Analysis of VIX-linked fee incentives in variable annuities via continuous-time Markov chain approximation

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Faculty of Actuarial Science and Insurance Research Seminar
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Motivation

1. Variable annuity (VA) liabilities are “mark-to-market".
2. Financial guarantees embedded in VAs financed through fee charges.
3. Constant fee rate leads to misalignment between insurer income streams and the market value of liabilities.
4. VIX index negatively correlated with equity, i.e. *leverage effect*.

Motivation

1. Financial guarantees in VAs similar to put option: increases when volatility increases.
2. VIX-linked fee structure: better alignment of VA guarantees with fees paid by policyholders.
3. Adverse selection with constant fee: policyholders lapse when market is stable, and refrain from lapsation when market is volatile.
4. VIX-linked fee: fee is low when market is stable, hence less incentive to lapse.

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NEXUS RISK MANAGEMENT

White Paper

VIX® for Variable Annuities - Part II

A study considering the advantages of tying a Variable Annuity fee to VIX

July 2013

February 3, 2010

Print This Article

SunAmerica Links VA Rider Fees to Volatility Index

By Editor Test *Wed, Feb 3, 2010* **SHARE ON:** Twitter Facebook LinkedIn

Designed for the Polaris series of variable annuities, the two new guaranteed minimum withdrawal benefits are called Income Plus 6% and Income Builder 8%.

SunAmerica, the AIG unit that bills itself as "The Retirement Specialist," has launched two new variable annuity living benefits whose rider fees fluctuate with the VIX, the index of S&P 500 equity volatility at the Chicago Board Options Exchange.

By sharing some of the hedging risk with the contract owner, the insurer hopes to maintain a relatively generous bonus during the accumulation stage and payout rate during the distribution phase. SunAmerica has apparently not chosen to simplify or strip down its variable annuities, but to offer benefits as rich as possible while still "de-risking."

Designed for the Polaris series of variable annuities, the two guaranteed minimum withdrawal benefits are called Income Plus 6% and Income Builder 8%. They have distinct but overlapping characteristics.

Both contracts encourage the contract owner to postpone withdrawals by promising to double the guaranteed income base (the purchase premium, initially, and the amount on which payouts will be based) if the contract is undisturbed for 12 years.

$250,000 but less than $500,000	2.50%	0.36%
$500,000 but less than $1,000,000	2.00%	0.29%
$1,000,000 or more	1.25%	0.18%

The initial Premium Based Charge is determined by the sum of Premiums received during the first contract quarter and the Accumulated Premium Breakpoint achieved by that amount. After the first contract Quarter Anniversary, the Premium Based Charge for each subsequent Premium is determined based on the sum of all Premiums (including the subsequent Premium) and the Accumulated Premium Breakpoint achieved by the sum of Premiums as of the Premium receipt date. **Please see EXPENSES below.**

[4] **Base Contract Expenses:** If you do not elect any optional features, your total Base Contract Expense would be 0.95% annually.

Beneficiary Expenses if Extended Legacy is Elected

If your Beneficiary elects to take the death benefit amount under the Extended Legacy Program, we will deduct an annual Base Contract Expense of 0.85% which is deducted daily from the average daily ending net asset value allocated to the Variable Portfolios. **Please see *Extended Legacy Program* under DEATH BENEFITS.**

[5] The fee is calculated as a percentage of the Income Base which determines the basis of the guaranteed benefit. The annual fee is deducted from your contract value at the end of the first quarter following election and quarterly thereafter. For a complete description of how the Income Base is calculated, please see **OPTIONAL LIVING BENEFIT** below.

[6] The current Initial Annual Fee Rate is set forth in the Rate Sheet Supplement and guaranteed not to change for the first Benefit Year. Subsequently, the fee rate may change quarterly subject to the parameters identified in the table below. Any fee adjustment is based on a non-discretionary formula tied to the change in the Volatility Index ("VIX®"), an index of market volatility reported by the Chicago Board Options Exchange. In general, as the average value of the VIX decreases or increases, your fee rate will decrease or increase accordingly, subject to the maximums identified in the Fee Table and the minimums described below. **Please see APPENDIX C — FORMULA AND EXAMPLES OF CALCULATIONS OF THE POLARIS INCOME BUILDER DAILY FLEX FEE.** If you purchased your contract prior to May 1, 2023, **please see APPENDIX F - LIVING BENEFITS FOR CONTRACTS ISSUED PRIOR TO MAY 1, 2023** for the Initial Annual Fee applicable to your contract.

Number of Covered Persons	Minimum Annual Fee Rate	Maximum Annualized Fee Rate Decrease or Increase Each Benefit Quarter*
One Covered Person	0.60%	±0.40%
Two Covered Persons	0.60%	±0.40%

* The fee rate can increase or decrease no more than 0.10% each quarter (0.40%/ 4).

Motivation

VIX-linked fee structure [Cui et al., 2017]:

$$c_t = c + m\,\mathrm{VIX}_t^2$$

- Valuation of maturity benefit
- Better alignment between fee and net liability
- Impact on surrender incentives? See also MacKay et al. [2017]
- Other ways to link the fee to the VIX index?

