The Transition from Interbank Offered Rates to Risk-Free Rates: Evolution in Pricing Models for Interest Rate Modelling

Vincenzo Russo

Bayes Business School, London

November 8^{st}, 2023

Contents

About this presentation

This presentation is based on the following two papers:

- Russo, V., and F.J. Fabozzi. 2023. "The Transition from Interbank Offered Rates to Risk-Free Rates: Evolution in Pricing Models for Interest Rate Derivatives." *The Journal of Fixed Income*, 32(4):45-59
- Russo, V., and F.J. Fabozzi. 2023. "Caplets/Floorlets with Backward-Looking Risk-Free Rates under the One- and Two-Factor Hull-White Models " *The Journal of Derivatives*, 31(1): 96-110

The following (main) references have been also considered under the papers above:

- Bianchetti, M., and Carlicchi, M. 2011. "Interest Rates After the Credit Crunch: Multiple Curve Vanilla Derivatives and SABR."Accessed April 23, 2020 from: https://ssrn.com/abstract=1783070
- Lyashenko, A., and F. Mercurio. 2019. "Looking Forward to Backward-Looking Rates: a Modeling Framework for Term Rates Replacing LIBOR."Available online at SSRN: https://ssrn.com/abstract=3330240.
- Russo, V., and F.J. Fabozzi. 2016. "Pricing Coupon Bond Options and Swaptions Under the One-Factor Hull–White Model." *The Journal of Fixed Income*, 25(4):76–82
- Russo, V., and F.J. Fabozzi. 2017. "Pricing Coupon Bond Options and Swaptions under the Two-Factor Hull–White Model." *The Journal of Fixed Income*, 27(2):30–36

Background

Historical events:

- Interbank offered rates (IBORs) widely used in the global interbank market as a benchmark for loans, notes, and derivatives: i.e. LIBOR and EURIBOR.
- Overnight rates (ORs) used by banks in the overnight market: US Federal Funds Rate (FFR) or fed funds, EONIA for the Eurozone.
- Despite IBORs and ORs have always been considered risk-free rates (RFRs), during the 2007–2009 credit crisis, the spread between IBORs and ORs started to exhibit large spreads, suggesting that LIBOR was not a purely RFR before the crisis.
- Moreover, during the crisis, LIBOR quotes were affected by a series of fraudulent actions with illicit manipulation of the rates.

In light of these events:

- In 2013–2014, FSB conducted fundamental reviews of major interest-rate benchmarks and recommended developing alternative.
- In July 2017, the UK's FCA announced that it intended to no longer support banks currently participating in setting LIBOR.
- In 2019 ISDA decided that reliance on LIBOR should cease, and a transaction-based overnight rate—a so-called RFR—should be introduced as the alternative.
- Central banks across the globe announced that the rates needed to be reformed

Consequently:

- LIBOR rates have been discontinued and RFRs have been selected to replace LIBOR: SOFR in US, SONIA in UK, SARON in Switzerland, TONA in Japan.
- The Eurozone selected €STR (Euro Short-Term Rate), to supersede EONIA. EURIBOR is to be reformed to make it a transaction-based rate and no cessation date has been proposed yet.
- However, substituting IBORs (i.e. dual-curve) with RFRs (single-curve) is not fully shared by market; quoting credit sensitive rates is still a crucial factor for the market players (i.e. AMERIBOR).

IBORs forward-looking vs RFRs backward-looking:

- IBOR rates are forward-looking term rates and they are quoted for maturities from 1 week up to 1 year as a simple compounded interest rate.
- RFR rates are backward-looking rates and they are referenced by overnight rates that are set daily and are not quoted for multiple tenors (i.e., there is no term structure)
- Forward-looking approach: expectation about the future (market-implied prediction); consequently, the rate is known at the beginning of the period.
- Backward-looking approach: overnight rates compounded over the application period (setting-in-arrears); consequently, the rate is known at the end of the period.

In a single-currency economy, we can refer to three different interest-rate environments:

1. **IBOR-based environment adopting a forward-looking approach under the single-curve approach**: IBORs rates used for both discounting and forwarding. This approach was adopted before the 2007-2009 credit crisis. We denote this environment with the suffix L.
2. **IBOR-based environment adopting a forward-looking approach under the dual-curve approach**: RFRs (i.e. EONIA, fed-fund or €STR, SOFR) used for discounting and IBORs (i.e. LIBOR, EURIBOR) used for forwarding. This approach is in place starting from the 2007-2009 credit crisis. We denote this environment with the suffix D for discounting and $\tilde{L}$ for forwarding. This approach is still in force for evaluating interest derivatives linked to €STR-EURIBOR.
3. **RFR-based environment adopting a backward-looking approach under the single-curve approach**: RFRs (i.e. €STR, SOFR) used for both discounting and forwarding. We denote this environment with the suffix R.

In the following, the RFR-based environment adopting a backward-looking approach under the single-curve approach is considered for evaluating interest rate derivatives.

