Mortality Dependence and Longevity Bond Pricing:

A Dynamic Factor Copula with the GAS Structure

Hua Chen, Temple University

Richard D. MacMinn, Illinois State University

Tao Sun, Temple University

Motivation

- Why multi-population mortality models?
  - Natural hedging is not perfect
  - Issues with a small population
  - Needs from mortality securitization
  - Hedge effectiveness and basis risk
- A case on Swiss Re
  - Transferred over $2.2 bn of mortality risk since 2003
  - Insured about $1 bn of pension liability for Royal County of Berkshire Pension Fund in 2009

Kortis Longevity Bond

- SPV: the Kortis Capital Ltd
- Time: issued in December 2010, will mature in January 2017
- Single tranche: 2010-I Class E Notes rated BB+ by S&P
- Size: USD 50 million
- Spread: 472 bps + six-month LIBOR
- Trigger: Longevity Divergence Index Value (LDIV)
- Reference populations: male aged 75-85 in E&W and male aged 55-65 in the US
- The bond hedges residual longevity risk

LDIV Computation

- Average mortality improvement over 8 years, for each age and country

$$\Delta m_x^j(2008,2016) = 1 - \left(\frac{m_{x,2016}^j}{m_{x,2008}^j}\right)^{1/8}$$

- Average mortality improvement over age groups, for each year and country

$$\Delta m_{2016}^j(x_1,x_2) = \frac{1}{1+x_2-x_1}\sum_{x=x_1}^{x_2}\Delta m_x^j(2008,2016)$$

- Compute the longevity divergence index value (LDIV)

$$\text{LDIV}_{2016} = \Delta m_{2016}^{\text{EW}}(75,85) - \Delta m_{2016}^{\text{US}}(55,65)$$

- Compute the principal reduction factor (PRF) based on LDIV, as a call option spread

$$\text{PRF} = \frac{[\text{LDIV}_{2016} - 3.4\%]_+ - [\text{LDIV}_{2016} - 3.9\%]_+}{3.9\% - 3.4\%}$$

Literature

- Two-population mortality model
  - Li and Lee (2005): augmented common factor model
  - Li and Hardy (2011): cointegrated Lee-Carter model
  - Dowd et al. (2011): gravity model (see also Jarnerand Kryger 2011)
  - Carins et al. (2011): mean-reverting process for mortality improvements
  - Zhou et al. (2013a): two-population Lee-Carter model with transitory jumps
  - Yang and Wang (2013) and Zhou et al. (2013b): co-integration process

- Hunt and Blake (2013)
  - "There is little evidence to suggest that there is mean revision to a constant difference in relative mortality rates in the shorter run".
  - M7 model in Carins et al. (2009) with a number of higher order age/period terms
  - Project the period functions based on co-integration
  - 401 and 525 free parameters for EW and US, respectively

A Glance at Our Model

- First Stage - Conditional distribution
  - Lee and Carter (1992): ARIMA with constant volatility
  - Lee and Miller (2001): volatility changing over time
  - Gao and Hu (2009), Giacometti et al. (2012), Chai et al. (2013): GARCH models
  - Giacometti et al. (2009, 2012) and Wang et al. (2013): heavy-tailed errors

- Second Stage – Copula model
  - Copulas used to model the bivariate survival function (Frees et al. 1996, etc)
  - Not used in the multiple-population analysis
  - Most in the Elliptical or Archimedean family

- Advantage: Separate development of the marginal distributions and the copula model

- Chen, MacMinn and Sun (2013)
- Wang, Yang and Huang (2013)

A Glance at Our Model

- Chen et al. (2013) v.s. Wang et al. (2013)

	Chen, MacMinn and Sun (2013)	Wang, Yang and Huang (2013)
Conditional Distributions	Best ARMA-GARCH Model	ARIMA (0,1,0)
Residual Assumptions	Gaussian or Student-t	Generalized Hyperbolic
Copula Models	Factor Copula	Gaussian, Student-t, Gumbel and Clayton
Dynamic Structure	No	Dynamic Conditional Correlation (DCC)

- We use a factor copula proposed by Oh and Patton (2012)
  - A simple linear, additive structure – attractive for high dimension applications.
  - More flexibility according to the number of variables and available data.

