DOUBLE CHAIN LADDER

BY

María Dolores Martínez Miranda, Jens Perch Nielsen
and Richard Verrall

Abstract

By adding the information of reported count data to a classical triangle of reserving data, we derive a suprisingly simple method for forecasting IBNR and RBNS claims. A simple relationship between development factors allows to involve and then estimate the reporting and payment delay. Bootstrap methods provide prediction errors and make possible the inference about IBNR and RBNS claims, separately.

Keywords

Bootstrapping, Chain Ladder, Claims Reserves, Reserve Risk.

1. Introduction

This paper presents an extension to the model formulated by Verrall et al. (2010) for forecasting outstanding claims liabilities. However, it also shows that the resulting model is closely related to the chain ladder method (CLM), in a rather remarkable way. Indeed, it is possible to produce exactly the same results as the CLM, if a particular choice is made about the way the estimates are obtained. This may then raise the question of why a new method is necessary if similar results can be obtained from the old CLM. There are many different answers to this, which will become clearer throughout this paper. However, in summary, we would say that the CLM is "ad hoc" in the sense that it was not based on any underlying theory about the way claims arise. This makes it difficult to justify it theoretically, and it also means that any extensions or alterations also have an ad hoc flavour, including the way in which tail factors are produced; the way in which data is adjusted for inflation; and the way in which other information from the company is incorporated. We believe that this paper will mark an important landmark in the theory of claims reserving, and allow many natural and desirable extensions to be properly formulated.

The chain ladder method is one of the most celebrated and well-known methods of estimating outstanding liabilities in non-life insurance. It was

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developed at a time when computers were not readily available and it was important to have simple closed form expressions. Since then, the CLM has retained its appeal because it is a simple method that is intuitively appealing, and which often gives reasonable results. Because of these strengths, there may have been some reluctance to adopt alternative methods of estimating outstanding liabilities. It should be noted that the CLM was originally only a method: it was a clever algorithm which calculated numbers rather than a well defined model based on sound mathematical statistics where the calculations are the process of estimating the parameters in the model. Later developments in actuarial science helped to clarify the connection between the CLM and the world of mathematical statistics. There have since been a number of articles showing how the estimates from the CLM can be related to classical maximum likelihood estimation. For example, Mack (1991) showed that the estimators of the CLM model are classical maximum likelihood estimators of a multiplicative Poisson model, and Renshaw and Verrall (1998) extended this to the over-dispersed Poisson model. (See also Verrall (2000), England and Verrall (2002 and 2006) and Wüthrich and Merz (2008) for reviews of chain ladder type methods.) This connection was a step in the direction of formalizing the CLM such that the insights of mathematical statistics could be taken into account without losing the original intuition and straightforwardness of the CLM. However, it is noteworthy that the rationale behind these papers was to formulate a statistical model that gives the same reserve estimates as the CLM. It was not the aim to start from basic risk theory and formulate a new model for the run-off triangle. This latter approach was adopted by Bühlmann et al. (1980) and Norberg (1986, 1993 and 1999), and it was also the basis for the model derived by Verrall et al. (2010).

The CLM operates on aggregate loss data, that is, on sums of individual paid (or incurred) claims. From a theoretical point of view this naturally gives rise to a compound Poisson distribution. In this paper we present a method — related to the CLM — that can be formulated as a model of mathematical statistics and which explicitly acknowledges that data, are in fact, compound Poisson distributed. While the classical CLM is incapable of dividing predicted outstanding liabilities into RBNS and IBNR claims, we show that our simple regression approach including counts data is able to do exactly this in a very simple and concise way. Thus, our approach allows a full model description of the entire cash flow of the outstanding RBNS liabilities. This might be of major importance when non-life insurance companies soon have to meet the requirements of the new regulatory regime of Solvency II.

The method in this paper takes as its starting point the recent papers of Verrall, Nielsen and Jessen (2010) and Martínez-Miranda, Nielsen, Nielsen and Verrall (2011), which combine the observed incurred count data with the observed paid data. Both these sets of data can be represented in a run-off triangle and represent well-defined and reliable information that we can expect any insurance company to be able to provide for any of their business lines. Verrall et al. (2010) and Martínez-Miranda et al. (2011) use a delay function to model the time lag from a claim being incurred to when it is actually paid out, and the parameters of this micro level model were then estimated from aggregated incurred counts and aggregated payments. It was assumed that only one payment could occur per claim and this payment was modelled in the micromodel with a constant average severity. In this paper we generalise this model such that the average severity is allowed to change in the underwriting year direction of the paid triangle. This could be interpreted as allowing for a claims inflation effect in the underwriting year direction. This is different from Verrall et al. (2010) and Martínez-Miranda et al. (2011) that did not allow for claims inflation of the severity in the underwriting year direction, resulting in a model where the row effects in the paid triangle were inherited from the row effects in the incurred counts triangle.

Although the model is a (relatively) straightforward extension to that of Verrall et al. (2010), there are two remarkable points about it which bring us full-circle back to the CLM. The first is that it is possible to perform all the estimation necessary for the outstanding claims using just the simple algorithm of the CLM. The algorithm has to be applied twice, once on the incurred count data and then on the paid claims data, but each time it is just the simple chain ladder algorithm that is used. Because of this — because the estimation method uses the CLM twice — we call the new method the "double chain ladder method". The second remarkable point is that if the fitted counts (rather than the actual counts) are used to produce the forecasts of outstanding claims in the double chain ladder method, the results are exactly the same as those from the straightforward CLM applied to the triangle of paid claims. For this reason, it is possible to view this model as a different stochastic model for the CLM, with the significant distinction that it is based on assumptions made at the micro claims level.

