Forward mortality rates

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Agenda

- Why forward mortality rates?
- Defining forward mortality rates
- Market consistent measure
- Hedging
- Discussion

Why forward mortality rates?

- Valuing technical provisions and pricing longevity-linked securities requires consistent expectations of future mortality rates
  - C.f. forward interest rates embedded in yield curve for bond pricing
- Other approaches to forward mortality rates
  - Continuous time – Bauer et al (2008,2012)
  - Non-parametric – Zhu and Bauer (2011a,b,2014)
  - Olivier-Smith model – Olivier and Jeffrey (2004), Smith (2005)

Defining forward mortality rates

- Hypothetical market in "longevity zeros" with price

$$\text{Price}(t,\tau) = B(\tau, \tau + t)\mathbb{E}_\tau\ {}_t p_{x,\tau}$$

- Define

$$\begin{aligned} {}_t P_{x,\tau}(\tau) &= \mathbb{E}_\tau\ {}_t p_{x,\tau} \\ &= \mathbb{E}_\tau \exp\left(-\sum_{u=0}^{t-1} \mu_{x+u,\tau+u}\right) \end{aligned}$$

- Forward mortality rates in discrete time

$$\nu_{x,t}(\tau) = -\ln\left(\frac{{}_{t-\tau+1}P_{x-t+\tau,\tau}(\tau)}{{}_{t-\tau}P_{x-t+\tau,\tau}(\tau)}\right)$$

$${}_t P_{x,\tau}(\tau) = \exp\left(-\sum_{u=0}^{t-1} \nu_{x+u,\tau+u}(\tau)\right)$$

Defining forward mortality rates

- We identify $\nu_{x,t}(\tau) = \mathbb{E}_\tau \mu_{x,t}$
  - Approximation due to Jensen's inequality but tested numerically and reasonable (within 0.1%) across most ages and years
- Assume that short mortality rates are modelled by an age/period/cohort mortality model – Hunt and Blake (2014d)

$$\ln(\mu_{x,t}) = \eta_{x,t} = \alpha_x + \boldsymbol{\beta}_x^\top \boldsymbol{\kappa}_t + \gamma_{t-x}$$

- Then

$$\nu_{x,t}(\tau) = \exp\left(\alpha_x + \boldsymbol{\beta}_x^\top \mathbb{E}_\tau \boldsymbol{\kappa}_t + \frac{1}{2}\boldsymbol{\beta}_x^\top Var_\tau(\boldsymbol{\kappa}_t)\boldsymbol{\beta}_x + \mathbb{E}_\tau \gamma_{t-x} + \frac{1}{2} Var_\tau(\gamma_{t-x})\right)$$

Defining forward mortality rates

- Assume random walk with drifts for the period functions

$$\kappa_t = \mu X_t + \kappa_{t-1} + \epsilon_t$$

  - Deterministic functions may be included in drift, X_t , for identifiability reasons – Hunt and Blake (2014b,c)

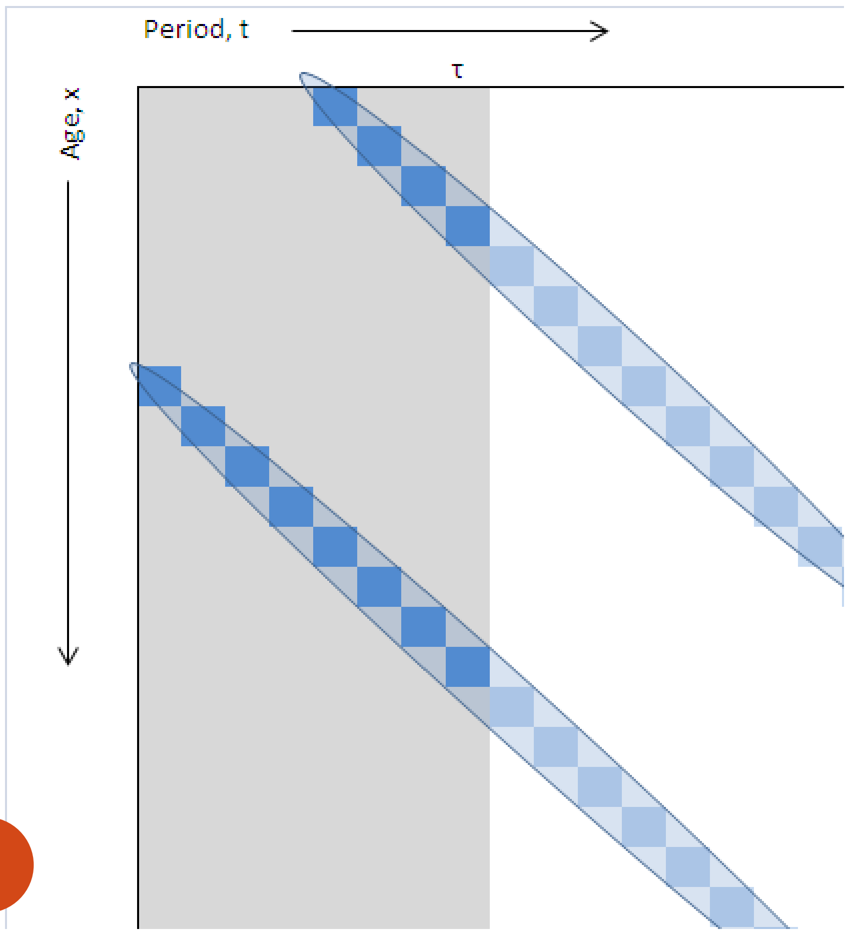
- Therefore

$$\mathbb{E}_\tau \kappa_t = \kappa_\tau + \mu \sum_{s=\tau+1}^{t} X_s$$

$$Var_\tau(\kappa_t) = (t-\tau)\Sigma$$

Defining forward mortality rates

- Use Bayesian approach to model and project the cohort parameters



- Fitted parameter estimates based on partial information
- Assume annual observations of each cohort providing new information
- Cohort parameter only known with certainty once observed over its entire life