- What's new?
  - Two-factor copula model - common factor and country factor
  - Dynamic dependence - generalized autoregressive score (GAS)

Methodology: Conditional Marginal Distribution

- We allow each series to have a time-varying conditional mean and conditional variance, each governed by parametric models.

$$r_{it} = \mu_i\left(Z_{t-1}\right) + \sigma_i\left(Z_{t-1}\right)\eta_{it} \text{ for } i = 1, 2, \ldots, N \text{ and } Z_{t-1} \in F_{t-1}$$

- The estimate of standardized residuals can be specified as

$$\hat{\eta}_{i,t} = \frac{r_{i,t} - \mu_i\left(Z_{t-1}; \hat{\alpha}\right)}{\sigma_i\left(Z_{t-1}; \hat{\alpha}\right)}$$

- We then estimate the distribution of the standardized residuals using EDF

$$\hat{F}_i\left(\eta\right) \equiv \frac{1}{T+1}\sum_{t=1}^{T} \mathbf{1}\left\{\hat{\eta}_{it} \le \eta\right\}$$

- The use of the EDF allows us to non-parametrically capture skewness and excess kurtosis in the residuals, if present, and allows these characteristics to differ across populations.

Methodology: Factor Copula

- Use a copula model to describe the joint distribution of innovations

$$\eta_t \equiv [\eta_{1t}, \cdots, \eta_{Nt}]^T \sim iid\ F_\eta = C\left(F_{\eta_1}, \cdots, F_{\eta_N}; \theta\right)$$

- The copula parameter is estimated using the simulation-based estimation approach (Oh and Patton, 2012).

- Their method is close to Simulated Method of Moments (SMM), but is not exactly the same as SMM, because the “moments” are functions of rank statistics.

- The SMM copula estimator is based on simulation from some joint distribution of a vector of latent variables *X*, with a implied copula *C*(*θ*).

Methodology: Factor Copula

- One-factor copula model

$$X_i = \lambda_i Z + \varepsilon_i, \;\; Z \perp \varepsilon_i \quad \forall i$$

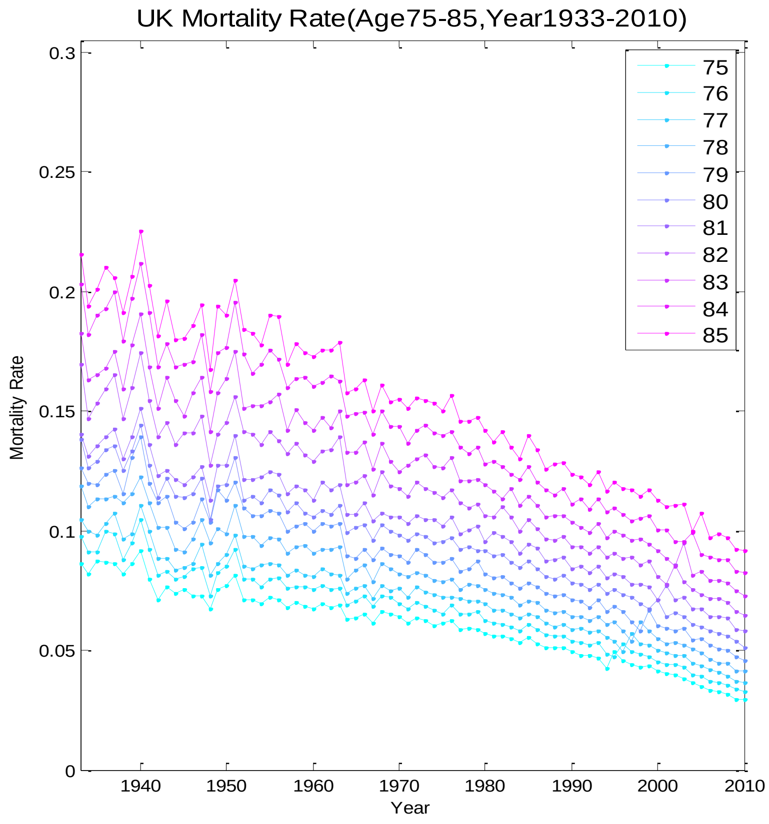
$$X \equiv \left[X_1, \cdots, X_N\right]' \sim F_X = C\left(F_{X_1}, \cdots, F_{X_N}\right)$$