▶ Continuous-time Markov chain (CTMC) approximation

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Plan of presentation

1. Overview of CTMC approximation method
2. Market model & variable annuity contract
3. Valuation of variable annuity contract via CTMC
4. Numerical examples

CTMC approximation

Selected literature on CTMC approximation

- Idea introduced in Kushner [1990]
- Approximation of one-dimensional Markov processes in Mijatović and Pistorius [2013], Lo and Skindilias [2014], Cai et al. [2019]
- Analysis of convergence and approximation error in Li and Zhang [2018],Zhang and Li [2019]
- Application to two-dimensional stochastic volatility models by Cai et al. [2015],Cui et al. [2018], Cui et al. [2019], Cai et al. [2019]

CTMC approximation in one dimension

- Let $S = \{S_t\}_{0 \le t \le T}$ be a time homogeneous diffusion process defined as the solution to

$$dS_t = \mu(S_t)\ dt + \sigma(S_t)\ dW_t$$

with state-space $\mathcal{S}$.

- Construct $S^N = \{S_t^N\}_{0 \le t \le T}$, a continuous time Markov chain with
  - finite support $\{s_1, \ldots, s_N\} \subset \mathcal{S}$, and
  - rate matrix (or generator) $\boldsymbol{Q} = (q_{ij})_{1 \le i,j \le N}$.

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More about S^N

- Elements of rate matrix $\boldsymbol{Q}$ satisfy

$$
\begin{aligned}
q_{i,i} &\leq 0, & 1 \leq i \leq N, \\
q_{i,j} &\geq 0, & 1 \leq i, j \leq N, i \neq j, \\
\sum_{j=1}^{N} q_{i,j} &= 0, & 1 \leq j \leq N.
\end{aligned}
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- Transition probability matrix is $\boldsymbol{P}(t) = (p_{i,j}(t))_{1 \leq i,j \leq N}$, where

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\boldsymbol{P}(t) = e^{t\boldsymbol{Q}} = \sum_{k=0}^{\infty} \frac{(t\boldsymbol{Q})^k}{k!},
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where $p_{i,j}(t) = P(S_t^N = s_j | S_0^N = s_i)$, $t > 0$.

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Choosing the rate matrix Q

- We want $S^N \Rightarrow S$ as $N \to \infty$.
- Local consistency conditions (see Kushner [1990]):

$$\mathbb{E}[S^N_{t+h} - S^N_t | \mathcal{F}_t] = E[S_{t+h} - S_t | \mathcal{F}_t] \approx \mu(S_t)h$$
$$\mathbb{E}[(S^N_{t+h} - S^N_t)^2 | \mathcal{F}_t] = E[(S_{t+h} - S_t | \mathcal{F}_t)^2] \approx \sigma(S_t)h$$

- Resulting rate matrix is tridiagonal:

$$\boldsymbol{Q} = \begin{bmatrix} q_{11} & q_{12} & 0 & 0 & \dots & 0 & 0 \\ q_{21} & q_{22} & q_{23} & 0 & \dots & 0 & 0 \\ 0 & q_{32} & q_{33} & q_{34} & \dots & 0 & 0 \\ \vdots & \ddots & & & & \vdots & \\ 0 & 0 & 0 & 0 & \dots & q_{N,N-1} & q_{NN} \end{bmatrix}$$

Approximation of vanilla option price

Price of an option with discounted payoff $\Phi : \mathcal{S} \mapsto \mathbb{R}^+$ can be approximated by

$$\begin{aligned}\mathbb{E}[\Phi(S_T)] &\approx \mathbb{E}[\Phi(S_T^N)] \\ &= \sum_{j=1}^{N} \Phi(s_j) P(S_T^N = s_j | S_0^N = s_{i^*}) \\ &= \boldsymbol{e}_{i^*}^\top e^{\boldsymbol{Q}T} \boldsymbol{\Phi}(\boldsymbol{S^N}),\end{aligned}$$

where $S_0 = S_0^N = s_{i^*}$, $\boldsymbol{\Phi}(\boldsymbol{S^N}) = (\Phi(s_1), \ldots, \Phi(s_N))^\top$ and $\boldsymbol{e}_{i^*}$ is the i^*-th canonical basis vector.

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CTMC approximation in two dimensions

Consider a (time-homogeneous) stochastic volatility model

$$dS_t = \mu_S(V_t)S_t\ dt + \sigma_S(V_t)S_t\ dW_t^{(1)}$$
$$dV_t = \mu_V(V_t)\ dt + \sigma_V(V_t)\ dW_t^{(2)},$$

with $d\langle W^{(1)}, W^{(2)}\rangle_t = \rho\ dt$.

How can we construct a CTMC approximating (S, V)?

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Main steps to approximating (S, V) via CTMC

1. V^m: CTMC approximation of V.
2. (S, V^m): Regime-switching diffusion process, remove the correlation between $W^{(1)}$ and $W^{(2)}$.
3. $(S^{m,N}, V^m)$: CTMC approximation of (S, V).