Under a single-currency economy, we define $C(t)$ to be the value of a cash account at time $t \geq 0$. We assume the cash account evolves as follows,

$$dC(t) = r(t)C(t)dt, \qquad C(0) = 1, \qquad C(t) = \exp\left\{ \int_0^t r(u)du \right\},$$

where $r(t)$ is the *instantaneous risk-free interest rate* often referred to as the *short-rate*. Assuming $\mathscr{F}_t$ as the filtration up to time t, it follows that

$$P(t, T) = \mathbb{E}^Q\left[\frac{C(t)}{C(T)} \middle| \mathscr{F}_t \right] = \mathbb{E}^Q\left[\exp\left\{ -\int_t^T r(u)du \right\} \middle| \mathscr{F}_t \right],$$

where $\mathbb{E}^Q$ is the expectation under the risk-neutral measure Q with numeraire $C(t)$.[1]

Denoting by $R(T_{i-1}, T_i)$ the daily-compounded setting-in-arrears risk-free rate (RFR) for the interval $[T_{i-1}, T_i)$, we have,

$$R(T_{i-1}, T_i) = \frac{1}{\tau(T_{i-1}, T_i)} \left[\prod_{j=1}^{d_i} \Big[1 + r(i,j)\delta(i,j) \Big] - 1 \right],$$ [2]

[1] The numeraire represents the market price of a traded asset.

[2] Where: (i) $\tau(T_{i-1}, T_i)$ denotes the time measure between T_{i-1} and T_i computed as a fraction of the year; (ii) d_i is the number of business days in the interval $[T_{i-1}, T_i)$; (iii) $r(i,j)$ is the RFR fixing on the business day j under the range $[T_{i-1}, T_i)$ with associated day-count fraction $\delta(i,j)$.

Taking the limit for the mesh of $\{\delta(i,1),...,\delta(i,j),...,\delta(i,d_i)\}$ going to zero, we have

$$R(T_{i-1},T_i)=\frac{1}{\tau(T_{i-1},T_i)}\left(e^{\int_{T_{i-1}}^{T_i} r(u)du}-1\right)=\frac{1}{\tau(T_{i-1},T_i)}\left(\frac{C(T_i)}{C(T_{i-1})}-1\right).$$

Replacing a discrete-time calculation (a product) with a continuous-time one (an integral) presents the advantage of more compact expressions and simpler calculations.

Following Lyashenko and Mercurio (2019), we calculate the expected value of $R(T_{i-1},T_i)$ at time $t \leq T_{i-1}$ under the T_i-forward risk-adjusted measure,

$$R(t,T_{i-1},T_i)=\mathbb{E}^{T_i}\left[R(T_{i-1},T_i)\middle|\mathscr{F}_t\right]=\frac{1}{\tau(T_{i-1},T_i)}\left[\frac{P(t,T_{i-1})}{P(t,T_i)}-1\right].$$

Note that the expected value computed at time t is calculated in the same way as the simple compounded forward interest rate under the IBOR framework

While, in the case $T_{i-1}<t<T_i$ where $\int_{T_{i-1}}^{t} r(u)du$ is known, we have that

$$R(t,T_{i-1},T_i)=\frac{1}{\tau(T_{i-1},T_i)}\left[e^{\int_{T_{i-1}}^{t} r(u)du}\frac{1}{P(t,T_i)}-1\right].$$

Consider the value of a forwad-start payer (or receiver) Overnight Indexed Swap (OIS) evaluated at time $t < T_0$, with maturity in T_n and notional N. Let's assume that the floating leg of the swap pays at each time T_i $(i = 1, 2, ..., n)$ and the fixed leg of the swap pays the fixed rate K at each date $T_j^{'}$ $(j = 1, 2, ..., m)$, with $T_0 = T_0^{'}$ and $T_n = T_m^{'}$.

Under the new RFRs environment, we refer to the OIS having a RFR as a reference floating rate (i.e. SOFR, €STER, SONIA) in place of the Interest Rate Swap (IRS) that typically has a IBOR-based floating rate.[3]

The value of a such payer OIS at time $t < T_0$, denoted by $OIS_R^{(p)}(t, T_0, T_n)$, is,

$$OIS_R^{(p)}(t, T_0, T_n) = N \sum_{i=1}^{n} P(t, T_i)\tau(T_{i-1}, T_i)R(t, T_{i-1}, T_i) - N \sum_{j=1}^{m} P(t, T_j^{'})\tau(T_{j-1}^{'}, T_j^{'})K.$$

Therefore, under the new RFRs environment, the swap's formula for $t < T_0$ is equal to the corresponding LIBOR-based one in a single-curve set up. This is not the case for $T_{i-1} < t < T_i$ for $i = 1, 2, ..., n$ where $\int_{T_{i-1}}^{t} r_R(u)du$ is known, so the formula above must be modified accordingly,

$$OIS_R^{(p)}(t, T_0, T_n) = NP(t, T_1)\left[e^{\int_{T_0}^{t} r_R(u)du} \frac{1}{P(t, T_1)} - 1\right] +$$

$$N \sum_{i=2}^{n} P(t, T_i)\tau(T_{i-1}, T_i)R(t, T_{i-1}, T_i) - N \sum_{j=1}^{m} P(t, T_j^{'})\tau(T_{j-1}^{'}, T_j^{'})K.$$

[3] OIS was also quoted before the IBOR fallback with overnight rates like FFR or EONIA.