Defining forward mortality rates

- Details get quite involved – see Hunt and Blake (2014a)

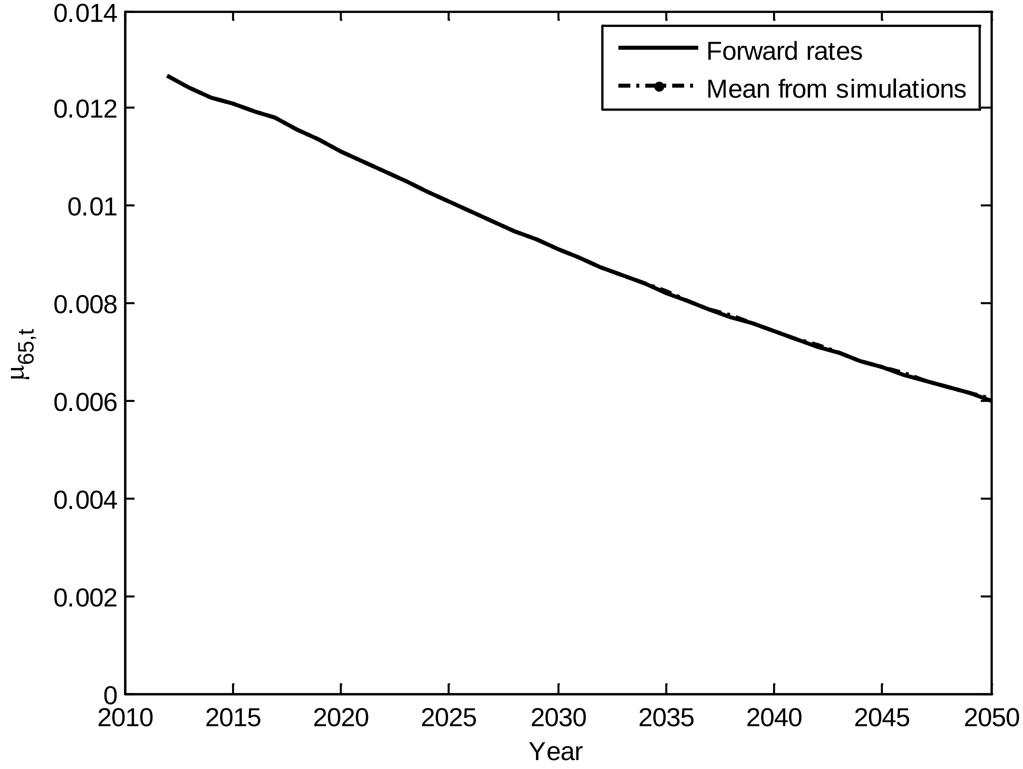
$$\mathbb{E}_\tau \gamma_y = M(y,\tau) = \sum_{s=0}^{\infty} \left[\prod_{r=0}^{s-1} (1 - D_{\tau-y+r}) \right] \rho^s \left[\underline{\gamma}_{y-s}(\tau) + (1 - D_{\tau-y+s}) \beta (Y_{y-s} - \rho Y_{y-s-1}) \right]$$

$$Var_\tau(\gamma_y) \equiv V(y,\tau) = \sum_{s=0}^{\infty} \left[\prod_{r=0}^{s-1} (1 - D_{t-y+r})^2 \right] (1 - D_{t-y+s}) \rho^{2s} \sigma^2$$

  - However, this approach is necessary for measuring risk, as discussed later

Defining forward mortality rates

- Together, these give the forward mortality surface
  - Difference < 0.1%, due to rounding errors in simulations



Market consistent measure

- In order to value liabilities or value securities, we need to convert the forward mortality surface from the historic to a market consistent measure
- Use Esscher transform, see Gerber and Shiu (1994)

$$\mathbb{E}^{\mathbb{Q}} exp(\eta) = \frac{\mathbb{E}^{\mathbb{P}} exp(Z\eta)}{\mathbb{E}^{\mathbb{P}} exp(Z)}$$