Remarks:

- $(S^{m,N}, V^m)$ has state-space $\{s_1, \ldots, s_N\} \times \{v_1 \ldots, v_m\} \subset \mathbb{R}^2$.
- Can also consider the process Y^{mN} taking value in $\{1, \ldots, mN\}$ with same generator as $(S^{m,N}, V^m)$.
- See Cui et al. [2018] for more details.

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Approximating the volatility process

- Let $V^m = \{V_t^m\}_{0 \le t \le T}$ be a CTMC with support $\mathcal{S}_V = \{v_1, \ldots, v_m\}$ and generator $\boldsymbol{Q^m}$.
- Choose $\boldsymbol{Q^m}$ as in the one-dimensional case, so that $V^m \Rightarrow V$ as $m \to \infty$.

Constructing a regime-switching diffusion process

- Define $X = \{X_t\}_{0 \le t \le T}$ by

$$X_t = \ln(S_t) - \rho f(V_t),$$

where $f(x) = \int_{\cdot}^{x} \frac{\sigma_S(u)}{\sigma_V(u)}\, du$.

- Then

$$dX_t = \mu_X(V_t)\, dt + \sigma_X(V_t)\, dW^*(t),$$

where $W_t^* = \frac{W_t^{(1)} - \rho W_t^{(2)}}{\sqrt{1-\rho^2}}$ is a Brownian motion independent of $W^{(2)}$.

Constructing a regime-switching diffusion process (cont'd)

- Define $X^m = \{X_t^m\}_{0 \le t \le T}$ as the solution of

$$dX_t^m = \mu_X(V_t^m)\ dt + \sigma_X(V_t^m)\ dW^*(t).$$

- Let $S^m = \{S_t^m\}_{0 \le t \le T}$ and

$$S_t^m = e^{X_t^m + \rho f(V_t^m)}.$$

Then S^m is the regime-switching diffusion process approximating S.

CTMC approximation of (S^m, V^m)

- Let $X^{m,N} = \{X_t^{m,N}\}_{0 \le t \le T}$ be the CTMC approximating X^m, with finite support $\mathcal{S}_X = \{x_1, \ldots, x_N\}$.
  - Transition probability depends on the state of V^m!
- Construct m generators G_k^N, $1 \le k \le m$ representing transition rates given each value v_k.
- $(X^{m,N}, V^m)$ has state-space $\mathcal{S}_X \times \mathcal{S}_V$ and transition rate matrix

$$G^{m,N} = \begin{pmatrix} q_{11} I_N + G_1^N & q_{12} I_N & \cdots & q_{1m} I_N \\ q_{21} I_N & q_{22} I_N + G_2^N & \cdots & q_{2m} I_N \\ \vdots & \vdots & \ddots & \vdots \\ q_{m1} I_N & q_{m2} I_N & \cdots & q_{mm} I_N + G_m^N \end{pmatrix}$$

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Application to VAs

Market model

(S, V) follows a stochastic volatility model

$$dS_t = rS_t dt + \sigma_S(V_t) S_t dW_t^{(1)},$$
$$dV_t = \mu_V(V_t) dt + \sigma_V(V_t) dW_t^{(2)},$$

with $r \geq 0$, $S_0 > 0$, $V_0 > 0$ and $d\langle W^{(1)}, W^{(2)} \rangle_t = \rho \, dt$.

Ex. of models: Heston, 3/2, α-Hypergeometric, Hull-White.

Variable annuity contract

- Contract maturity: $T > 0$.
- VA account process $F = \{F_t\}_{0 \le t \le T}$, with

$$\frac{dF_t}{F_t} = \frac{dS_t}{S_t} - dC(t, V_t),$$

  where $C(t, V_t)$ is continuous or bounded.
- Maturity benefit: $\max(G, F_T)$, $G > 0$.
- Early surrender payout $g(t, V_t)F_t$, with $g : [0, T] \times \mathcal{S}_V \mapsto [0, 1]$, non-decreasing in t.

Why call $C(t, V_t)$ "VIX-linked"?

- Can write the VIX index as

$$\mathrm{VIX}_t^2 = \mathbb{E}_t\left[\frac{1}{\tau}\int_t^{t+\tau} \sigma_s^2(V_s)\, ds\right] = h(V_t),$$

  with $\tau = 30/365$ for some function $h : \mathbb{R}^+ \mapsto \mathbb{R}^+$ (see Cui et al. [2024]).

- Heston model: $\mathrm{VIX}_t^2 = A + BV_t$, with A and B constants depending on the model parameters (see Zhu and Zhang [2007]).

- Examples of fee functions:

$$C(t, V_t) = c + m\ \mathrm{VIX}_t^2$$
$$C(t, V_t) = \min(c + m\ \mathrm{VIX}_t^2, K)$$
$$C(t, V_t) = c + m\ \mathrm{VIX}_t$$

Pricing the VA contract

- Optimal stopping problem because of early surrenders
- Reward function:

$$\varphi(t,x,y) = \begin{cases} g(t,y)x, & t < T \\ \max(G,x), & t = T. \end{cases}$$

- Value of the contract at t, $0 \le t \le T$:

$$\sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_{t,x,y}[e^{-r(T-t)}\varphi(\tau, F_\tau, V_\tau)], \tag{1}$$

where $\mathbb{E}_{t,x,y}[\cdot] = \mathbb{E}[\cdot | F_t = x, V_t = y]$.

