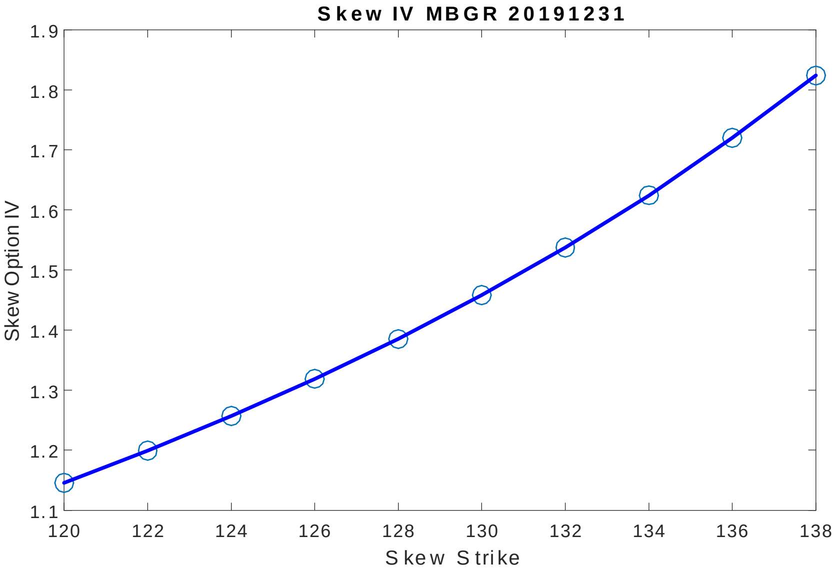
Sample of Skew Index Option Prices

Date	SPY	Skew Index	Strike	Gaussian	MBGIR	MBG
20190725	300.01	121.88	130	13.0074	16.5882	16.0711
			135	1.7434	2.8679	2.2357
			140	0.2120	0.4588	0.2537
20190808	292.87	113.96	120	25.7779	33.8614	33.8702
			125	1.4053	5.3283	4.4689
			130	0.0233	0.6924	0.3926
20190920	298.60	118.77	125	20.7941	30.3751	30.3892
			130	0.5884	4.1542	3.5363
			135	0.0040	0.3901	0.2642
20191018	298.75	128.01	135	35.3036	34.2741	34.0262
			140	5.2503	6.2899	5.2605
			145	0.5155	1.0189	0.5967
20191115	311.21	126.28	135	9.7250	16.9236	19.1172
			140	0.2176	1.8398	2.0862
			145	0.0018	0.1230	0.1147
20191231	321.59	139.52	150	11.1739	33.6734	29.2192
			155	0.5777	10.6390	8.1769
			160	0.0087	3.2909	1.9925

VIX and Skew of the Skew Index itself

- One may fit a bilateral gamma model to out of the money skew options to infer the VIX and Skew Index of the Skew Index.
- We present a graph of such a fit.
- The VIX of the Skew was 5.72% and the skew index was 101.56.
- Also presented is the implied volatility curve for the Skew options.


Skew Option Price 20191231 MBGR S5 K2.5 Stress 0.025
bp = 0.0156
cp = 41.4629
bn = 0.0262
cn = 42.5175
Skew Option Price
Skew Strike
3
2.5
2
1.5
1
0.5
0
120
125
130
135
140
145
150
155
160


Skew IV MBGR 20191231
Skew Option IV
Skew Strike
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
120
122
124
126
128
130
132
134
136
138

Conclusion

- Joint laws are formulated for the pair of returns on an asset and a related asset with an active option market.
- The joint laws are used to simulate the conditional return on the asset to be hedged given the return of the hedging asset.
- This residual risk is priced using pricing to acceptability to generate a function of the hedging asset price that must be earned to cover the residual risk exposures.
- The function of the hedging asset price is earned using a portfolio of bond, stocks and options.
- Options of the hedged asset are then priced at the cost of this hedge.
- The methodology is illustrated by pricing JPM options using XLF and VIX options using SPY.
- The methods are then applied to price options on CBOE Skew Index that currently does not have an active option market.