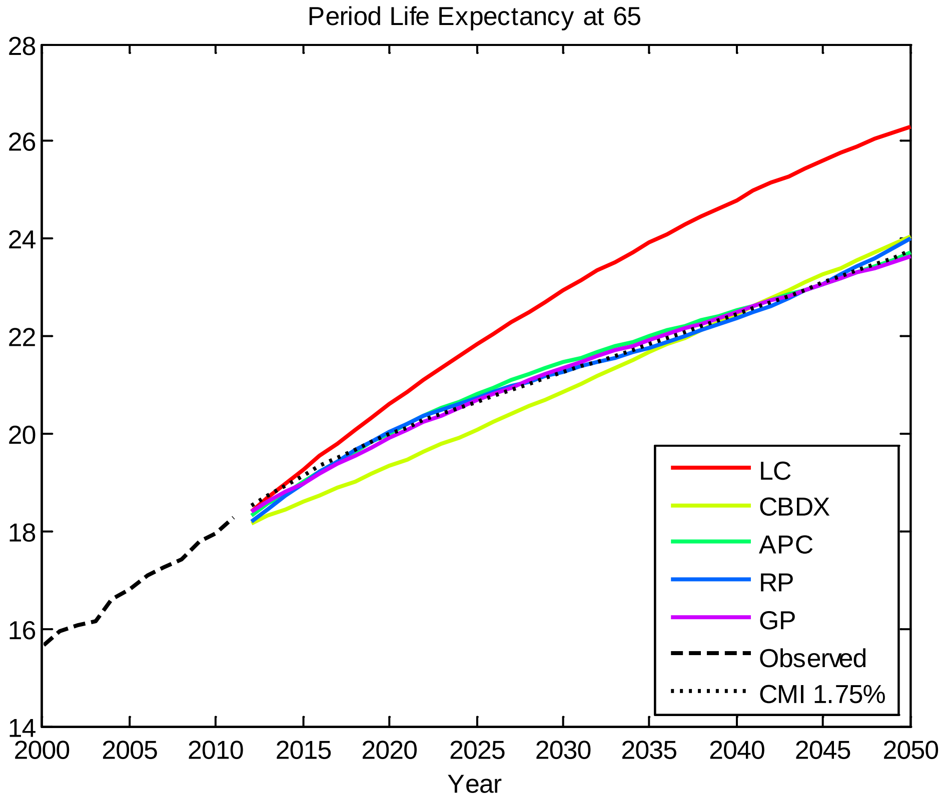
$$Z_{x,t} = \boldsymbol{\beta}_x^{\top} \Lambda \boldsymbol{\kappa}_t + \lambda^{\gamma} \gamma_{t-x}$$

$$\Lambda = \begin{pmatrix} \lambda^{(1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda^{(N)} \end{pmatrix}$$

$$\nu_{x,t}^{\mathbb{Q}}(\tau) = \exp\left(\boldsymbol{\beta}_x^{\top} \Lambda Var_{\tau}^{\mathbb{P}}(\boldsymbol{\kappa}_t) \boldsymbol{\beta}_x + \lambda^{\gamma} Var_{\tau}^{\mathbb{P}}(\gamma_{t-x})\right) \nu_{x,t}^{\mathbb{P}}(\tau)$$

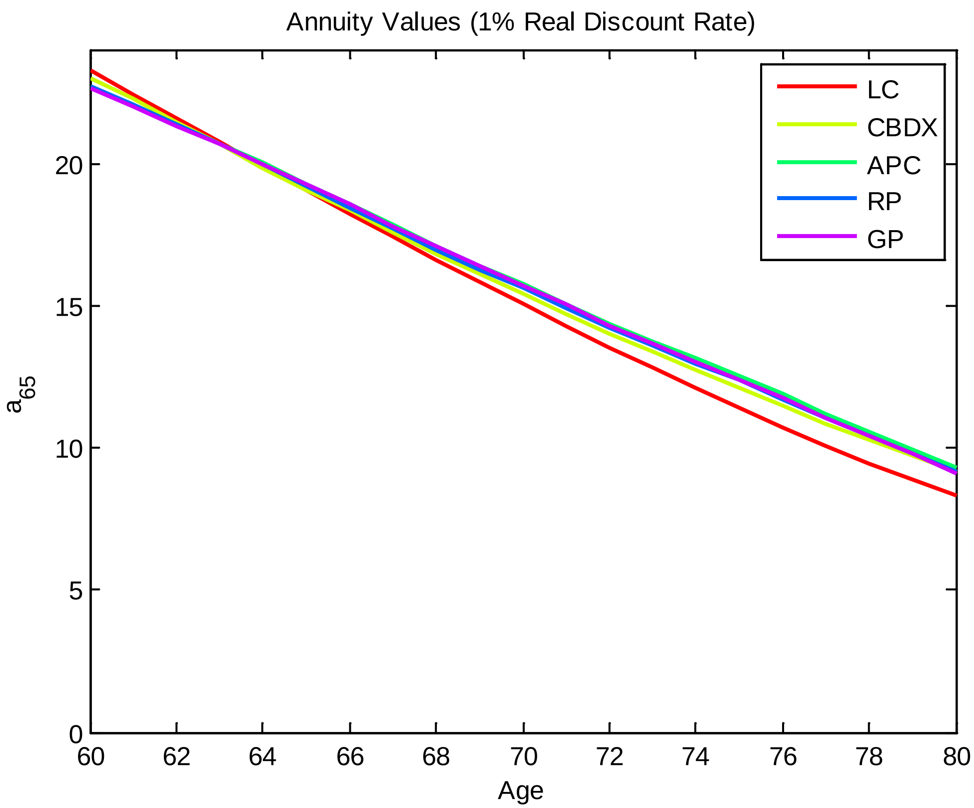
Market consistent measure

- Values of market prices of longevity risk, $\lambda^{(j)}$ found from:
  - prices of traded longevity securities (if they exist) or
  - deterministic projection of mortality (e.g., CMI Projection Model)