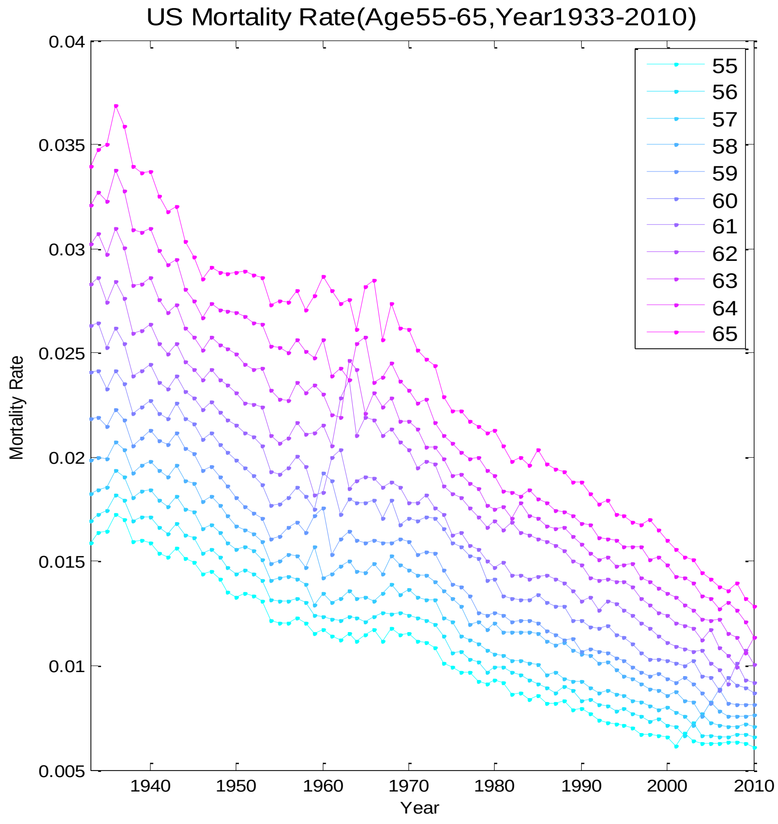
- Flexibility
  - A fat-tailed common factor – correlated mortality jumps
  - A skewed common factor – asymmetric tail dependence structure
  - The same loading – equidependence
  - Different loading – heterogeneity
  - A subset of variables have the same loading: same region, same country
  - Multiple factors
- Extensions
  - Two-factor copula
  - Dynamic structure

Sample and Data

- Data source: Human Mortality Database
- Country: US and UK (England & Wales)
- Period: 1933-2010.
- Age groups: UK males with ages 75 to 85 and US males with ages 55 to 65