Thus, all parameters of our model can be back-calculated from the two sets of well-known chain ladder development factors. It is also possible to compare directly the difference between the chain ladder estimator (stemming from theoretically estimated incurred counts) and the prediction of our model using the observed incurred counts for estimation. The approach of this paper also has the other advantages in common with Verrall et al. (2010) and Martínez-Miranda et al. (2011) that it includes a full stochastic cash flow approach; the full run-off is split between RBNS and IBNR reserves; and the micro statistical model allows the inclusion of tail factors in a completely consistent way.

The rest of the paper is set out as follows. Section 2 defines the data used in the method, and sets out the basic first moments assumptions of the model. Section 3 describes the estimation of the first moment parameters and explains the reason for calling this paper "Double Chain Ladder". In Section 4 we define how to obtain first moment forecasts of outstanding claims and thereby construct the reserves. Note that we use the terminology "RBNS reserve" and "IBNR reserve" throughout this paper as simplified way to denote the corresponding estimates of outstanding claims. Sections 2, 3 and 4 use very weak assumptions concerning only the first moments, in a very similar way to the crude chain ladder technique. When considering prediction errors and predictive distributions, it is necessary to make further assumptions about the second-moment properties of the underlying distributions. Thus, the remaining sections consider a less general case, with the assumptions for a particular model set out in Section 5. These assumptions are for the most simple case which is that there is one payment per claim. Although this is the most simple case that could be considered, we believe that the results will probably be satisfactory in most cases, for reasons set out in Section 5. Section 6 contains an illustration of the application of the method to the data used in Verrall et al. (2010) and Martínez-Miranda et al. (2011). Finally, Section 7 contains the conclusions.

2. Data and first moment assumptions

We assume that two data run-off triangles are available: aggregated payments and incurred counts defined as follows.

Aggregated incurred counts: $\aleph_m = \{N_{ij} : (i,j) \in \mathcal{I}\}$, with N_{ij} being the total number of claims of insurance incurred in year i which have been reported in year $i + j$ i.e. with j periods delay from year i; and $\mathcal{I} = \{(i,j) : i = 1, \ldots, m, j = 0, \ldots, m - 1; i + j \leq m\}$.

Aggregated payments: $\Delta_m = \{X_{ij} : (i,j) \in \mathcal{I}\}$, with X_{ij} being the total payments from claims incurred in year i and paid with j periods delay from year i.

Note that both data triangles are usually available in practice, and also that the methods can be applied to other shapes of data. We now outline the Double Chain Ladder model.

The counts and payments triangles $(\aleph_m, \Delta_m)$ are observed real data, but the settlement delay (or RBNS delay) is a stochastic component modelled by considering the micro-level unobserved variables, N_{ijl}^{paid}, which are the number of the future payments originating from the N_{ij} reported claims, which were finally paid with l periods delay, with $l = 0, \ldots, m - 1$.

Also, let $Y_{ijl}^{(k)}$ denote the individual settled payments which arise from N_{ijl}^{paid} $(k = 1, \ldots, N_{ijl}^{paid}, (i,j) \in \mathcal{I}, l = 0, \ldots, m - 1)$. Using these components, it is possible to estimate the RBNS reserve. For the IBNR reserve, it is necessary to model the IBNR delay.

With these definitions, the first moment conditions of the DCL model are formulated below.

M1. The counts N_{ij} are random variables with mean having a multiplicative parametrization $E[N_{ij}] = \alpha_i \beta_j$ and identification (Mack 1991), $\sum_{j=0}^{m-1} \beta_j = 1$.

M2. The mean of the RBNS delay variables is $E[N_{ijl}^{paid} | \aleph_m] = N_{i,j} \tilde{\pi}_l$, for each $(i,j) \in \mathcal{I}$, $l = 0, \ldots, m - 1$.

M3. Conditional on the number of payments, the mean of the individual payments size is given by $E[Y_{ijl}^{(k)} | N_{ijl}^{paid}] = \tilde{\mu}_l \gamma_i$.

These assumptions are very similar to those used in Verrall et al. (2010) and Martínez-Miranda et al. (2011), apart from M3. Note that the assumptions are written in terms of the first moments, rather than in terms of basic distributional assumptions. Note also that the mean in M3 depends on the accident year and the payment delay, but not on the reporting delay, so that $\mathrm{E}\big[Y^{(k)}_{i,j-l,l}\big] = \tilde{\mu}_l\,\gamma_i$ as well. It is possible to make M3 slightly simpler by replacing μ_l by μ: in which case, the only difference with Verrall et al. (2010) and Martínez-Miranda et al. (2011) would be that the mean claim size depends on the accident year through γ_i. This is the approach taken in Section 5, but we use the slightly more general assumption here.