Consider a European swaption that gives a counterparty the right to enter a payer (or receiver) OIS at time $t < T_0$ with fixed-rate equal to K and notional N. The underlying OIS is intended to be indexed to the new RFR as a reference floating rate.
Assuming that the swap rate $S_R(t, T_0, T_n)$ is a log-normal martingale under the swap measure denoted by $\mathbb{S}_R$ with the related numeraire, we have,

$$\frac{dS_R(t, T_0, T_n)}{S_R(t, T_0, T_n)} = \sigma_{S_R} dW_R^{\mathbb{S}_R}(t), \qquad \sum_{j=1}^{m} P(t, T_i^{'})\tau(T_{j-1}^{'}, T_j^{'}),$$

where σ_{S_R} is the volatility of the swap rate and $dW_R^{\mathbb{S}_R}(t)$ is a standard Brownian motion. Under this assumption, applying the Black (1976) model, the price of a payer swaption at time t is,

$$PSWT_R(t, T_0, T_n) = N \sum_{j=1}^{n} P(t, T_j^{'})\tau(T_{j-1}^{'}, T_j^{'})\left[S_R(t, T_0, T_n)\Phi(d_1) - K\Phi(d_2)\right].$$

where $d_1 = \dfrac{\log\left[\frac{S_R(t,T_0,T_n)}{K}\right] + \frac{1}{2}\sigma_{S_R}^2(T_0 - t)}{\sigma_{S_R}\sqrt{T_0 - t}}$ and $d_2 = \dfrac{\log\left[\frac{S_R(t,T_0,T_n)}{K}\right] - \frac{1}{2}\sigma_{S_R}^2(T_0 - t)}{\sigma_{S_R}\sqrt{T_0 - t}}$.

The valuation of swaptions is unaffected by whether or not the underlying rate is backward- or forward-looking.

Let's consider a caplet (or floorlet) evaluated at time t with maturity T_i, strike equal to K, and notional equal to N. Moreover, let's assume that it is written on a RFR backward-looking rate (i.e., €STR, SOFR).

Assuming $R(T_{i-1}, T_i)$ as reference rate fixing at time T_i under the backward-looking approach, the payoff of the caplet is,

$$N\tau(T_{i-1}, T_i)\left[R(T_{i-1}, T_i) - K\right]^{+} =$$

$$N\tau(T_{i-1}, T_i)\left[\frac{1}{\tau(T_{i-1}, T_i)}\left(e^{\int_{T_{i-1}}^{T_i} r(u)du} - 1\right) - K\right]^{+} =$$

$$N\left[e^{\int_{T_{i-1}}^{T_i} r(u)du} - [1 + K\tau(T_{i-1}, T_1)]\right]^{+} = N\left[\frac{C(T_i)}{C(T_{i-1})} - [1 + K\tau(T_{i-1}, T_1)]\right]^{+}.$$

Contents

Caplets with Backward-Looking RFRs: Black '76

Assuming that $R(t, T_{i-1}, T_i)$ is a martingale under the $\mathbb{T}_R^i$-forward measure, we have[4]

$$\frac{dR(t, T_{i-1}, T_i)}{R(t, T_{i-1}, T_i)} = \sigma_R(t, T_{i-1}, T_i) dW_R^{\mathbb{T}_R^i}(t),$$

where $\sigma_R(t, T_{i-1}, T_i)$ is the volatility of $R(t, T_{i-1}, T_i)$ and $dW_R^{\mathbb{T}_R^i}(t)$ is a standard Brownian motion under $\mathbb{T}_R$.

Following Lyashenko and Mercurio (2019), the volatility's function $\sigma_R(t, T_{i-1}, T_i)$ can be defined as a piece-wise differentiable deterministic function such that,

$$\sigma_R(t, T_{i-1}, T_i) = \sigma_R \times \min\left\{\frac{(T_i - t)^+}{(T_i - T_{i-1})}, 1\right\} = \sigma_R \times \begin{cases} 1, & \textit{for} \quad t \leq T_{i-1}, \\ \frac{(T_i - t)}{(T_i - T_{i-1})}, & \textit{for} \quad T_{i-1} < t < T_i, \\ 0, & \textit{for} \quad t \geq T_i. \end{cases}$$

The volatility's function as reported above ensures that the dynamic of $R(t, T_{i-1}, T_i)$ can be defined for any time t, including $t \geq T_i$.

[4] Specifically, Lyashenko and Mercurio (2019) refer to the extended T_i-forward measure where the numeraire is the extended price $P(t, T_i)$. The extended ZCB price is such that its definition can be extended to times $t > T_i$ in force of a self-financing strategy that consists of buying the ZCB with maturity T_i and reinvesting the proceeds of the bond's unit notional received at time $t > T_i$ at the risk-free rate $r_R(t)$ from time T_i onwards.