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When are early surrenders suboptimal?

Proposition

If $C(t,y)$ and $g(t,y)$ satisfy

$$g_t(t,y) + [\rho\sigma_S(y)\sigma_V(y) + \mu_V(y)]g_y(t,y) + \frac{\sigma_V^2(y)}{2}g_{yy}(t,y) - C(t,y)g(t,y) \geq 0,$$

for all $(t,y) \in [0,T] \times \mathcal{S}_V$, T is an optimal stopping time for (1).

- Then $\{e^{-rt}F_t\}_{0\leq t\leq T}$ is a submartingale.
- "Always better to wait."
- Extends the result of MacKay et al. [2017].

Pricing the VA contract with early surrenders

- Discretize $[0, T]$ in M subintervals of length $h = T/M$ and use dynamic programming.
- Price process of the (Bermudan) contract $B = \{B_i\}_{i=0,\dots,M}$, with $B_i = B(ih, F_{ih}, V_{ih})$ obtained by recursion:

$$\begin{cases} B_M = \varphi(T, F_T, V_T) \\ B_i = \max\left(\varphi(ih, F_{ih}, V_{ih}), e^{-rh}\mathbb{E}_{ih,F_{ih},V_{ih}}[B_{i+1}]\right), \quad i = 0, \dots, M-1. \end{cases}$$

Approximating the price of the contract via CTMC

- Replace (F, V) by a CTMC approximation $(F^{m,N}, V^m)$ with state-space $\{(f_1, v_1), \ldots, (f_{mN}, v_{mN})\}$.
- Define the vector-valued price process $\boldsymbol{B}^{m,N}$ via recursion

$$\begin{cases} \boldsymbol{B}_M^{m,N} = \boldsymbol{\Phi}^{(1)}, \\ \boldsymbol{B}_i^{m,N} = \max\{\boldsymbol{\Phi}_i^{(2)}, e^{-rh} e^{h\boldsymbol{G}^{m,N}} \boldsymbol{B}_{i+1}^{m,N}\} \quad i = 0, \ldots, M-1, \end{cases}$$

where

$$\begin{aligned} \boldsymbol{\Phi}^{(1)} &= (\max(G, f_j))_{j=1,\ldots,mN}^\top \\ \boldsymbol{\Phi}_i^{(2)} &= (g(ih, v_j) f_j)_{j=1,\ldots,mN}^\top, \qquad i = 0, \ldots, M-1. \end{aligned}$$

- Then $B(0, F_0, V_0) \approx \boldsymbol{e}_{i^*} \boldsymbol{B}_0^{m,N}$, where $(F_0, V_0) = (f_{i^*}, v_{i^*})$.

Approximating the price of the contract via CTMC

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$$\boldsymbol{\Phi}_i^{(2)} = (g(ih, v_j) f_j)_{j=1,\ldots,mN}^\top, \qquad i = 0, \ldots, M-1.$$

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where

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- Then $B(0, F_0, V_0) \approx \boldsymbol{e}_{i^*} \boldsymbol{B}_0^{m,N}$, where $(F_0, V_0) = (f_{i^*}, v_{i^*})$.

CTMC approximation of the optimal surrender strategy

- Continuation region:

$$\mathcal{C}^{m,N} = \left\{ (i, f_j, v_j) \middle| B_i^{m,N}(ih, f_j, v_j) > g(ih, v_j) f_j \right\},$$

with $B_i^{m,N}(ih, f_j, v_j) = \boldsymbol{e}_i \boldsymbol{B}_i^{mN}$.

- If applicable, the optimal surrender surface is defined as $\{s^{m,N}(ih, v_j)\}_{i=0\ldots,M-1, j=1\ldots,mN}$, with

$$s^{m,N}(ih, v_j) := \inf \left\{ f_j \in \{f_1, \ldots, f_{mN}\} \middle| B_i^{m,N}(ih, f_j, v_j) \leq g(ih, v_j) f_j \right\}.$$

Numerical examples

Market model & VA contract

- Heston model:
  - $\sigma_S(y) = \sqrt{y}$;
  - $\mu_V(y) = \kappa(\theta - y)$, $\sigma_V(y) = \sigma\sqrt{y}$;
  - $V_0 = 0.03$, $\kappa = 2$, $\theta = 0.04$, $\sigma = 0.2$, $\rho = -0.75$, $r = 0.03$.
- VA contract:
  - $T = 10$;
  - $G = F_0 = 100$;
  - $g(t, y) = e^{-0.002(T-t)}$.
- VIX-linked fee structures:
  - $c(V_t) = c + m\mathrm{VIX}_t^2$;
  - $c(V_t) = \min(c + m\mathrm{VIX}_t^2, K)$;
  - Fee structure is "actuarially fair" if $F_0 = \mathbb{E}[e^{-rT}\max(G, F_T)]$.