Using M1 to M3 we have that

$$\mathrm{E}\left[\sum_{k=1}^{N^{paid}_{i,j-l,l}} Y^{(k)}_{i,j-l,l} \,|\, \aleph_m\right] = \mathrm{E}\left[\sum_{k=1}^{N^{paid}_{i,j-l,l}} \mathrm{E}[Y^{(k)}_{i,j-l,l} \,|\, \aleph_m, N^{paid}_{i,j-l,l}] \,|\, \aleph_m\right]$$
$$= \mathrm{E}[N^{paid}_{i,j-l,l}\,\tilde{\mu}_l\gamma_i \,|\, \aleph_m] = N_{i,j-l}\,\tilde{\pi}_l\,\tilde{\mu}_l\gamma_i$$

Note that the observed aggregated payments can be written as

$$X_{ij} = \sum_{l=0}^{j} \sum_{k=1}^{N^{paid}_{i,j-l,l}} Y^{(k)}_{i,j-l,l}, \text{ for each } (i,j) \in \mathcal{I}.$$

Therefore

$$\mathrm{E}[X_{ij} \,|\, \aleph_m] = \sum_{l=0}^{j} N_{i,j-l}\,\tilde{\pi}_l\,\tilde{\mu}_l\gamma_i = \sum_{l=0}^{j} N_{i,j-l}\,\pi_l\,\mu\gamma_i, \tag{1}$$

with $\mu = \sum_{l=0}^{m-1} \tilde{\pi}_l\tilde{\mu}_l$ and $\pi_l = \tilde{\pi}_l\,\tilde{\mu}_l/\mu$. Also the unconditional mean is

$$\mathrm{E}[X_{ij}] = \alpha_i\,\mu\gamma_i \sum_{l=0}^{j} \beta_{j-l}\pi_l. \tag{2}$$

It would be possible to use either (1) or (2) to construct the RBNS reserve. For the IBNR reserves, it is obviously necessary to use (2), with estimates of future numbers of incurred claims.

So for the RBNS reserve we would recommend that it is more appropriate to use (1), with the actual numbers of incurred claims. The exception to this is when we show that it is possible to produce exactly the standard chain ladder forecasts, when we will use (2). To do this, consider the over-dispersed Poisson stochastic model for chain ladder applied to the aggregated payments Δ_m. The CLM assumes that the X_{ij}'s are independent random variables with multiplicative parametrization

$$\mathrm{E}[X_{ij}] = \tilde{\alpha}_i\,\tilde{\beta}_j. \tag{3}$$

We use the identification from Mack (1991): $\sum_{j=0}^{m-1} \tilde{\beta}_j = 1$. Similarly, the CLM applied to the triangle of the incurred counts is defined by

$$E[N_{ij}] = \alpha_i \beta_j \tag{4}$$

with the identification $\sum_{j=0}^{m-1} \beta_j = 1$.

We will show in Section 4 that the standard chain ladder method arises from (1) as follows:

$$\alpha_i \gamma_i \mu = \tilde{\alpha}_i \tag{5}$$

$$\sum_{l=0}^{j} \beta_{j-l} \pi_l = \tilde{\beta}_j. \tag{6}$$

Therefore while other micro-structure formulations might exist, the specified by (5) and (6), is only one of several possible. In other words, we could consider the above model as a detailed specification of the CLM which allows to provide the full cash flow.

3. The estimation of the first moment parameters

To estimate the outstanding claims and thereby construct RBNS and IBNR reserves we need to estimate the parameters involved in assumptions M1 to M3 in Section 2 above and in this section we use the simple chain-ladder algorithm for this purpose. In fact, as implied by the name Double Chain Ladder (DCL), the classical chain ladder technique is applied twice and from this everything needed to estimate the first moments of the outstanding claims is available.

Denote the estimates from applying the chain-ladder algorithm to the triangles of paid claims, Δ_m, and incurred counts, $\aleph_m$, respectively, for $i = 1, \ldots, m$, $j = 0, \ldots, m-1$, by $(\hat{\alpha}_i, \hat{\beta}_j)$ and $(\hat{\tilde{\alpha}}_i, \hat{\tilde{\beta}}_j)$,

From these estimates the parameters $\pi = \{\pi_l : l = 0, \ldots, m-1\}$ can be estimated by solving the following linear system:

$$\begin{pmatrix} \hat{\tilde{\beta}}_0 \\ \vdots \\ \vdots \\ \hat{\tilde{\beta}}_{m-1} \end{pmatrix} = \begin{pmatrix} \hat{\beta}_0 & 0 & \cdots & 0 \\ \hat{\beta}_1 & \hat{\beta}_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \hat{\beta}_{m-1} & \cdots & \hat{\beta}_1 & \hat{\beta}_0 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \vdots \\ \vdots \\ \pi_{m-1} \end{pmatrix}. \tag{7}$$

Let $\hat{\pi}$ denote the solution of (7), with the individual elements denoted by $\hat{\pi}_l$, $l = 0, \ldots, m-1$.

Now we consider the estimation of the parameters involved in the means of individual payments. From the relationship (5) it can be seen that

$$\hat{\gamma}_i = \frac{\widehat{\widehat{\alpha}}_i}{\widehat{\alpha}_i \mu} \quad i = 1, \ldots, m. \tag{8}$$

Of course, the model is technically over-parameterised since there are too many inflation parameters. The simplest way to ensure identifiability is to set $\gamma_1 = 1$, and then the estimate of μ, $\hat{\mu}$ can be obtained from

$$\hat{\mu} = \frac{\widehat{\widehat{\alpha}}_1}{\widehat{\alpha}_1}. \tag{9}$$

Using $\hat{\mu}$, the estimates of the remaining parameters can be found from equation (8).

3.1. Estimating the DCL parameters from classical chain-ladder forward factors

As mentioned above we use the simple chain-ladder algorithm applied to the reserve triangles to estimate the parameters in (2). This makes it possible to estimate the outstanding claims and thereby construct RBNS and IBNR reserves, as described in Section 4.