Caplets with Backward-Looking RFRs: Black '76

Under the RFR environment, the reference rate for the caplet becomes the RFR rate. Under the assumptions reported above, the caplet's price at time t becomes,

$$CPLT_R(t, T_{i-1}, T_i) = \tau(T_{i-1}, T_i) P(t, T_i) \mathbb{E}^{\mathbb{T}^i_R}\left[\left(R(T_{i-1}, T_{i-1}, T_i) - K\right)^+\right] =$$
$$\tau(T_{i-1}, T_i) P(t, T_i) \left[R(t, T_{i-1}, T_i)\Phi(d_1) - K\Phi(d_2)\right].$$

with,

$$d_1 = \frac{\log\left[\frac{R(t, T_{i-1}, T_i)}{K}\right] + \frac{1}{2}\Sigma_R(t, T_{i-1}, T_i)^2}{\Sigma_R(t, T_{i-1}, T_i)}, \qquad d_2 = \frac{\log\left[\frac{R(t, T_{i-1}, T_i)}{K}\right] - \frac{1}{2}\Sigma_R(t, T_{i-1}, T_i)^2}{\Sigma_R(t, T_{i-1}, T_i)}.$$

Under the RFR environment, the variance $\Sigma_R(t, T_{i-1}, T_i)^2$ is such that,

$$\Sigma_R(t, T_{i-1}, T_i)^2 = \sigma_R^2 \int_t^{T_i} \left(\min\left\{\frac{(T_i - s)^+}{(T_i - T_{i-1})}, 1\right\}\right)^2 ds =$$
$$\sigma_R^2 \left[\max(T_{i-1} - t, 0) + \frac{1}{3}\frac{(T_i - \max(T_{i-1}, t))^3}{(T_i - T_{i-1})^2}\right].$$

It is worth noting that, by definition, $\Sigma_R(t, T_{i-1}, T_i) \geq \Sigma_L(t, T_{i-1}, T_i)$. This is because a forward-looking caplet is known at the beginning of the period T_{i-1} while a backward-looking caplet is known at the end of the period T_i.

Contents

One- and Two-Factor Hull-White: stochastic dynamic for the short rate

In the case of the two-factor Hull-White model, the short-rate $r(t)$ at time $t \geq 0$, under the risk-neutral measure Q, is defined as follows:

$$r(t) = \alpha(t) + x(t) + y(t),$$

where $x(t)$ and $y(t)$ are state variables while $\alpha(t)$ is a deterministic function of time.[5] The variables $x(t)$ and $y(t)$ are such that,

$$dx(t) = -a_x x(t)dt + \sigma_x dW_x(t), \qquad x(0) = 0,$$
$$dy(t) = -a_y y(t)dt + \sigma_y dW_y(t), \qquad y(0) = 0,$$

where a_x, a_y, σ_x, and σ_y are model parameters while $dW_x(t)$ and $dW_y(t)$ are correlated Brownian motions such that $dW_x(t)dW_y(t) = \rho dt$ with $-1 \leq \rho \leq 1$.

The one-factor Hull-White model can be obtained by simply setting $\sigma_y \to 0^+$.

[5] Under the two-factor Hull-White model, the function $\alpha(t)$ is calculated as follows:

$$\alpha(t) = f^M(0,t) + \frac{\sigma_x^2}{2a_x^2}\left(1 - e^{-a_x t}\right)^2 + \frac{\sigma_y^2}{2a_y^2}\left(1 - e^{-a_y t}\right)^2 + \rho\frac{\sigma_x \sigma_y}{a_x a_y}\left(1 - e^{-a_x t}\right)\left(1 - e^{-a_y t}\right),$$

where $f^M(0,t)$ is the market instantaneous forward interest rate prevailing at time 0 for maturity t.

One- and Two-Factor Hull-White: stochastic dynamic for the ZCB price

In the case of the two-factor Hull-White model, the price of a ZCB with maturity T satisfies the following stochastic differential equation,

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt - \sigma_{P_x}(t,T)dW_x(t) - \sigma_{P_y}(t,T)dW_y(t),$$

where,

$$\sigma_{P_x}(t,T) = \sigma_x D_{P_x}(t,T),$$
$$\sigma_{P_y}(t,T) = \sigma_y D_{P_y}(t,T).$$

The quantities $D_{P_x}(t,T)$ and $D_{P_y}(t,T)$ are stochastic durations of the ZCB with respect to the factors x and y respectively,

$$D_{P_x}(t,T) = \frac{1}{a_x}\left[1 - e^{-a_x(T-t)}\right],$$
$$D_{P_y}(t,T) = \frac{1}{a_y}\left[1 - e^{-a_y(T-t)}\right].$$

Also in this case, the one-factor Hull-White model can be obtained by simply setting $\sigma_y \to 0^+$ in the formulation reported above.

Assuming $R(T_{i-1}, T_i)$ as reference rate fixing at time T_i under the backward-looking approach, the payoff of the caplet is,

$$N\tau(T_{i-1},T_i)\big[R(T_{i-1},T_i)-K\big]^+ = N\tau(T_{i-1},T_i)\left[\tfrac{1}{\tau(T_{i-1},T_i)}\left(e^{\int_{T_{i-1}}^{T_i} r(u)du}-1\right)-K\right]^+ =$$

$$N\left[e^{\int_{T_{i-1}}^{T_i} r(u)du}-[1+K\tau(T_{i-1},T_1)]\right]^+ = N\left[\tfrac{C(T_i)}{C(T_{i-1})}-[1+K\tau(T_{i-1},T_1)]\right]^+.$$