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  - $\mu_V(y) = \kappa(\theta - y)$, $\sigma_V(y) = \sigma\sqrt{y}$;
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  - $c(V_t) = c + m\mathrm{VIX}_t^2$;
  - $c(V_t) = \min(c + m\mathrm{VIX}_t^2, K)$;
  - ▶ Fee structure is "actuarially fair" if $F_0 = \mathbb{E}[e^{-rT}\max(G, F_T)]$.

CTMC parameters

- Non-uniform grid of Tavella and Randall [2000].
  - Improved fit to transition density of V with fewer points.
  - Non-uniform grid increase stability and improve convergence, less sensitive to choice of boundary values (see Lo and Skindilias [2014],Leitao Rodriguez et al. [2021]).
- Volatility process:
  - $m = 50$
  - $v_1 = V_0/100$, $v_m = 7V_0$
- Fund process
  - $N = 2000$
  - $x_1 = X_0/10^6$, $x_N = 1.95X_0$, with $X_0 = \ln(S_0) - V_0\rho/\sigma$
- Number of time steps $M = 500T$ (for dynamic programming), i.e. 500 time steps per year.

Fair fee structures

$m^* =$	0.000	0.1500	0.3000	0.4345
$c^* =$	1.5338%	1.0036%	0.4741%	0.000%

Table: Fair fee vector (c^*, m^*), $C(V_t) = c + m\text{VIX}_t^2$.

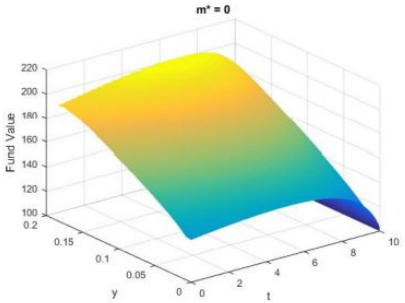
$m^* =$	0.000	0.1500	0.3000	0.4927
$c^* =$	1.5338%	1.0112%	0.5415%	0.000%

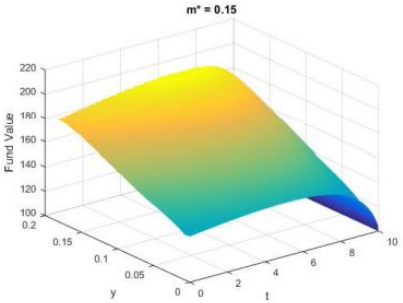
Table: Fair fee vector (c^*, m^*), $C(V_t) = \min(c + m\text{VIX}_t^2, K)$, $K = 2\%$.

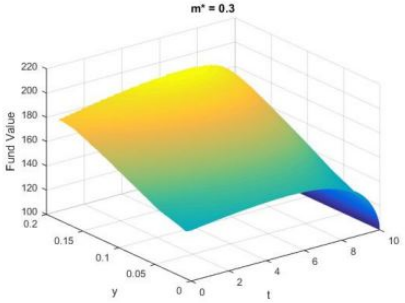
Value of early surrenders – Uncapped fee

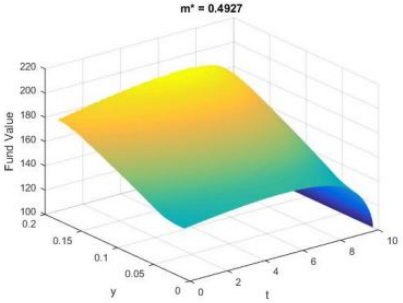
m^*	0.0000	0.1500	0.3000	0.4345
c^*	1.5338%	1.0036%	0.4741%	0.000%
VA without ES	100.00	100.00	100.00	100.00
VA with ES	103.02	103.01	103.00	103.00
Value of ES	3.02	3.01	3.00	3.00

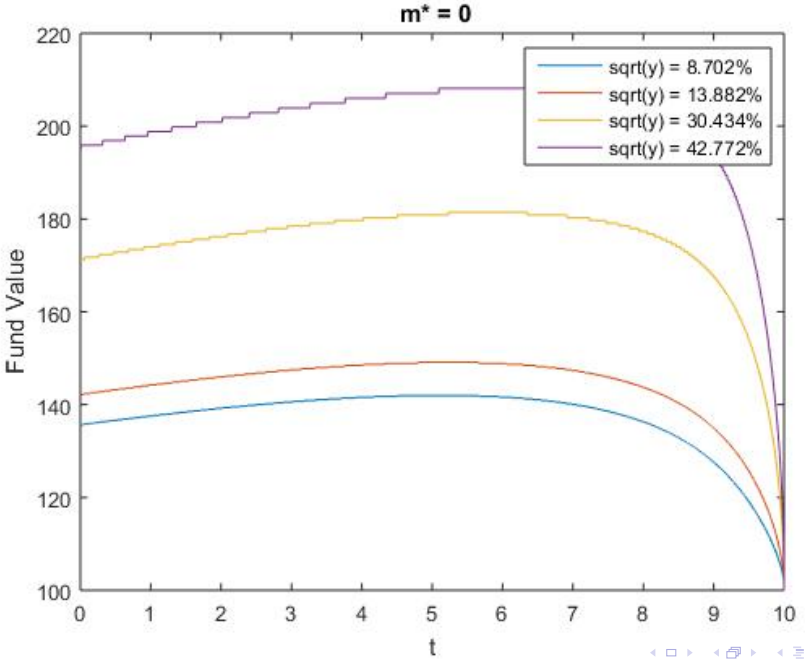
Table: Value of VA contract with and without early surrenders using CTMC approximation.