As implied by the name Double Chain Ladder (DCL), the classical chain ladder technique is applied twice and from this everything needed to estimate the outstanding claims is available. Thus, the DCL estimation method uses the estimates of the chain ladder parameters from the triangle of counts and the triangle of payments. The two sets of estimators are denoted by $(\hat{\alpha}_i, \hat{\beta}_j)$ and $(\hat{\hat{\alpha}}_i, \hat{\hat{\beta}}_j)$, respectively, for $i = 1, \ldots, m, j = 0, \ldots, m - 1$.

There are various methods for obtaining these estimators: including using the straightforward chain ladder algorithm. The chain ladder algorithm will produce estimates of development factors, $\lambda_j, j = 1, 2, \ldots, m - 1$, which can be converted into estimates of β_j for $j = 0, \ldots, m - 1$ using the following identities which were derived in Verrall (1991).

$$\hat{\beta}_0 = \frac{1}{\prod_{l=1}^{m-1} \hat{\lambda}_i} \tag{10}$$

and

$$\hat{\beta}_j = \frac{\hat{\lambda}_j - 1}{\prod_{l=j}^{m-1} \hat{\lambda}_l} \tag{11}$$

for $j = 1, \ldots, m - 1$.

The estimates of the parameters for the accident years can be obtained by "grossing-up" the latest cumulative entry in each row. So, for example, the estimate of α_i can be obtained using

$$\widehat{\alpha}_i = \sum_{j=0}^{n-i} N_{ij} \prod_{j=m-i+1}^{m-1} \hat{\lambda}_j. \tag{12}$$

Similar expressions can be used for the parameters of the paid claims triangle. Alternatively, analytical expressions for the estimators can also be derived directly (rather than using the chain ladder algorithm) and further details can be found in Kuang, Nielsen and Nielsen (2009). Note that these will all give the same parameter estimates, and whatever method is used to obtain these estimates.

4. DCL estimates of the RBNS and IBNR reserves

The estimated parameters $\hat{\theta} = (\hat{\pi}, \hat{\mu}, \hat{\gamma})$ can be used to calculate a point forecast of the RBNS and IBNR components of the reserve. For the RBNS reserve, our recommendation is to condition on the actual numbers of claims, and use (1). For the IBNR reserve it is necessary first to construct predictions of future numbers of reported claims (using the CLM). Using the notation of Verrall et al. (2010) and Martínez-Miranda et al. (2011), we consider predictions over the following triangles (which are illustrated in Figure 1):

$$\mathcal{J}_1 = \{i = 2, \ldots, m;\ j = 0, \ldots, m-1 \text{ so } i + j = m+1, \ldots, 2m-1\}$$
$$\mathcal{J}_2 = \{i = 1, \ldots, m;\ j = m, \ldots, 2m-1 \text{ so } i + j = m+1, \ldots, 2m-1\}$$
$$\mathcal{J}_3 = \{i = 2, \ldots, m;\ j = m, \ldots, 2m-1 \text{ so } i + j = 3m, \ldots, 3m-2\}.$$

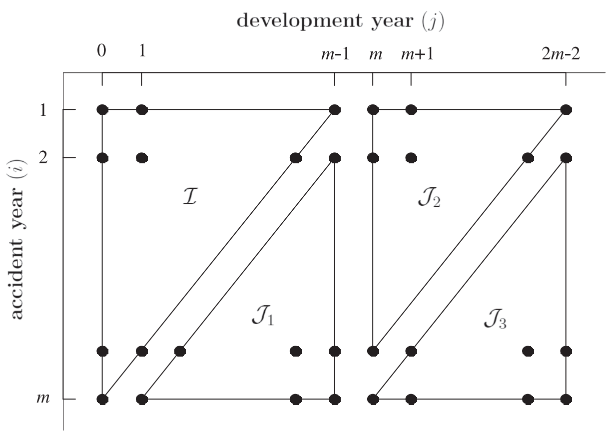

Figure 1: Index sets for aggregate claims data, assuming a maximum delay $m-1$.

Note that the standard CLM would produce forecasts over only $\mathcal{J}_1$. If the CLM is being used, it is therefore necessary to construct tail factors in some way. For example, this is sometimes done by assuming that the run-off will follow a set shape, thereby making it possible to extrapolate the development factors. In contrast, DCL provides also the tail over $\mathcal{J}_2 \cup \mathcal{J}_3$ using the same underlying assumptions about the development. Thus, DCL is consistent over all parts of the data, and uses the same assumptions concerning the delay mechanisms producing the data throughout.

In Section 4.1 we set out the way the outstanding claims can be estimated, ignoring the tail, and in Section 4.2 we consider also the tail.