In order to derive the pricing formula for the caplet at time $t \leq T_{i-1}$, we set $X = \frac{C(T_i)}{C(T_{i-1})}$ and $K^{'} = 1 + K\tau(T_{i-1}, T_i)$. By applying standard no-arbitrage pricing theory under the risk-neutral measure, the price of the caplet with backward-looking rates is,

$$CPLT_R(t, T_{i-1}, T_i, K^{'}, N) = N\mathbb{E}^{Q_R}\left[e^{-\int_t^{T_i} r(u)du}\left(X-K^{'}\right)^+\right].$$

However, in order to solve the expectation under stochastic interest rates, we need to know the joint distribution of X and $e^{-\int_t^{T_i} r(u)du}$, with the two quantities that are correlated. Because in general this is not easy, we make a change of measure switching from the risk-neutral measure Q_R to the T_R^i-forward risk-adjusted measure.[6]

[6] With ZCB price as numeraire and forward measure T_i defined by the Radon-Nikodym derivative.

Under the T_i-forward risk-adjusted measure, the conditional expectation of $C(T_i)/C(T_{i-1})$ is,

$$\mathbb{E}^{T_i}\left[\frac{C(T_i)}{C(T_{i-1})}\middle|\mathscr{F}_t\right] = \frac{P(t, T_{i-1})}{P(t, T_i)},$$

while the variance of the logarithm of $C(T_i)/C(T_{i-1})$ is given by,

$$\mathbb{V}ar^{T_i}\left[\ln\frac{C(T_i)}{C(T_{i-1})}\middle|\mathscr{F}_t\right] = \Sigma_R(t, T_{i-1}, T_i)^2.$$

Consequently, we have that,

$$CPLT_R(t, T_{i-1}, T_i, K^{'}, N) = N\mathbb{E}^Q\left[e^{-\int_t^{T_i} r(u)du}\left(X - K^{'}\right)^+\right] = NP(t, T_i)\mathbb{E}^{T_i}\left[\left(X - K^{'}\right)^+\right].$$

So, the pricing of the caplet at time $t \leq T_{i-1}$ is,

$$CPLT_R(t, T_{i-1}, T_i, K^{'}, N) = NP(t, T_{i-1})\Phi(d_1) - NP(t, T_i)K^{'}\Phi(d_2),$$

where,

$$d_1 = \frac{\log\left[\frac{P(t,T_{i-1})}{P(t,T_i)}\frac{1}{K^{'}}\right] + \frac{1}{2}\Sigma_R(t, T_{i-1}, T_i)^2}{\Sigma_R(t, T_{i-1}, T_i)}, \quad d_2 = \frac{\log\left[\frac{P(t,T_{i-1})}{P(t,T_i)}\frac{1}{K^{'}}\right] - \frac{1}{2}\Sigma_R(t, T_{i-1}, T_i)^2}{\Sigma_R(t, T_{i-1}, T_i)}.$$

Two-Factor Hull-White: In particular, in the case of the variance we have that,[7]

$$\Sigma_R(t, T_{i-1}, T_i)^2 = \int_t^{T_{i-1}} \Big[\sigma_P(u, T_i) - \sigma_P(u, T_{i-1})\Big]^2 du + \int_{T_{i-1}}^{T_i} \sigma_P(u, T_i)^2 du,$$

where the first integral is the variance of the ZCB's forward price under the T-forward risk-adjusted measure while the second integral represents the additional variance, related to the time interval from T_{i-1} to T_i, considered in the case of backward-looking caplets. Consequently, we have that,

$$\Sigma_R(t, T_{i-1}, T_i)^2 = \int_t^{T_{i-1}} \sigma_P(u, T_i)^2 du + \int_t^{T_{i-1}} \sigma_P(u, T_{i-1})^2 du$$
$$-2\int_t^{T_{i-1}} \sigma_P(u, T_i)\sigma_P(u, T_{i-1}) du + \int_{T_{i-1}}^{T_i} \sigma_P(u, T_i)^2 du,$$

where $\sigma_P(t, T)^2 = \sigma_{P_x}(t, T)^2 + \sigma_{P_y}(t, T)^2 + 2\rho\sigma_{P_x}(t, T)\sigma_{P_y}(t, T)$.
In the specific case of the two-factor Hull-White model, we have that,

$$\sigma_P(t, T)^2 = \frac{\sigma_x^2}{a_x^2}\Big[1 - e^{-a_x(T-t)}\Big]^2 + \frac{\sigma_y^2}{a_y^2}\Big[1 - e^{-a_y(T-t)}\Big]^2 +$$
$$2\rho\frac{\sigma_x\sigma_y}{a_x a_y}\Big[1 - e^{-a_x(T-t)}\Big]\Big[1 - e^{-a_y(T-t)}\Big].$$

[7] A similar approach is adopted in Brigo and Mercurio (2006) for the pricing of Inflation-Indexed Caplets.