Sample and Data





Summary Statistics: Mortality Improvement

Age	No. of Obs	Mean	Std. Dev	Median	Min	Max	Skewness	Kurtosis
UK75	77	-0.0141	0.0499	-0.0160	-0.1398	0.1538	0.3582	4.5500
UK76	77	-0.0143	0.0543	-0.0179	-0.1541	0.1265	0.0068	3.6428
UK77	77	-0.0137	0.0523	-0.0167	-0.1457	0.1352	0.0631	3.8443
UK78	77	-0.0137	0.0508	-0.0102	-0.1516	0.1400	0.1781	3.9124
UK79	77	-0.0133	0.0550	-0.0159	-0.1524	0.1396	0.2660	3.6663
UK80	77	-0.0129	0.0515	-0.0163	-0.1501	0.1262	-0.2525	3.5902
UK81	77	-0.0115	0.0512	-0.0111	-0.1415	0.1222	-0.1110	3.0261
UK82	77	-0.0125	0.0553	-0.0159	-0.1527	0.0958	-0.3692	3.1232
UK83	77	-0.0120	0.0590	-0.0152	-0.1535	0.1108	-0.3055	3.0358
UK84	77	-0.0117	0.0517	-0.0053	-0.1396	0.1029	-0.2580	3.0568
UK85	77	-0.0111	0.0572	-0.0078	-0.1501	0.1476	-0.1128	2.9797
US55	77	-0.0125	0.0301	-0.0144	-0.0702	0.0954	0.4832	4.0686
US56	77	-0.0122	0.0279	-0.0116	-0.0862	0.0858	0.1399	4.3162
US57	77	-0.0123	0.0278	-0.0150	-0.0737	0.0782	0.2449	3.6865
US58	77	-0.0124	0.0315	-0.0164	-0.0988	0.0861	0.2061	4.2160
US59	77	-0.0128	0.0313	-0.0125	-0.1383	0.0773	-0.4744	5.6521
US60	77	-0.0133	0.0319	-0.0132	-0.0912	0.0950	0.4094	4.2988
US61	77	-0.0136	0.0322	-0.0152	-0.0936	0.1053	0.6199	5.3628
US62	77	-0.0135	0.0337	-0.0148	-0.1214	0.1067	0.4639	5.4813
US63	77	-0.0127	0.0302	-0.0170	-0.0911	0.1173	1.1282	6.8815
US64	77	-0.0135	0.0260	-0.0126	-0.0862	0.0704	0.1689	4.0406
US65	77	-0.0126	0.0288	-0.0134	-0.1047	0.0761	0.2808	4.5029

Conditional Distributions

- Conditional mean ARMA(r,m) $$Y_t = c + \sum_{i=1}^{r} \varphi_i Y_{t-i} + \sum_{j=1}^{m} \theta_j \varepsilon_{t-j} + \varepsilon_t$$
- Conditional variance GARCH(p,q) $$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \sigma_{t-i}^2 + \sum_{j=1}^{q} \beta_j \varepsilon_{t-j}^2$$
- Model selection based on BIC