4.1. Estimation of outstanding claims ignoring the tail

The estimates of outstanding claims using the CLM can be constructed using $\widehat{X}_{ij}^{CL} = \widehat{\widehat{\alpha}}_i \widehat{\widehat{\beta}}_j$ for $(i,j) \in \mathcal{J}_1$. There are a number of possibilities that could be used to estimate X_{ij} (for $(i,j) \in \mathcal{J}_1$) using the assumptions in section 2. For these assumptions, the estimates will be the sum of an RBNS component and an IBNR component. We consider first using (1) and (2) in order to show the connection with the CLM. It is possible to use either the actual numbers of claims or the fitted values for the RBNS component. Thus, there are two possible estimates, which we denote by $\widehat{X}_{ij}^{rbns(1)}$, based on (1), and $\widehat{X}_{ij}^{rbns(2)}$, based on (2):

$$\widehat{X}_{ij}^{rbns(1)} = \sum_{l=i-m+j}^{j} N_{i,j-l}\,\hat{\pi}_l \hat{\mu} \hat{\gamma}_i \tag{13}$$

and

$$\widehat{X}_{ij}^{rbns(2)} = \sum_{l=i-m+j}^{j} \widehat{N}_{i,j-l}\,\hat{\pi}_l \hat{\mu} \hat{\gamma}_i \tag{14}$$

where $\widehat{N}_{ij} = \hat{\alpha}_i \hat{\beta}_j$. The IBNR component always uses (2):

$$\widehat{X}_{ij}^{ibnr} = \sum_{l=0}^{i-m+j-1} \widehat{N}_{i,j-l}\,\hat{\pi}_l \hat{\mu} \hat{\gamma}_i . \tag{15}$$

The following theorem shows that using (14) and (15) gives exactly the same estimates of outstanding claims as the CLM.

Theorem 1. *For* $(i,j) \in \mathcal{J}_1$, *define*

$$\widehat{X}_{ij}^{CL} = \widehat{\widehat{\alpha}}_i \widehat{\widehat{\beta}}_j$$

$$\widehat{X}_{ij}^{rbns(2)} = \sum_{l=i-m+j}^{j} \widehat{N}_{i,j-l}\,\hat{\pi}_l \hat{\mu} \hat{\gamma}_i$$

$$\widehat{X}_{ij}^{ibnr} = \sum_{l=0}^{i-m+j-1} \widehat{N}_{i,j-l}\,\hat{\pi}_l \hat{\mu} \hat{\gamma}_i$$

where

$$\hat{\alpha}_i \hat{\mu} \hat{\gamma}_i = \widehat{\widehat{\alpha}}_i,$$

$$\sum_{l=0}^{j} \hat{\beta}_{j-l} \hat{\pi}_l = \widehat{\widehat{\beta}}_j .$$

Then $\widehat{X}_{ij}^{CL} = \widehat{X}_{ij}^{rbns(2)} + \widehat{X}_{ij}^{ibnr}$.

Proof

$$\begin{aligned}
\widehat{X}_{ij}^{rbns(2)} + \widehat{X}_{ij}^{ibnr} &= \sum_{l=i-m+j}^{j} \widehat{N}_{i,j-l}\,\hat{\pi}_l\,\hat{\mu}\,\hat{\gamma}_i + \sum_{l=0}^{i-m+j-1} \widehat{N}_{i,j-l}\,\hat{\pi}_l\,\hat{\mu}\,\hat{\gamma}_i \\
&= \sum_{l=0}^{j} \widehat{N}_{i,j-l}\,\hat{\pi}_l\,\hat{\mu}\,\hat{\gamma}_i \\
&= \sum_{l=0}^{j} \widehat{\alpha}_i\,\widehat{\beta}_{j-l}\,\hat{\pi}_l\,\hat{\mu}\,\hat{\gamma}_i \\
&= \sum_{l=0}^{j} (\widehat{\alpha}_i\,\hat{\mu}\,\hat{\gamma}_i)\,\widehat{\beta}_{j-l}\,\hat{\pi}_l \\
&= \widehat{\widehat{\alpha}}_i \sum_{l=0}^{j} \widehat{\beta}_{j-l}\,\hat{\pi}_l \\
&= \widehat{\widehat{\alpha}}_i\,\widehat{\widehat{\beta}}_j = \widehat{X}_{ij}^{CL}.
\end{aligned}$$

Thus, with these choices of estimators, it is possible to reproduce the CLM using DCL. Hence, the estimate of outstanding claims using DCL without the tail (defined over $\mathcal{J}_2 \cup \mathcal{J}_3$) will be exactly the same as the standard CLM estimate. Thus, we have shown that this can be considered as another specification of a stochastic model for the CLM. Note it is close to the first moment specifications defined in Section 2, which is based on detailed assumptions about the mechanisms generating the data, and is not simply defined in order to provide the same estimates as the CLM.

While it would be possible to simply use the specification used in the above theorem (i.e. use equation (2) together with the fitted numbers of claims), we believe that this is not the best thing to do. We believe that it is better to use the actual numbers of claims for the RBNS reserve estimate, rather than the fitted values. Thus, although our preferred model is similar in structure to the (detailed) CLM, it will not give the same results. We believe that the estimates from our model are preferable, and this could be seen as a (mild) criticism of the CLM, although the differences in the estimates will probably not be large. More importantly, we believe that our model is also superior to the basic CLM since all the parameters have a real interpretation. For this reason, when it is necessary to make alterations to the parameter estimates, or to move on to more sophisticated models within the same basic framework, we believe that our model will be preferable. When setting reserves, assessing capital requirements or proving adequate solvency conditions, we believe that it is easier to justify expert intervention on parameters that relate to real underlying factors. The development factors for the CLM applied to aggregated payments represent a complex combination of these underlying factors, and it is therefore more difficult to show that intervention to alter their values is based on well-formulated arguments and is not simply ad hoc, or (even worse) designed simply to get the “right” answer.