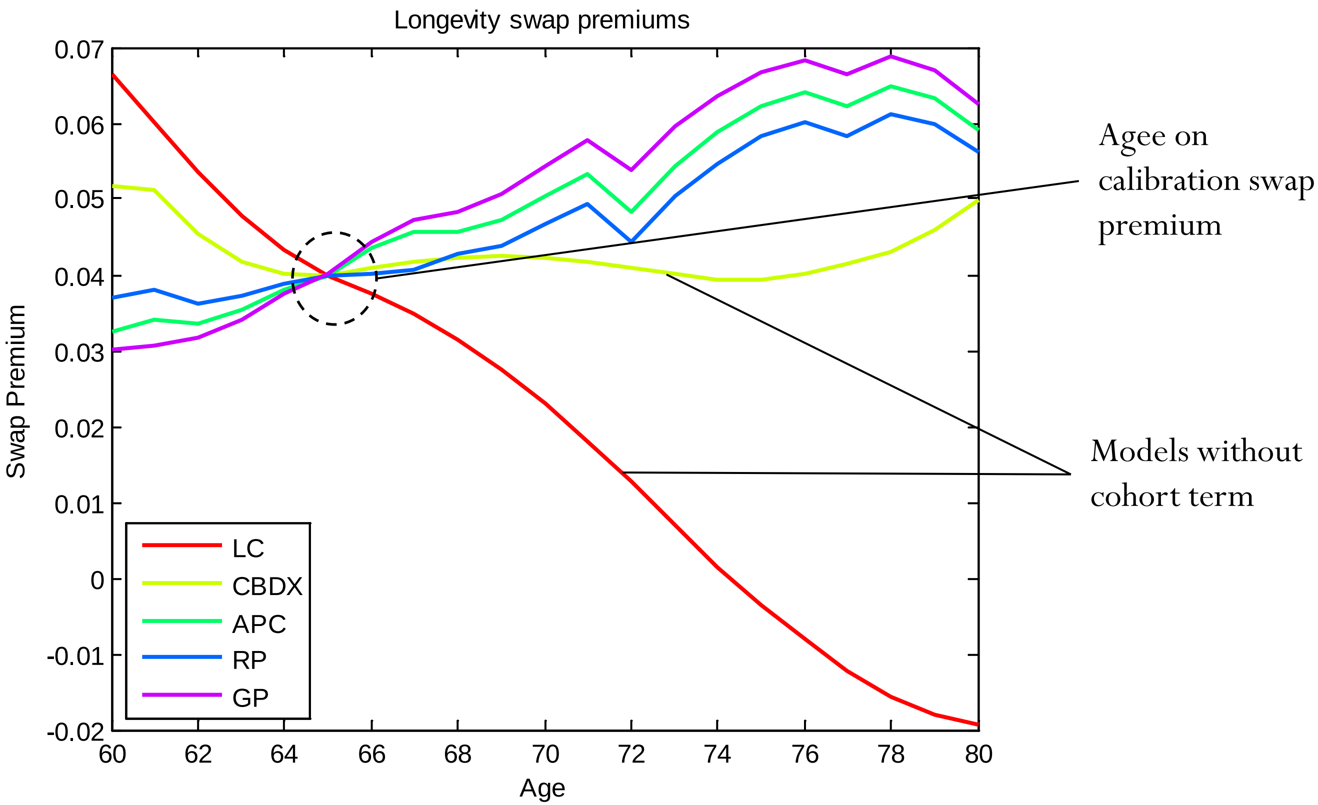
Market consistent measure

- Consistent prices for liabilities, e.g., annuity values, can now be found using same forward mortality surface



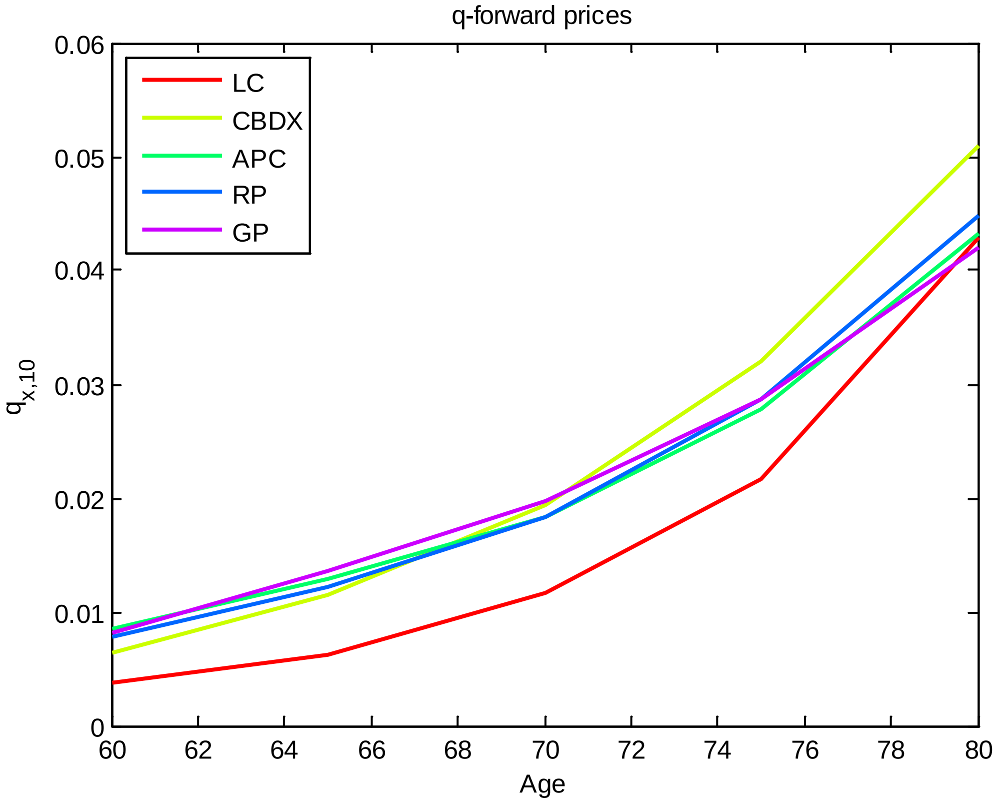
Market consistent measure

- Can also find prices for other longevity linked securities consistent with few market prices observed
  - For example, premiums on index longevity swaps at different ages