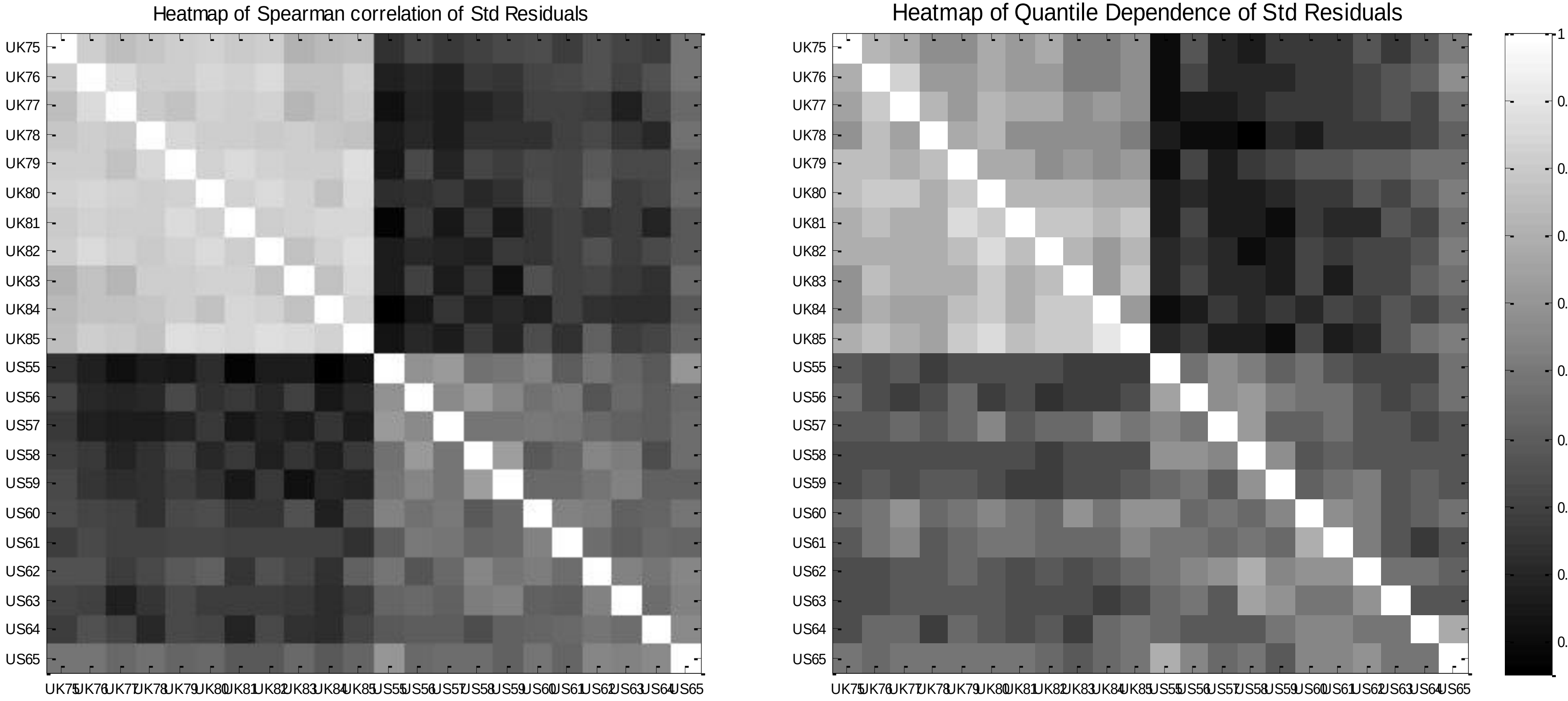
Model Fitting

	Student t Innovation					Gaussian Innovation				
Group	AR	MA	ARCH	GARCH	BIC	AR	MA	ARCH	GARCH	BIC
UK75	0	1	1	0	-255.42	0	1	1	0	-259.74
UK76	1	3	0	0	-228.86	0	1	0	0	-230.26
UK77	0	1	0	0	-237.85	0	1	0	0	-238.47
UK78	0	1	0	0	-241.38	0	1	0	0	-242.02
UK79	0	1	0	0	-227.12	0	1	0	0	-230.22
UK80	1	2	0	0	-250.37	0	1	0	0	-250.88
UK81	0	1	0	0	-247.15	0	1	0	0	-250.12
UK82	0	1	0	0	-236.01	0	1	0	0	-238.05
UK83	0	1	1	0	-241.91	0	1	1	0	-246.37
UK84	0	1	0	0	-250.76	0	1	0	0	-249.53
UK85	0	1	0	0	-240.71	0	1	0	0	-243.81
US55	0	1	0	0	-315.22	0	1	0	0	-318.37
US56	1	0	0	0	-325.69	1	0	0	0	-328.23
US57	1	0	0	0	-328.59	1	0	0	0	-329.99
US58	0	1	0	0	-315.25	1	0	0	0	-317.01
US59	4	0	1	0	-310.82	2	0	1	0	-315.12
US60	0	1	0	0	-311.98	0	1	0	0	-313.81
US61	0	1	1	0	-310.11	0	1	1	0	-311.15
US62	2	0	0	0	-310.03	0	1	0	0	-305.54
US63	1	2	1	0	-314.15	0	1	1	0	-313.56
US64	0	1	0	0	-336.46	0	1	0	0	-337.21
US65	2	2	1	0	-321.72	0	1	1	0	-325.98