4.2. DCL including the run-off

Although the CLM does not include estimates for development years beyond the maximum already observed, it is necessary to include these when setting a reserve. In the context of the CLM, this is often done by fitting a curve of some form to the development year parameters. The tail consists of estimates over $\mathcal{J}_2 \cup \mathcal{J}_3$ and these are given quite naturally by DCL. The estimate of outstanding claims for the tail, for (2), is given by

$$\widehat{R}^{tail} = \sum_{(i,j) \in \mathcal{J}_2 \cup \mathcal{J}_3} \sum_{l=0}^{\min(j,d)} \widehat{N}_{i,j-l}\, \hat{\pi}_l \hat{\mu} \hat{\gamma}_i.$$

When estimating the tail we have implicitly assumed that we have seen a full run-off of the first underwriting year. If this is not the case our tail estimation underestimates the real tail and further adjustments might be necessary.

5. One statistical model with the DCL first moments

Many mathematical statistical models exist with the first moment structure of the DCL. In this section we go through the simplest and perhaps most important one: the one payment per reported claim model. The purpose of introducing a mathematical statistical DCL model is to be able to understand the distribution of the outstanding liabilities. With the selected estimation procedure of the first moment parameters of DCL, the statistical model will not affect the best estimate of these outstanding liabilities, but only their distribution. Of course, it is often the case that insurance claims give rise to more than one payment, or even to zero payments. However, the distributional assumptions in this section will be an approximation to the underlying true distribution. The one payment assumption shoud give a good approximation to the true underlying distribution that will be difficult to improve upon in practice. Even if full information on the historical payment process was available, the incorporation of this information into the statistical model would imply understanding the non-trivial time series correlation between payments in the payment process. If this correlation is not well modelled, the payment process (even when observed) in the statistical model might not improve the approximation to the underlying distribution. And also, in general when more information is included in an attempt to improve on the approximation to the DCL, care should be taken to ensure that the extra information is not counterweighted by the often unavoidable added model error introduced from modelling this extra information. In short, we introduce the simple one payment per claim model believing this to be a good first approximation to the underlying distribution, because the payment process in practice often is dominated by one of the payments.

A second assumption which is used for illustrative purposes is that the payments are gamma distributed. This would be simple to adjust if necessary, but the gamma distribution is a convenient distribution to start with. The approach can be easily generalised to other distributions, for example those with heavier pareto tails that might be appropriate for some data sets. Another adjustment of the severity distribution could be to consider a mixed model of for example a gamma distribution and the point measure at zero in order to allow for the distributional properties stemming from the possibility that some of the reported claims are indeed zero claims. However, such a zero-claim approach would include an extra parameter in the volatility estimation process below complicating our estimation procedure and exposition and we have therefore decided not to include it this study.

The model of Verrall et al. (2010) and Martínez-Miranda et al. (2011) was constructed by considering three stochastic components: the settlement delay, the individual payments and the reported counts. Here we consider a very similar model which is formulated under the assumptions given below.

D1. *The counts*: The counts N_{ij} are independent random variables from a Poisson distribution with multiplicative parametrization $\mathrm{E}[N_{ij}] = \alpha_i \beta_j$ and identification (Mack 1991), $\sum_{j=0}^{m-1} \beta_j = 1$.

D2 *The RBNS delay*. Given N_{ij}, the distribution of the numbers of paid claims follows a multinomial distribution, so that the random vector $\left(N_{ij0}^{paid}, \ldots, N_{ijd}^{paid}\right) \sim \mathrm{Multi}\,(N_{ij};\ p_0, \ldots, p_d)$, for each $(i,j) \in \mathcal{I}$, where d denotes the maximum delay ($d \leq m-1$). Let $\mathrm{p} = (p_0, \ldots, p_d)$ denote the delay probabilities such that $\sum_{l=0}^{d} p_l = 1$ and $0 < p_l < 1$, $\forall l$.

D3. *The payments*. The individual payments $Y_{ijl}^{(k)}$ are mutually independent with distributions f_i. Let μ_i and σ_i^2 denote the mean and the variance for each $i = 1, \ldots, m$. Assume that $\mu_i = \mu\gamma_i$, with μ being a mean factor and γ_i the inflation in the accident years. Also the variances are $\sigma_i^2 = \sigma^2 \gamma_i^2$ with σ^2 being a variance factor. Note that we are considering a more general situation than Verrall et al. (2010) by assuming that the distribution depends on the accident year, but a slightly less general case than in Section 2 where the mean also depended on the payment delay. In fact, the model of Verrall et al. (2010) assumes that $\gamma_i = 1$, $\forall i = 1, \ldots, m$.

D4. *Independence*: We assume also that the variables $Y_{ijl}^{(k)}$ are independent of the counts N_{ij}, and also of the RBNS and IBRN delays. Also, it is assumed that the claims are settled with a single payment or maybe as "zero-claims" (to deal with such a situation, it is necessary to consider a mixed-type distribution for the individual payments following the arguments in Verrall et al. (2010)).

Under the above assumptions the conditional mean of X_{ij} becomes

$$\mathrm{E}[X_{ij} \,|\, \aleph_m] = \sum_{l=0}^{\min(j,d)} N_{i,j-l}\, p_l \mu \gamma_i, \tag{16}$$

and therefore the unconditional mean is

$$E[X_{ij}] = \alpha_i \mu \gamma_i \sum_{l=0}^{\min(j,d)} \beta_{j-l} p_l. \tag{17}$$

Note that these first moments has the DCL mean structure defined in (1) and (2) by replacing the parameters $\pi = \{\pi_l : l = 0, \ldots, m-1\}$ with no restrictions on the values of π_l by the probabilities $p = \{p_l : l = 0,\ldots,d\}$, where $\sum_{l=0}^{d} p_l = 1$ and $0 < p_l < 1$, $\forall l$.