Substituting and integrating, we obtain the following expression for the variance to be used in the formula of a caplet with backward-looking rates,

$$
\begin{aligned}
&\Sigma_R(t, T_{i-1}, T_i)^2 = \frac{\sigma_x^2}{2a_x^3}\left[1 - e^{-a_x(T_i - T_{i-1})}\right]^2\left[1 - e^{-2a_x(T_{i-1}-t)}\right] + \\
&\frac{\sigma_y^2}{2a_y^3}\left[1 - e^{-a_y(T_i - T_{i-1})}\right]^2\left[1 - e^{-2a_y(T_{i-1}-t)}\right] + \\
&2\rho\frac{\sigma_x\sigma_y}{a_x a_y(a_x + a_y)}\left[1 - e^{-a_x(T_i - T_{i-1})}\right]\left[1 - e^{-a_y(T_i - T_{i-1})}\right]\left[1 - e^{-(a_x+a_y)(T_{i-1}-t)}\right] + \\
&\frac{\sigma_x^2}{a_x^2}\left[T_i - T_{i-1} + \frac{2}{a_x}e^{-a_x(T_i - T_{i-1})} - \frac{1}{2a_x}e^{-2a_x(T_i - T_{i-1})} - \frac{3}{2a_x}\right] + \\
&\frac{\sigma_y^2}{a_y^2}\left[T_i - T_{i-1} + \frac{2}{a_y}e^{-a_y(T_i - T_{i-1})} - \frac{1}{2a_y}e^{-2a_y(T_i - T_{i-1})} - \frac{3}{2a_y}\right] + \\
&2\rho\frac{\sigma_x\sigma_y}{a_x a_y}\left[T_i - T_{i-1} - \frac{1 - e^{-a_x(T_i - T_{i-1})}}{a_x} - \frac{1 - e^{-a_y(T_i - T_{i-1})}}{a_y} + \frac{1 - e^{-(a_x+a_y)(T_i - T_{i-1})}}{a_x + a_y}\right].
\end{aligned}
$$

One-Factor Hull-White: In the case of the one-factor Hull-White model, the pricing model reported above remains the same with the exception of the variance that is calculated as follows,

$$\Sigma_R(t, T_{i-1}, T_i)^2 = \frac{\sigma_x^2}{2a_x^3}\left[1 - e^{-a_x(T_i - T_{i-1})}\right]^2\left[1 - e^{-2a_x(T_{i-1}-t)}\right] +$$
$$\frac{\sigma_x^2}{a_x^2}\left[T_i - T_{i-1} + \frac{2}{a_x}e^{-a_x(T_i - T_{i-1})} - \frac{1}{2a_x}e^{-2a_x(T_i - T_{i-1})} - \frac{3}{2a_x}\right].$$

The variance we derive in the case of the one-factor Hull-White model is consistent with the results obtained by Turfus (2020) and coincides with the results obtained by Hofmann (2020) and Hasegawa (2021).

Contents

It can be noted that $\Sigma_R(t, T_{i-1}, T_i)^2$ is the sum of the variance of a European ZCB option related to IBORs ($\Sigma_L(t, T_{i-1}, T_i)^2$) and an additional term,

$$
\begin{aligned}
&\Sigma_R(t, T_{i-1}, T_i)^2 = \Sigma_L(t, T_{i-1}, T_i)^2 + \\
&\frac{\sigma_x^2}{a_x^2}\left[T_i - T_{i-1} + \frac{2}{a_x}e^{-a_x(T_i - T_{i-1})} - \frac{1}{2a_x}e^{-2a_x(T_i - T_{i-1})} - \frac{3}{2a_x}\right] + \\
&\frac{\sigma_y^2}{a_y^2}\left[T_i - T_{i-1} + \frac{2}{a_y}e^{-a_y(T_i - T_{i-1})} - \frac{1}{2a_y}e^{-2a_y(T_i - T_{i-1})} - \frac{3}{2a_y}\right] + \\
&2\rho\frac{\sigma_x\sigma_y}{a_x a_y}\left[T_i - T_{i-1} - \frac{1 - e^{-a_x(T_i - T_{i-1})}}{a_x} - \frac{1 - e^{-a_y(T_i - T_{i-1})}}{a_y} + \frac{1 - e^{-(a_x + a_y)(T_i - T_{i-1})}}{a_x + a_y}\right].
\end{aligned}
$$

Under the two-factor Hull-White model, the quantity,

$$
\begin{aligned}
&\Sigma_L(t, T_{i-1}, T_i)^2 = \frac{\sigma_x^2}{2a_x^3}\left[1 - e^{-a_x(T_i - T_{i-1})}\right]^2\left[1 - e^{-2a_x(T_{i-1} - t)}\right] + \\
&\frac{\sigma_y^2}{2a_y^3}\left[1 - e^{-a_y(T_i - T_{i-1})}\right]^2\left[1 - e^{-2a_y(T_{i-1} - t)}\right] + \\
&2\rho\frac{\sigma_x\sigma_y}{a_x a_y(a_x + a_y)}\left[1 - e^{-a_x(T_i - T_{i-1})}\right]\left[1 - e^{-a_y(T_i - T_{i-1})}\right]\left[1 - e^{-(a_x + a_y)(T_{i-1} - t)}\right],
\end{aligned}
$$

is the variance of a European ZCB option evaluated at time t, written on a ZCB with maturity T_i, expiring in T_{i-1} and strike equal to $K'' = 1/K'$.