Market consistent measure

- Can also find prices for other longevity linked securities consistent with few market prices observed
  - For example, of 10 year q-forwards



Hedging

- For many purposes, we need to know how the forward mortality surface updates
  - E.g., Value at Risk, hedging
- This depends upon how the period and cohort functions update with one year's extra observations
- NB – by tower property of conditional expectations, have

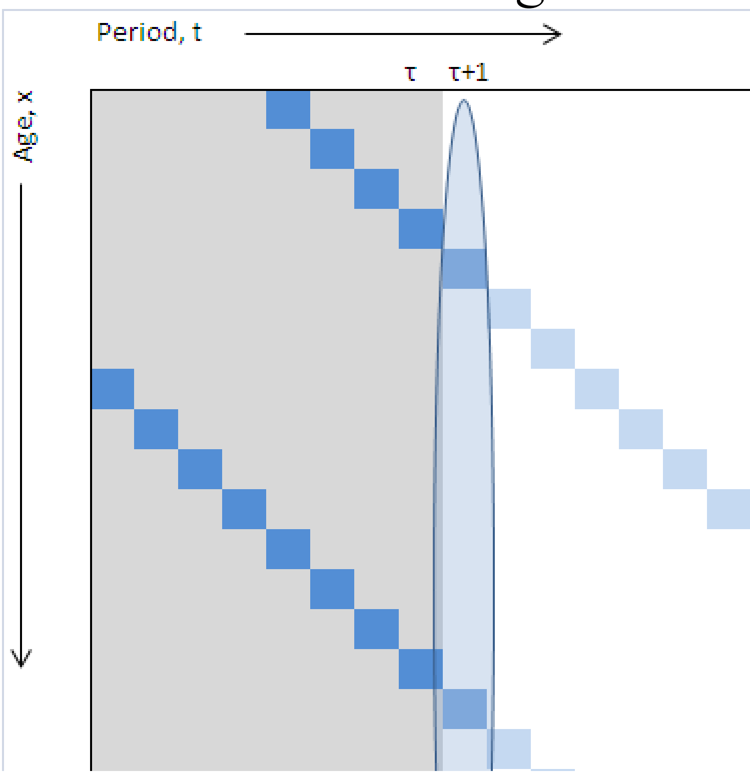
$$\nu_{x,t}(\tau) = \mathbb{E}_\tau \nu_{x,t}(\tau + 1)$$

- Period functions are straightforward

$$\mathbb{E}_{\tau+1}\kappa_t = \mathbb{E}_\tau \kappa_t + \epsilon_{\tau+1}$$
$$Var_{\tau+1}(\kappa_t) = (t - \tau - 1)\Sigma$$

Hedging

- Cohort functions, need to use Bayesian approach and assumed data generating process