In general, we would expect the values of these parameters, p and π, to be very similar and that the predictions from the models would also be similar. Although these parameter estimates ($\hat{p}_j$ and $\hat{\pi}_j$) will be very similar (by definition), there may be differences for the longer reporting delays, which will affect the estimates of the reserves. This is illustrated in the example in Section 6.

Now using assumptions D1-D4 and arguments from Verrall et al. (2010) we can deal with higher moments calculations and provide the variance of the aggregated payments. Specifically the conditional variance of X_{ij} is approximately proportional to the mean. Since we have introduced the parameters γ_i, the dispersion parameter in this case depends on i.

$$V[X_{ij} | \aleph_m] \approx \frac{\sigma_i^2 + \mu_i^2}{\mu_i} E[X_{ij} | \aleph_m] \tag{18}$$

$$= \gamma_i \frac{\sigma^2 + \mu^2}{\mu} E[X_{ij} | \aleph_m] \tag{19}$$

$$= \varphi_i E[X_{ij} | \aleph_m]. \tag{20}$$

where $\varphi_i = \gamma_i \varphi$ and $\varphi = \frac{\sigma^2 + \mu^2}{\mu}$. This means that an over-dispersed Poisson model can be used to estimate the parameters.

5.1. Estimation of the reporting delay

We consider first the mean specification given in (2), and then discuss how to modify the results in order to provide estimates for the parameters in (17). So first we estimate the parameters $\pi = \{\pi_l : l = 0, \ldots, m-1\}$ by solving the linear system defined in (7). Let $\hat{\pi}$ denote such solution with the individual elements denoted by $\hat{\pi}_l$, $l = 0, \ldots, m-1$. Note that the values $\hat{\pi}_l$ could be negative and they could also sum to more than 1.

Considering the parameters in (17), there are a number of ways in which the parameters could be estimated including a constrained estimation procedure. However, we use a simple method, which we believe will provide reasonable estimates in most cases. For this, we estimate the maximum delay period, d, by counting the number of successive $\hat{\pi}_l \geq 0$ such that $\sum_{l=0}^{d-1} \hat{\pi}_l < 1 \leq \sum_{l=0}^{d} \hat{\pi}_l$. Then the estimated delay parameters in (2) are defined as

$$\hat{p}_1 = \hat{\pi}_1, \; l = 0, \ldots, d-1, \tag{21}$$

$$\hat{p}_d = 1 - \sum_{l=0}^{d-1} \hat{p}_l. \tag{22}$$

Thus, $\hat{p} = (\hat{p}_0, \ldots, \hat{p}_{d-1}, 1 - \sum_{l=0}^{d-1} \hat{p}_l)$.

5.2. Estimation of the parameters of the distribution of individual payments

The estimation of the mean of the distribution of individual payments, including the parameters which measure the inflation in the accident years comes from equations (8) and (9) above.

The estimation of the variances, $\sigma_i^2 (i = 1, \ldots, m)$ can be provided using the estimator proposed by Verrall et al. (2010). Specifically we estimate the over-dispersion parameter φ by

$$\hat{\varphi} = \frac{1}{n-(d+1)} \sum_{i,j \in \mathcal{I}} \frac{(X_{ij} - \widehat{X}_{ij}^{DCL})^2}{\widehat{X}_{ij}^{DCL} \hat{\gamma}_i}, \tag{23}$$

where $n = m(m+1)/2$ and $\widehat{X}_{ij}^{DCL}$ is the DCL estimate of $E[X_{ij} | \aleph_m]$ defined by $\widehat{X}_{ij}^{DCL} = \sum_{l=0}^{\min(j,d)} N_{i,j-l} \hat{p}_l \hat{\mu} \hat{\gamma}_i$. Then the variance factor of individual payment can be estimated by

$$\hat{\sigma}_i^2 = \hat{\sigma}^2 \hat{\gamma}_i^2 \tag{24}$$

for each $i = 1, \ldots, m$, where $\hat{\sigma}^2 = \hat{\mu}\hat{\varphi} - \hat{\mu}^2$.

6. Empirical illustration

This paper uses the same motor data as Verrall et al. (2010) and Martínez-Miranda et al. (2011), which originates from the general insurer RSA and is based on a portfolio of motor third party liability policies. The data available

TABLE 1

Run-off triangle of number of reported claims, N_{ij}.

i \ j	0	1	2	3	4	5	6	7	8	9
1	6238	831	49	7	1	1	2	1	2	3
2	7773	1381	23	4	1	3	1	1	3	
3	10306	1093	17	5	2	0	2	2		
4	9639	995	17	6	1	5	4			
5	9511	1386	39	4	6	5				
6	10023	1342	31	16	9					
7	9834	1424	59	24						
8	10899	1503	84							
9	11954	1704								
10	10989									

TABLE 2

RUN-OFF TRIANGLE OF AGGREGATED PAYMENTS, X_{ij}.

i \ j	0	1	2	3	4	5	6	7	8	9
1	451288	339519	333371	144988	93243	45511	25217	20406	31482	1729
2	448627	512882	168467	130674	56044	33397	56071	26522	14346	
3	693574	497737	202272	120753	125046	37154	27608	17864		
4	652043	546406	244474	200896	106802	106753	63688			
5	566082	503970	217838	145181	165519	91313				
6	606606	562543	227374	153551	132743					
7	536976	472525	154205	150564						
8	554833	590880	300964							
9	537238	701111								
10	684944									

consists of two incremental run-off triangles of dimension $m = 10$, one for reported counts, N_{ij}, and one for aggregated payments, X_{ij}, where $i = 1, \ldots, m$ denotes the accident year and $j = 0, \ldots, m - 1$ is the development year. The data are shown in Tables 1 and 2, respectively.