Relation with Caplets under Forward-Looking Rates

By applying standard no-arbitrage pricing theory, denoting by $ZCBC(t, T_{i-1}, T_i, K^{''}, 1)$ the price of a ZCB call option with notional equal to 1, under the T_{i-1}-forward risk-adjusted measure[8] we have,

$$ZCBC(t, T_{i-1}, T_i, K^{''}, 1) = P(t, T_{i-1})\mathbb{E}^{T_{i-1}}\left[\left(P(T_{i-1}, T_i) - K^{''}\right)^+ \middle| \mathscr{F}_t\right] =$$

$$P(t, T_i)\Phi(d_1^*) - K^{''} P(t, T_{i-1})\Phi(d_2^*),$$

where,

$$d_1^* = \frac{\log\left[\frac{P(t,T_i)}{P(t,T_{i-1})}\frac{1}{K^{''}}\right] + \frac{1}{2}\Sigma_L(t, T_{i-1}, T_i)^2}{\Sigma_L(t, T_{i-1}, T_i)},$$

and

$$d_2^* = \frac{\log\left[\frac{P(t,T_i)}{P(t,T_{i-1})}\frac{1}{K^{''}}\right] - \frac{1}{2}\Sigma_L(t, T_{i-1}, T_i)^2}{\Sigma_L(t, T_{i-1}, T_i)}.$$

The corresponding ZCB put option can be obtained by put-call parity,

$$ZCBP(t, T_{i-1}, T_i, K^{''}, 1) = ZCBC(t, T_{i-1}, T_i, K^{''}, 1) - P(t, T_i) + XP(t, T_{i-1}).$$

[8] Also in this case, for sake of convenience, a change of measure is applied switching from the risk-neutral measure Q to the T_{i-1}-forward risk-adjusted measure.

Relation with Caplets under Forward-Looking Rates

Let's consider now a caplet evaluated at time t with maturity T_i, strike equal to K, and notional equal to N. Let's also assume that it is written on a LIBOR-based forward-looking rate, denoted $L(T_{i-1}, T_i)$, fixing at time T_{i-1}. The payoff is,

$$N\tau(T_{i-1}, T_i)\big[L(T_{i-1}, T_i) - K\big]^+.$$

As the price of a Caplet with forward-looking rates, denoted by $CPLT_L(t, T_{i-1}, T_i, K^{'}, N)$, is equivalent to a multiple of the price of a European ZCB put option we have,

$$CPLT_L(t, T_{i-1}, T_i, K^{'}, N) = N[1 + K\tau(T_{i-1}, T_i)]ZCBP(t, T_{i-1}, T_i, K^{''}, 1).$$

Rearranging the formula we have,

$$CPLT_L(t, T_{i-1}, T_i, K^{'}, N) = NP(t, T_{i-1})\Phi(d_1^{**}) - NP(t, T_i)K^{'}\Phi(d_2^{**}),$$

where,

$$d_1^{**} = -d_2^{*} = \frac{\log\left[\frac{P(t,T_{i-1})}{P(t,T_i)}\frac{1}{K'}\right] + \frac{1}{2}\Sigma_L(t, T_{i-1}, T_i)^2}{\Sigma_L(t, T_{i-1}, T_i)},$$

and,

$$d_2^{**} = -d_1^{*} = \frac{\log\left[\frac{P(t,T_{i-1})}{P(t,T_i)}\frac{1}{K'}\right] - \frac{1}{2}\Sigma_L(t, T_{i-1}, T_i)^2}{\Sigma_L(t, T_{i-1}, T_i)}.$$

Relation with Caplets under Forward-Looking Rates

Note that the pricing formula of a caplet with forward-looking rates is the same as in the backward-looking case except for the variance that, in the case of caplet with forward-looking rates, is equal to the variance of the related ZCB put option.

Finally, also note that under the same parameterization of the one- and two-factor Hull-White model, we have that $\Sigma_R(t, T_{i-1}, T_i)^2 \geq \Sigma_L(t, T_{i-1}, T_i)^2$ and, consequently, $CPLT_R(t, T_{i-1}, T_i, K^{'}, N) \geq CPLT_L(t, T_{i-1}, T_i, K^{'}, N)$.

This is because a backward-looking caplet is known at the end of the period T_i while a forward-looking caplet is known at the beginning of the period T_{i-1}. This implies that a backward-looking caplet is always more expensive than a forward-looking caplet under the same values for the parameters featuring the one- and two-factor Hull-White model.

However, it is worth noting that the relation between backward-looking and forward-looking caplets/floorlets ($CPLT_R(t, T_{i-1}, T_i, K^{'}, N) \geq CPLT_L(t, T_{i-1}, T_i, K^{'}, N)$) holds assuming the same parameterization, namely (i) same strike, (ii) same underlying interest rate curve, and (iii) same value for the parameters underlying the variance's function. When backward-looking and forward-looking caplets/floorlets are quoted in the financial market they refer to different yield curves and implied volatilities. Consequently, the relation $CPLT_R(t, T_{i-1}, T_i, K^{'}, N) \geq CPLT_L(t, T_{i-1}, T_i, K^{'}, N)$ could not be always verified.

Contents

Numerical results on Caplets

Some numerical results obtained by running the proposed model for the pricing of caplets/floorlets with backward-looking risk-free rates under the two-factor Hull-White model are reported in the following.