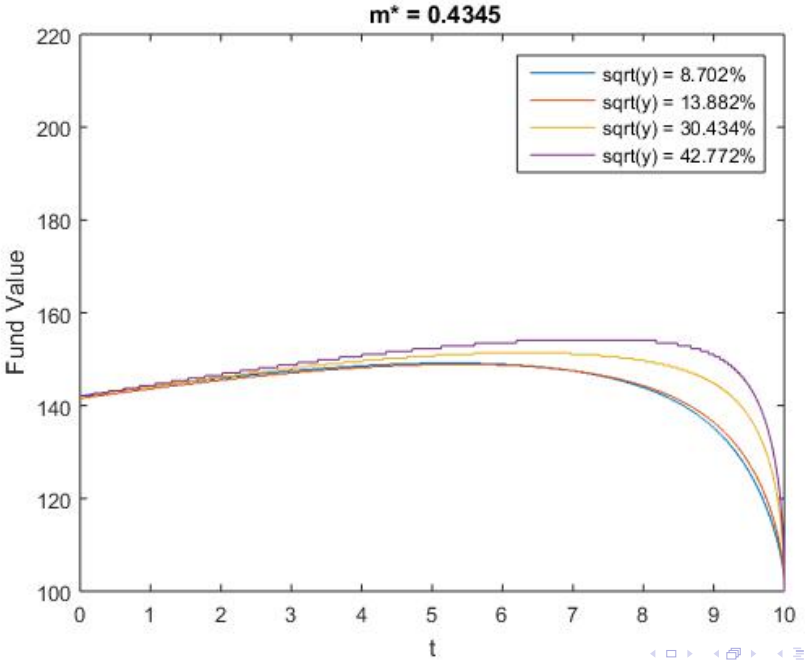

m* = 0
Fund Value
220
200
180
160
140
120
100
0.2
0.15
0.1
0.05
0
y
0
2
4
6
8
10
t

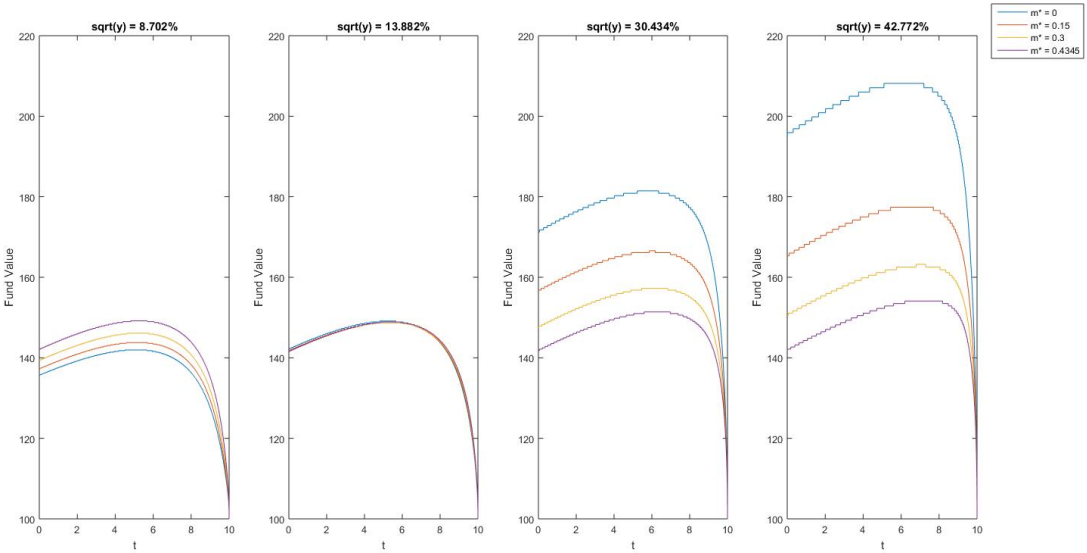

m* = 0.15
Fund Value
220
200
180
160
140
120
100
0.2
0.15
0.1
0.05
0
y
0
2
4
6
8
10
t


m* = 0.3
Fund Value
220
200
180
160
140
120
100
0.2
0.15
0.1
0.05
0
y
0
2
4
6
8
10
t


m* = 0.4927
Fund Value
220
200
180
160
140
120
100
0.2
0.15
0.1
0.05
0
y
0
2
4
6
8
10
t


m* = 0
sqrt(y) = 8.702%
sqrt(y) = 13.882%
sqrt(y) = 30.434%
sqrt(y) = 42.772%
Fund Value
t
220
200
180
160
140
120
100
0
1
2
3
4
5
6
7
8
9
10