Table 3 gives the estimates of the parameters from the motor data for the model (D1-D4).

Point forecasts of the reported but not settled (RBNS) reserve and the incurred but not reported (IBNR) reserve can now be constructed along the lines of Verrall et al. (2010) Martínez-Miranda et al. (2011). The cash flow by calendar year is computed by summing the point forecasts $\widehat{X}_{ij}$ along the diagonals o $\mathcal{J}_1$. Table 4 shows the RBNS and IBNR reserve and also the total (RBNS+IBNR) forecasts. As a benchmark for comparison purposes, the

TABLE 3

ESTIMATED PARAMETERS FOR MOTOR DATA: THE PARAMETERS $\hat{\pi}_l$ ($l = 0, \ldots, 9$), THE DELAY PROBABILITIES $\hat{p}_l$ ($l = 0, \ldots, d = 8$), THE INFLATION PARAMETERS $\hat{\gamma}_i$ AND THE MEAN AND VARIANCE FACTORS μ AND σ^2 RESPECTIVELY.

$\hat{\pi}_l$	$\hat{p}_l$	$\hat{\gamma}_i$
0.3649	0.3649	1
0.2924	0.2924	0.7562
0.1119	0.1119	0.7350
0.0839	0.0839	0.8908
0.0630	0.0630	0.7840
0.0332	0.0332	0.7790
0.0245	0.0245	0.6605
0.0121	0.0121	0.7370
0.0158	0.0141	0.6990
–0.0012		0.8198
$\hat{\mu} = 208.3748$		
$\hat{\sigma}^2 = 2055944$		

TABLE 4

POINT FORECASTS OF CASHFLOW BY CALENDAR YEAR, IN THOUSANDS.
COLUMNS 2-4 SHOW THE PREDICTION FROM THE DOUBLE CHAIN LADDER METHOD (DCL).
COLUMN 5 SHOWS THE STANDARD CHAIN LADDER PREDICTIONS (CLM).

	DCL			
Future	RBNS	IBNR	Total	CLM
1	1260	97	1357	1354
2	672	83	754	754
3	453	35	489	489
4	292	26	319	318
5	165	20	185	185
6	103	12	115	115
7	54	9	63	63
8	30	5	36	36
9	0	5	5	2
10		1	1	
11		0.6	0.6	
12		0.4	0.4	
13		0.2	0.2	
14		0.1	0.1	
15		0.06	0.06	
16		0.03	0.03	
17		0.01	0.01	
Total	3030	296	3326	3316

predicted chain ladder reserve (denoted by CLM) is also shown in the last column of Table 4.

To derive the predictive distribution of the RBNS and IBNR reserves we consider bootstrap methods as proposed by Martínez-Miranda et al. (2011).

TABLE 5

DISTRIBUTION FORECASTS OF RBNS, IBNR AND TOTAL RESERVE, IN THOUSANDS. THE THREE FIRST COLUMN GIVE THE SUMMARY OF THE DISTRIBUTION FROM THE PROPOSED BOOTSTRAP METHOD WHICH TAKES INTO ACCOUNT THE UNCERTAINTY OF THE PARAMETERS. THE LAST COLUMN PROVIDES THE RESULTS FOR THE TOTAL RESERVE FOR THE BOOTSTRAP METHOD OF ENGLAND AND VERRALL (1999) AND ENGLAND (2002).

	Bootstrap predictive distribution			
	DCL			CLM
	RBNS	IBNR	Total	Total
mean	3013	294	3307	3314
pe	279	52	300	345
1%	2415	198	2661	2588
5%	2575	215	2821	2780
50%	2995	289	3291	3287
95%	3505	389	3813	3911
99%	3649	425	4020	4061

The bootstrap technique allows us to take into account the uncertainty of the parameters in the assumed model. The summary statistics from the RBNS and IBNR cash-flows, estimated by these bootstrap method are shown in Table 5. The root mean square error of prediction, commonly known as the prediction error, is denoted by "pe". We also compare the cash-flows derived from the proposed DCL method with the results from the BCL package in R by Gesmann et al. (2011) which implements the bootstrap method of England and Verrall (1999) and England (2002) for the CLM in Table 4.

7. Conclusions

This paper has presented a new model for outstanding claims, which is very closely connected with the chain ladder method. The estimation method employed is in fact the basic chain ladder algorithm, applied to two triangles. The predictive distribution of outstanding claims can also be found using the methods of Martínez-Miranda et al. (2011). We believe that this method provides a better approach to (approximating) the CLM than other stochastic models, since it is based on quantities that have a real interpretation in the context of insurance data. Thus, although it is possible to use DCL to reproduce the results of the CLM, we believe that it is better to use it in its purer form, where the assumptions are based on the underlying risk theory.

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María Dolores Martínez Miranda
University of Granada, Spain
E-Mail: mmiranda@ugr.es

Jens Perch Nielsen
Cass Business School, City University London, U.K.
E-Mail: Jens.Nielsen.1@city.ac.uk

Richard Verrall
Cass Business School, City University London, U.K.
E-Mail: R.J.Verrall@city.ac.uk