Standardized Residual

Group	No. of Obs	Mean	Std Dev	Median	Min	Max	Skewness	Kurtosis
UK75	77	-0.0365	1.0059	-0.1038	-3.1116	1.8940	-0.0848	3.0609
UK76	77	-0.0047	1.0065	-0.1274	-2.6796	2.8051	0.3805	3.7236
UK77	77	-0.0034	1.0065	0.0218	-2.5167	2.9362	0.5854	3.9953
UK78	77	-0.0062	1.0065	-0.0005	-2.6398	3.0046	0.2890	4.0395
UK79	77	-0.0040	1.0065	-0.1538	-2.1980	2.6826	0.5620	3.4419
UK80	77	-0.0107	1.0065	-0.0905	-2.8497	3.2326	0.3609	4.2594
UK81	77	-0.0037	1.0066	0.0774	-2.2761	3.2594	0.5403	3.7864
UK82	77	-0.0223	1.0063	-0.0642	-2.8195	2.8292	0.1795	3.8767
UK83	77	-0.0027	1.0066	-0.1750	-2.2317	2.2632	0.2317	2.4017
UK84	77	-0.0158	1.0064	-0.0520	-2.6640	3.5607	0.5670	4.8633
UK85	77	-0.0163	1.0064	0.0400	-2.1232	2.9289	0.2442	3.5872
US55	77	0.0092	1.0065	0.0023	-2.1761	3.1860	0.4014	3.6188
US56	77	0.0000	1.0066	-0.0135	-2.6135	3.1041	0.0381	3.7244
US57	77	0.0000	1.0066	-0.1011	-2.4630	2.8015	0.1064	3.7129
US58	77	0.0000	1.0066	-0.0969	-2.6777	2.3411	0.1093	3.4469
US59	77	-0.0075	1.0065	0.1162	-2.6654	2.5952	-0.1233	3.0720
US60	77	0.0028	1.0066	-0.0702	-2.1547	3.4938	0.3497	4.0414
US61	77	-0.0066	1.0066	-0.0130	-2.5206	3.7093	0.4542	4.4881
US62	77	-0.0561	1.0050	-0.1249	-2.1874	4.0170	0.8119	5.2592
US63	77	0.0072	1.0065	-0.1496	-2.2099	4.6148	1.3338	7.5363
US64	77	0.0070	1.0065	0.0639	-2.4140	3.2976	0.4130	4.0859
US65	77	0.0690	1.0042	-0.0555	-2.3199	2.5927	0.3396	3.0321

Dependence Measures



Factor Copula Model

- Two-factor copula model:

$$X_i = \lambda_i^0 Z_0 + \lambda_i^c Z_c + \varepsilon_i$$

$$Z_c \perp Z_0,\ c = \text{US or UK};$$

$$\varepsilon_i \perp Z_0 \text{ and } \varepsilon_i \perp Z_c, \forall i;$$

$$X \equiv \left[X_1, \cdots, X_N\right]' \sim F_X = C\left(F_{X_1}, \cdots, F_{X_N}\right)$$

- Generalized Autoregressive Score (GAS) (see Creal et al. 2013)

$$\log \lambda_{i,t}^j = \omega_i + \beta \log \lambda_{i,t-1}^j + \alpha s_{i,t-1},\ j \in \{0, c\}$$