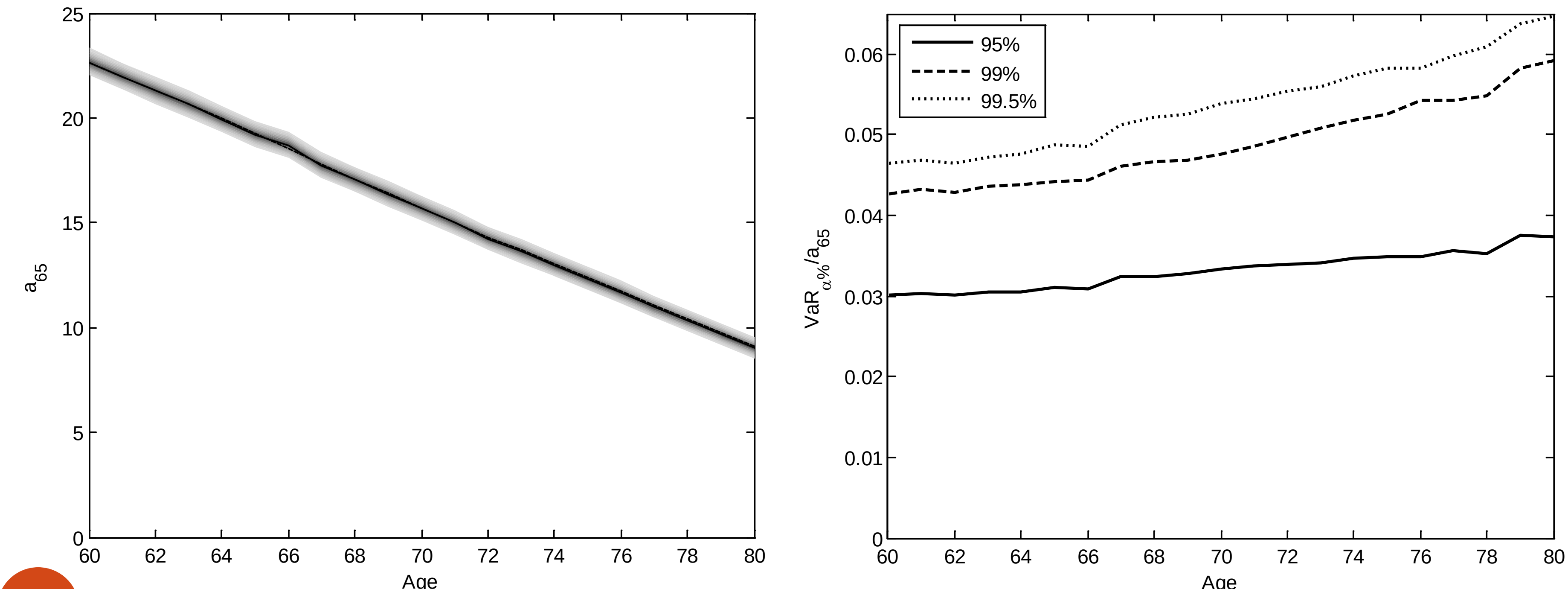


$$\underline{\gamma}_y(\tau+1) = \underline{\gamma}_y(\tau) + d_{\tau+1-y}\gamma_y^{\tau+1-y}$$

$$\gamma_y^{\tau+1-y} | \mathcal{F}_{\tau,y}, \beta, \rho, \sigma^2 \sim N\left(\beta Y_y + \rho(M(y-1,\tau) - \beta Y_{y-1}), V(y-1,\tau) + \frac{\sigma^2}{d_{\tau+1-y}}\right)$$

Hedging

- Using this framework, we can update the forward mortality surface by one year and recalculate liability values or securities prices
  - Value at Risk

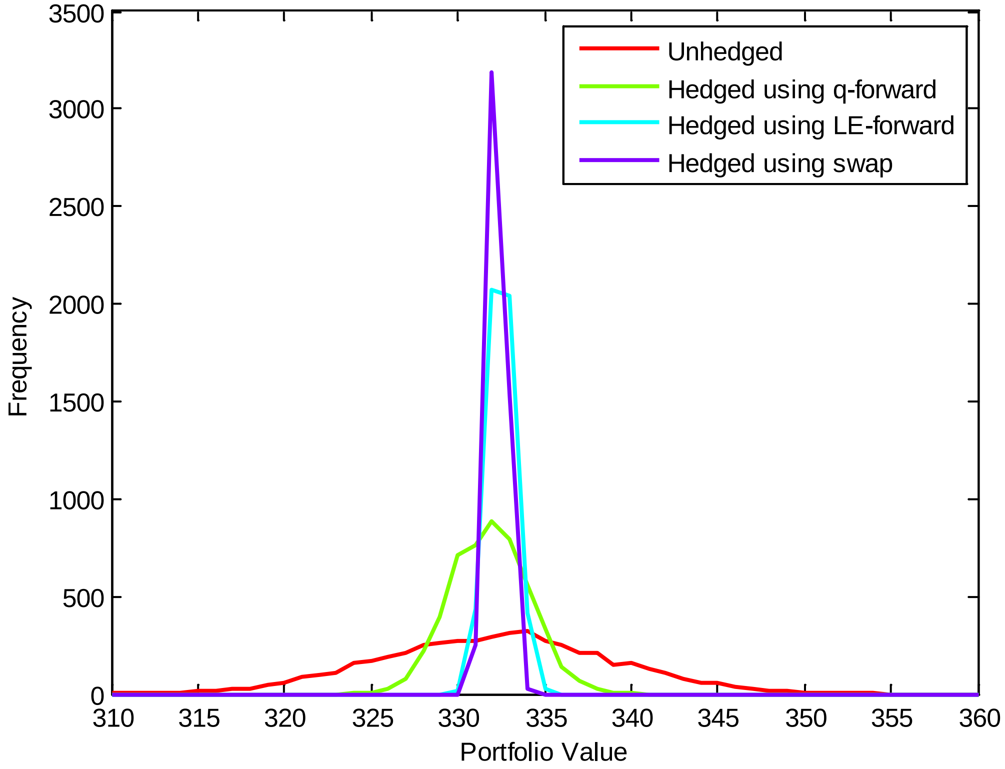


Hedging

- Because liabilities and securities prices are valued from the same forward mortality surface, they will be updated consistently with one another
- Useful for investigating hedge effectiveness for different value hedging strategies
  - Single hedging instrument to hedge annuity portfolio
  - Hedge ratio chosen to minimise variance

Hedging

- Empirical distribution of liability value for different hedging instruments



Hedging

Risk measure (as % of liabilities)	Unhedged	Q-forward	LE-forward	Swap
VaR(95%)	2.75%	0.79%	0.24%	0.05%
% reduction	-	71%	91%	98%
TVaR(95%)	3.56%	1.06%	0.33%	0.11%
% reduction	-	70%	91%	97%

- Relatively high reductions in risk for very simple (single instrument) hedging strategies
  - Model dependent (though valuation will be mark-to-model for foreseeable future)
  - No allowance for basis risk

Discussion

- Forward mortality rates provide a useful framework for many of the issues with the valuation / risk management of longevity risk
- We have introduced a discrete time forward mortality rate framework which:
  - Is consistent with models of the short mortality rate
  - Can be calibrated easily to available data
  - Can be used with a variety of individual short rate models
  - Can be extended for different processes governing period and (more difficult) cohort functions

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Questions?