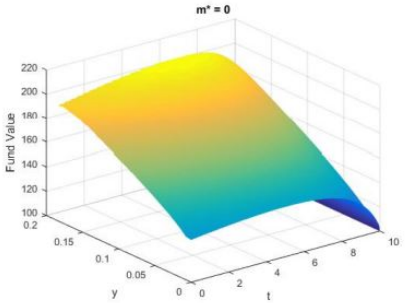

m* = 0.4345
sqrt(y) = 8.702%
sqrt(y) = 13.882%
sqrt(y) = 30.434%
sqrt(y) = 42.772%
Fund Value
t
220
200
180
160
140
120
100
0
1
2
3
4
5
6
7
8
9
10

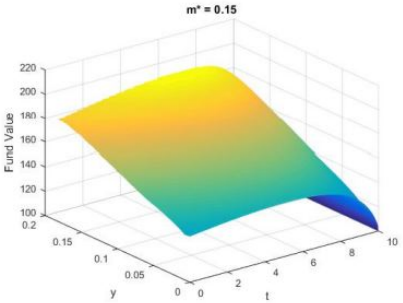

sqrt(y) = 8.702%
sqrt(y) = 13.882%
sqrt(y) = 30.434%
sqrt(y) = 42.772%
Fund Value
t
m* = 0
m* = 0.15
m* = 0.3
m* = 0.4345

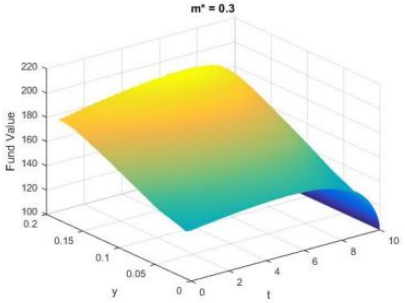
Value of early surrenders – Capped fee

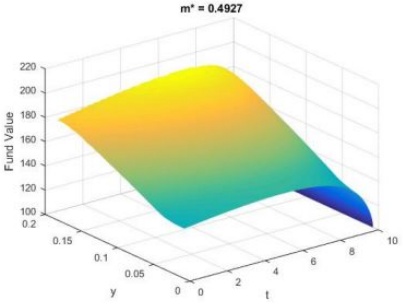
m^*	**0.0000**	**0.1500**	**0.3000**	**0.4927**
c^*	**1.5338%**	**1.0112%**	**0.5415%**	**0.000%**
VA without ES	100.00	100.00	100.00	100.00
VA with ES	103.02	103.01	102.99	102.98
Value of ES	3.02	3.01	2.99	2.98

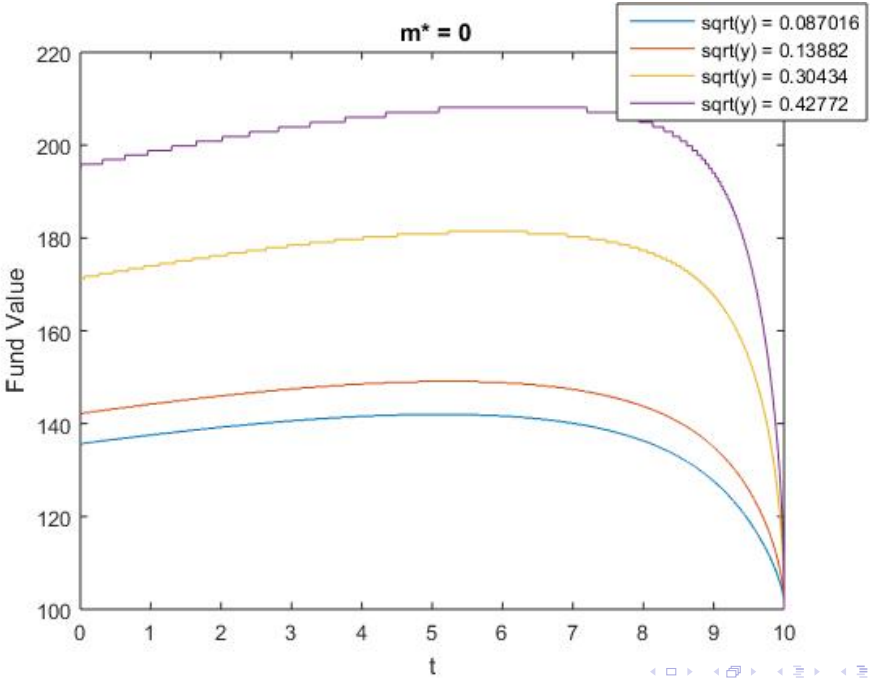
Table: Value of VA contract with and without early surrenders using CTMC approximation, $K = 2\%$.