In order to perform the calculations, we refer to a unique set of parameters to run the two-factor Hull-White model. This is a deliberate choice because starting from the same model allows us to appreciate the relation between backward-looking and forward-looking caplets/floorlets; in particular, it is possible to appreciate the fact that backward-looking are, theoretically, always more expensive than the forward-looking caplets/floorlets. For the scope of the numerical analysis, we assume that the initial yield curve is flat at a level of 3% (annually compounded) for all maturities. Moreover, we assume the following values for the model parameters: $a_x = 4.0\%$, $\sigma_x = 1.5\%$, $a_y = 5.0\%$, $\sigma_y = 0.5\%$, $\rho = -20\%$.

Numerical results for at-the-money (ATM), in-the-money (ITM), and out-of-the-money (OTM) caplets/floorlets are provided, comparing, in terms of price and volatility obtained under the two-factor Hull-White model, caplets/floorlets with backward-looking rates with caplets/floorlets with forward-looking rates.

Exhibit 1: Price and volatility of ATM caplets/floorlets with $K = 3\%$.

		Forward-looking approach		**Backward-looking approach**	
Description	Maturity:	Volatility:	Price:	Volatility:	Price:
Caplet	1Y	0.55%	21.52	0.63%	24.93
Caplet	2Y	0.93%	35.46	0.98%	37.51
Caplet	3Y	1.18%	43.57	1.22%	45.16
Caplet	4Y	1.36%	49.08	1.40%	50.41
Caplet	5Y	1.52%	53.00	1.55%	54.16
Floorlet	1Y	0.55%	21.52	0.63%	24.93
Floorlet	2Y	0.93%	35.46	0.98%	37.51
Floorlet	3Y	1.18%	43.57	1.22%	45.16
Floorlet	4Y	1.36%	49.08	1.40%	50.41
Floorlet	5Y	1.52%	53.00	1.55%	54.16

Exhibit 3: Price and volatility of OTM caplets/floorlets with $K = 4\%$.

		Forward-looking approach		Backward-looking approach	
Description:	Maturity:	Volatility:	Price:	Volatility:	Price:
Caplet	1Y	0.55%	5.45	0.63%	7.87
Caplet	2Y	0.93%	16.84	0.98%	18.64
Caplet	3Y	1.18%	24.56	1.22%	26.02
Caplet	4Y	1.36%	30.15	1.40%	31.40
Caplet	5Y	1.52%	34.32	1.55%	35.43
Floorlet	1Y	0.55%	54.00	0.63%	56.41
Floorlet	2Y	0.93%	63.97	0.98%	65.77
Floorlet	3Y	1.18%	70.32	1.22%	71.78
Floorlet	4Y	1.36%	74.57	1.40%	75.83
Floorlet	5Y	1.52%	77.45	1.55%	78.57

Numerical results on Caplets

Exhibit 2: Price and volatility of ITM caplets/floorlets with $K = 2\%$.

		Forward-looking approach		Backward-looking approach	
Description:	Maturity:	Volatility:	Price:	Volatility:	Price:
Caplet	1Y	0.55%	53.93	0.63%	56.32
Caplet	2Y	0.93%	63.82	0.98%	65.61
Caplet	3Y	1.18%	70.12	1.22%	71.58
Caplet	4Y	1.36%	74.35	1.40%	75.59
Caplet	5Y	1.52%	77.21	1.55%	78.31
Floorlet	1Y	0.55%	5.38	0.63%	7.78
Floorlet	2Y	0.93%	16.69	0.98%	18.48
Floorlet	3Y	1.18%	24.37	1.22%	25.82
Floorlet	4Y	1.36%	29.92	1.40%	31.17
Floorlet	5Y	1.52%	34.08	1.55%	35.18

There are three key findings from the results reported in the three exhibits.

First, as expected, the greater the volatility, the higher the price. Moreover, consistent with expectations, the longer the time to maturity, the greater the volatility. Furthermore, we can see that ATM caplets (floorlets) have a greater (lower) value than the OTM caplets (floorlets) and ITM caplets (floorlets) values greater (lower) than ATM caplets (floorlets).

The second main finding is that not only in the case of the forward-looking approach but also under the backward-looking approach, the standard relationship in terms of put-call parity, between ATM caplets and floorlet is respectfully:

$$CPLT_{L/R}(t, T_{i-1}, T_i, K^{'}, N) + NK\tau(T_{i-1}, T_i)P(t, T_i) =$$
$$FLRT_{L/R}(t, T_{i-1}, T_i, K^{'}, N) + NP(t, T_{i-1}) - NP(t, T_i).$$

Finally, looking at the overall results, the price/volatility of backward-looking caplets/floorlets are more expensive/higher than the forward-looking caplets/floorlets, given the same parametrization of the two-factor Hull-White model.

Vincenzo Russo

Head of Group Life & Health Risk Capital Modelling
Assicurazioni Generali S.p.A., Milan, Italy
russovincent@gmail.com

Disclaimer

This presentation represents only the opinion of Vincenzo Russo and it is not necessarely meant to represent the position or opinion of his employer. Vincenzo Russo and not his employer is solely responsible for any errors.