- Thank you very much for your attention and your feedback

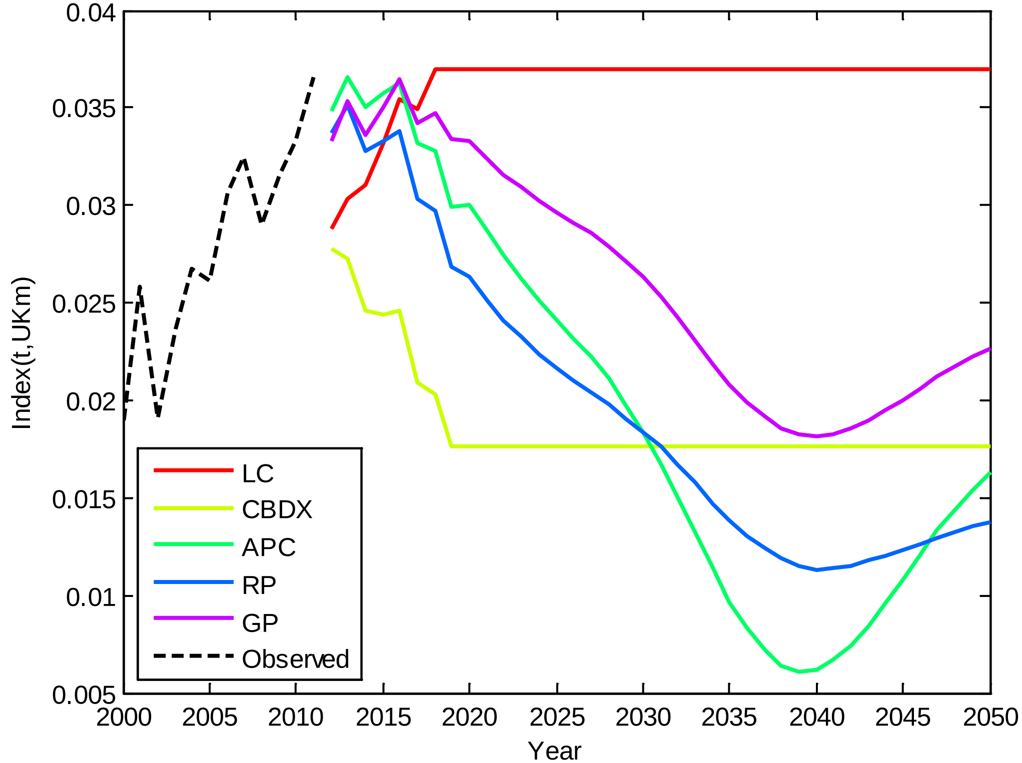
Addendum

	$\lambda^{(1)}$	$\lambda^{(2)}$	$\lambda^{(3)}$	$\lambda^{(\gamma)}$
LC	-41.9			
CBDX	-0.5	-32.6		
APC	-10.6			281.3
RP	3.9	-5.4		138.2
GP	-23.9	20.7	-155.2	165.5

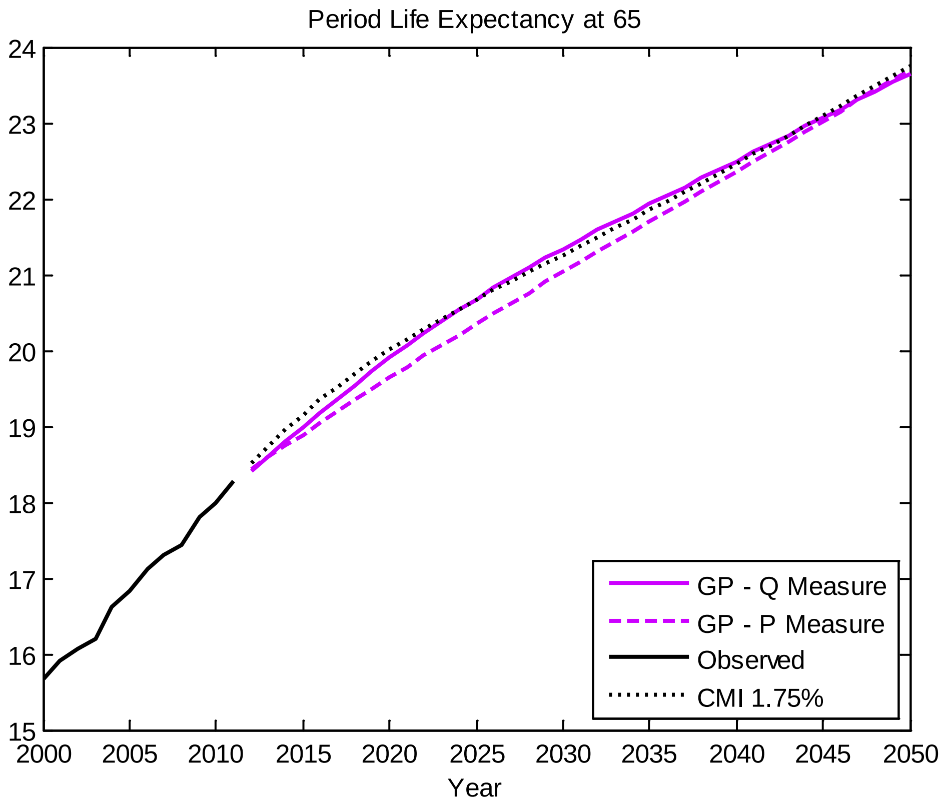
- Market prices of risk are dimensionless and not directly comparable across models