m* = 0
Fund Value
220
200
180
160
140
120
100
0.2
0.15
0.1
0.05
0
y
0
2
4
6
8
10
t


m* = 0.15
Fund Value
220
200
180
160
140
120
100
0.2
0.15
0.1
0.05
0
y
0
2
4
6
8
10
t


m* = 0.3
Fund Value
220
200
180
160
140
120
100
0.2
0.15
0.1
0.05
0
y
0
2
4
6
8
10
t


m* = 0.4927
Fund Value
220
200
180
160
140
120
100
0.2
0.15
0.1
0.05
0
y
0
2
4
6
8
10
t


m* = 0
sqrt(y) = 0.087016
sqrt(y) = 0.13882
sqrt(y) = 0.30434
sqrt(y) = 0.42772
Fund Value
t
220
200
180
160
140
120
100
0
1
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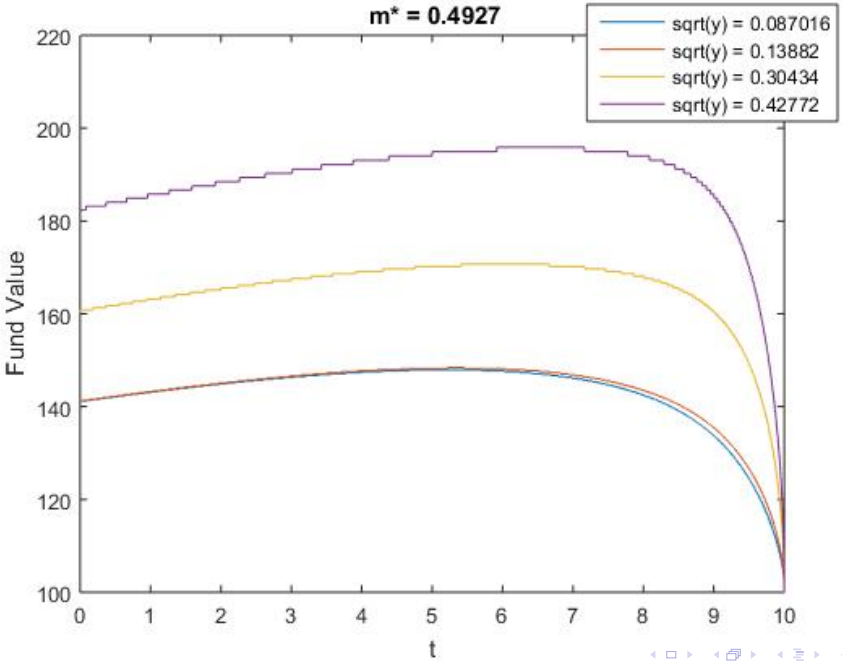
m* = 0.4927
sqrt(y) = 0.087016
sqrt(y) = 0.13882
sqrt(y) = 0.30434
sqrt(y) = 0.42772
Fund Value
t
220
200
180
160
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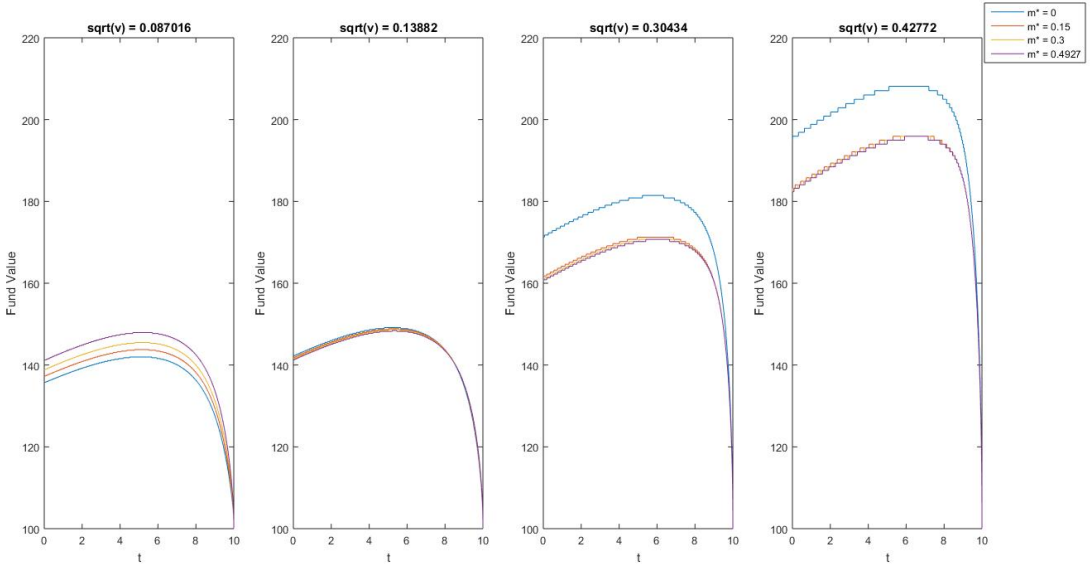
sqrt(v) = 0.087016
sqrt(v) = 0.13882
sqrt(v) = 0.30434
sqrt(v) = 0.42772
m* = 0
m* = 0.15
m* = 0.3
m* = 0.4927
Fund Value
t
220
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180
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140
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Concluding remarks

- CTMC approximations are fast and accurate.
- Interplay between fee rate and surrender charge can reduce surrender incentives.
- VIX-linked fees can mitigate exposure to volatility and surrender risk.

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