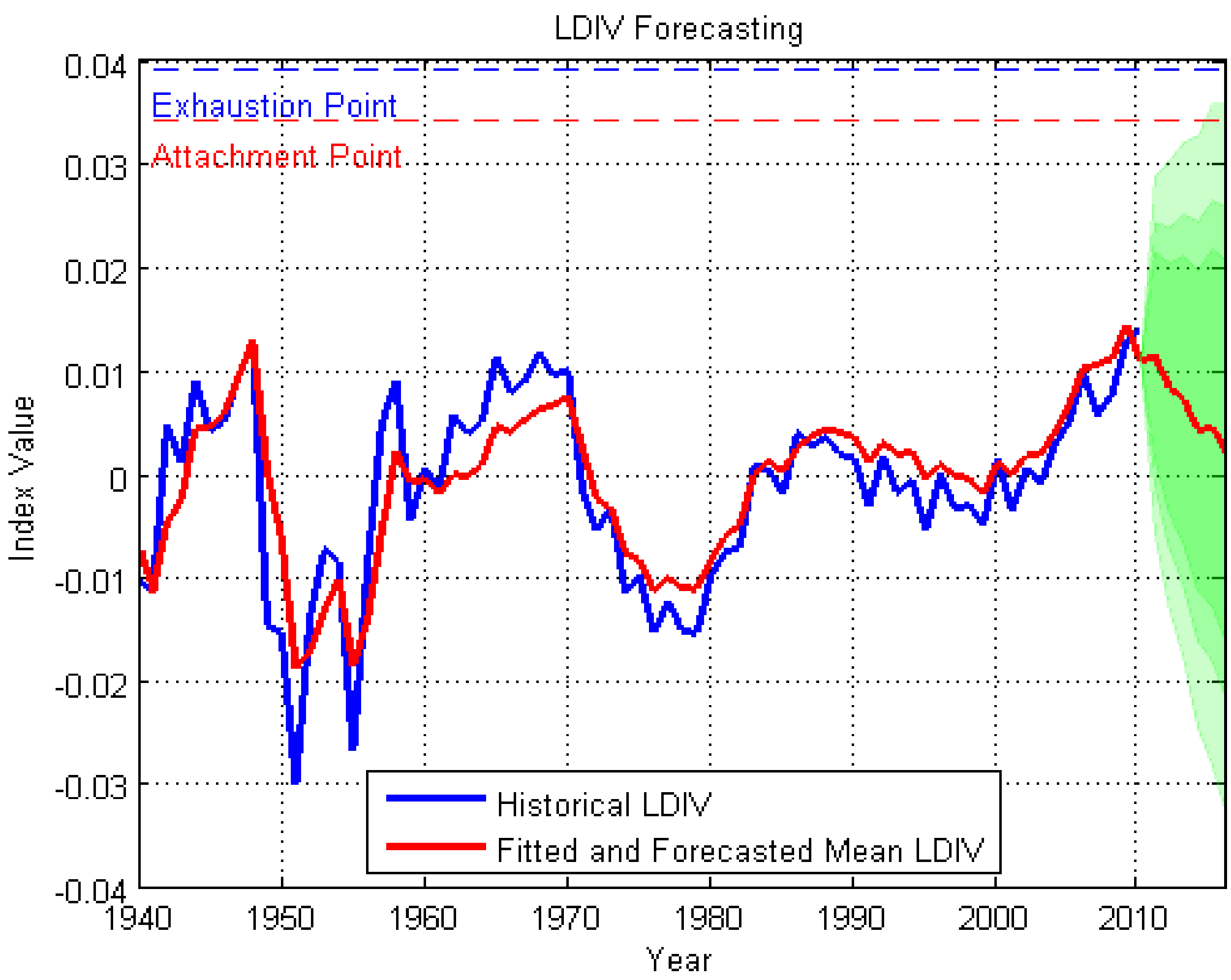
$$\text{where } \omega_i = (1-\beta) E\left[\log \lambda_{it}^j\right] \text{ and } s_{it} = \frac{\partial \log c(u_t; \boldsymbol{\lambda}_t)}{\partial \lambda_{it}^j}$$

Factor Copula Model

- Parameter estimates for the factor copula model

Student *t* (Common Factor)-Student *t* (Country Factor)-Normal (Idiosyncratic Factor) model				
	Estimate	Std Error	t-statistics	p-value
1/DoF (Common Factor)	0.0546	0.0116	4.6911	0.0000
1/DoF (Country Factor)	0.0549	0.0049	11.1775	0.0000
α (GAS)	0.0947	0.0031	30.2260	0.0000
β (GAS)	0.9305	0.0020	464.9544	0.0000
Log Likelihood		-3206.6		

LDIV Fitting and Forecasting



Probability Distribution of Principal Loss

LDIV ≥	PRF>=	RMS Estimation	Our Estimation
3.4%	0.00%	0.88%	1.08%
3.5%	20.00%	0.72%	0.92%
3.6%	40.00%	0.58%	0.72%
3.7%	60.00%	0.47%	0.58%
3.8%	80.00%	0.38%	0.48%
3.9%	100.00%	0.30%	0.36%

Pricing Methodologies

- Mortality risk pricing
  - Arbitrage free pricing (Cairns et al. 2006b, Bauer et al. 2010).
  - Wang transform (Dowd et al. 2006, Denuit et al. 2007, Lin and Cox 2008, Chen and Cox 2009)
  - Esscher transform (Chen et al. 2010, Li et al. 2010).
  - Maximum entropy principle (Li 2010, Kogure and Kurachi 2010, Li and Ng 2011).
  - CAT bond pricing (Lane, 2000; Chen and Cummins, 2010)

- CAT bond pricing
  - Expected loss principle
  - Variance or standard deviation principle
  - Risk cubic pricing (Lane 2000)
  - Multi-linear or log-linear model (Major and Krups, 2003, Berge 2005; Lane and Beckwith, 2008, Lane and Mahul, 2008; Dieckmann, 2008)

CAT Bond Pricing Model

Variables	(1) Ln(Spread)	(2) Ln(Spread)	(3) Ln(Spread)	(4) Ln(Spread)	(5) Ln(Spread)
Ln(CEL)	0.188***	0.165***	0.191***	0.142**	0.141**
	(0.0413)	(0.0502)	(0.0615)	(0.0616)	(0.0678)
Ln(PFL)	0.658***	0.666***	0.651***	0.585***	0.571***
	(0.0364)	(0.0293)	(0.0310)	(0.0254)	(0.0283)
Size			0.000284	0.000453*	0.000451*
			(0.000298)	(0.000242)	(0.000251)
Term			-0.00409**	-0.00118	-0.000333
			(0.00165)	(0.00156)	(0.00146)
Indemnity			0.0523	-0.0342	-0.0311
			(0.0500)	(0.0438)	(0.0438)
US				0.240***	0.277***
				(0.0439)	(0.0782)
Mortality				-0.341	-0.106
				(0.233)	(0.361)
Wind				0.189***	0.225***
				(0.0381)	(0.0566)
Earthquake				0.0804**	0.0388
				(0.0323)	(0.0519)
US*Mortality					-0.524
					(0.429)
US*Wind					-0.0447
					(0.0754)
US*Earthquake					0.0522
					(0.0664)
Δ ROL		1.204***	1.178***	1.425***	1.268***
		(0.347)	(0.348)	(0.313)	(0.282)
BB_Spread		4.095***	4.447***	3.438***	3.364***
		(0.852)	(0.844)	(0.729)	(0.729)
Swiss Re		-0.243***	-0.231***	-0.135**	-0.103**
		(0.0414)	(0.0535)	(0.0553)	(0.0462)
Constant	-0.0365	-0.0880	-0.0577	-0.786***	-0.909***
	(0.144)	(0.120)	(0.124)	(0.157)	(0.136)
Observations	277	277	277	277	277
R-squared	0.770	0.821	0.827	0.868	0.873
Adj R-squared	0.769	0.818	0.822	0.862	0.866

Kortis Bond Pricing

LDIV ≥	PRF ≥	RMS Estimation	Our Estimation
3.4%	0	0.88%	1.08%
3.5%	20%	0.72%	0.92%
3.6%	40%	0.58%	0.72%
3.7%	60%	0.47%	0.58%
3.8%	80%	0.38%	0.48%
3.9%	100%	0.30%	0.36%
PFL		0.88%	1.08%
CEL		62.05%	68.04%
Premium		472bps	215bps

Conclusions

- Uses factor copula models to analyze mortality dependence
- Focus on the residual risk (tail dependence)
- Efficient for high dimensional data
- Future research
  - Other heavy tailed distributions?
  - Cohort effect?
  - More accurate bond fitting?