Addendum

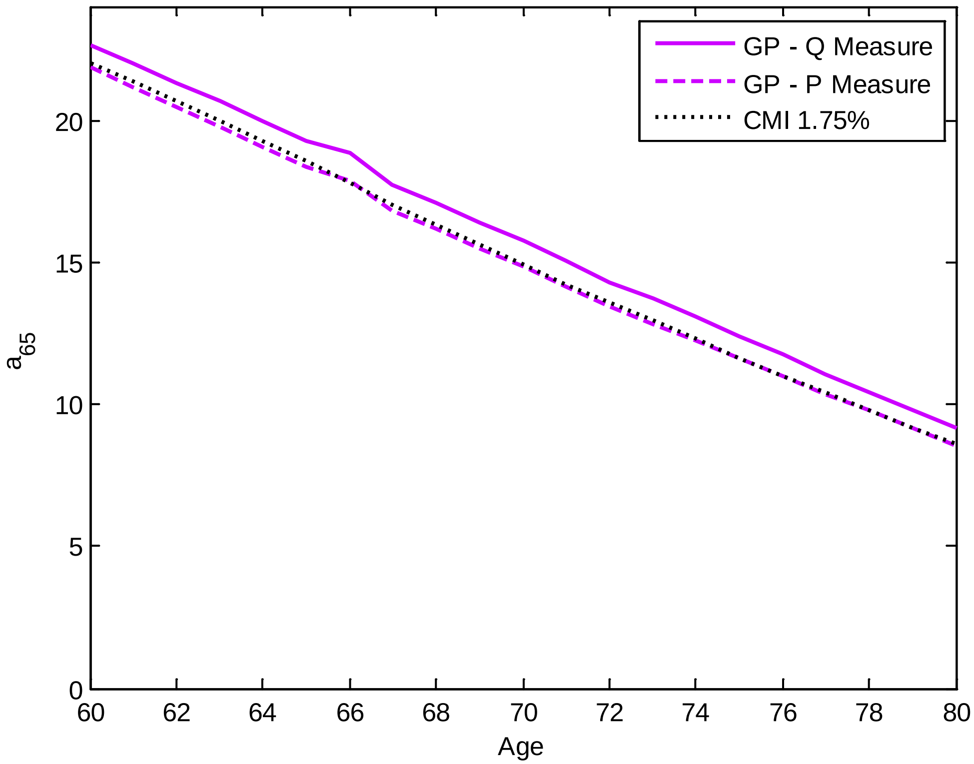
- Can also find prices for other longevity linked securities consistent with few market prices observed
  - For example, index of mortality improvement rates used in construction of the Kortis bond



Addendum


Period Life Expectancy at 65
24
23
22
21
20
19
18
17
16
15
2000
2005
2010
2015
2020
2025
2030
2035
2040
2045
2050
Year
GP - Q Measure
GP - P Measure
Observed
CMI 1.75%

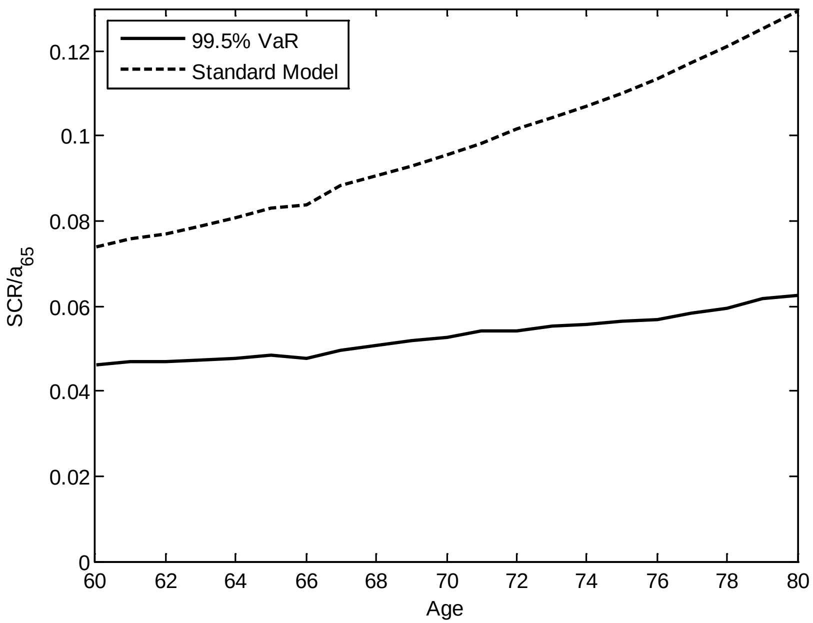
Addendum



GP - Q Measure
GP - P Measure
CMI 1.75%
a_{65}
Age
0
5
10
15
20
60
62
64
66
68
70
72
74
76
78
80

Addendum

- Solvency II SCR is the 99.5% VaR of the technical provisions
- Therefore, forward rate model can calculate SCR by repeated updates of forward mortality surface
  - Avoids nested sims for SCR
- Compare with Solvency II standard model - 20% shock to mortality to proxy for VaR
  - C.f., Börger (2010)



Addendum

$$\text{Risk Margin} = CoC \times \sum_{s=0}^{\infty} SCR(s)(1+r_s)^{-s}$$

- Calculation of risk margin suffers from calculation problems
- Short rate approach:
  - Needs simulations (to give liabilities at s+1) within simulations (to give VaR at s) within simulations (to model the run off of liabilities to s)
- Forward mortality rate approach
  - Needs simulations (to give VaR at s) within simulations (to model the run off of liabilities to s)
  - Progress, but not the complete answer

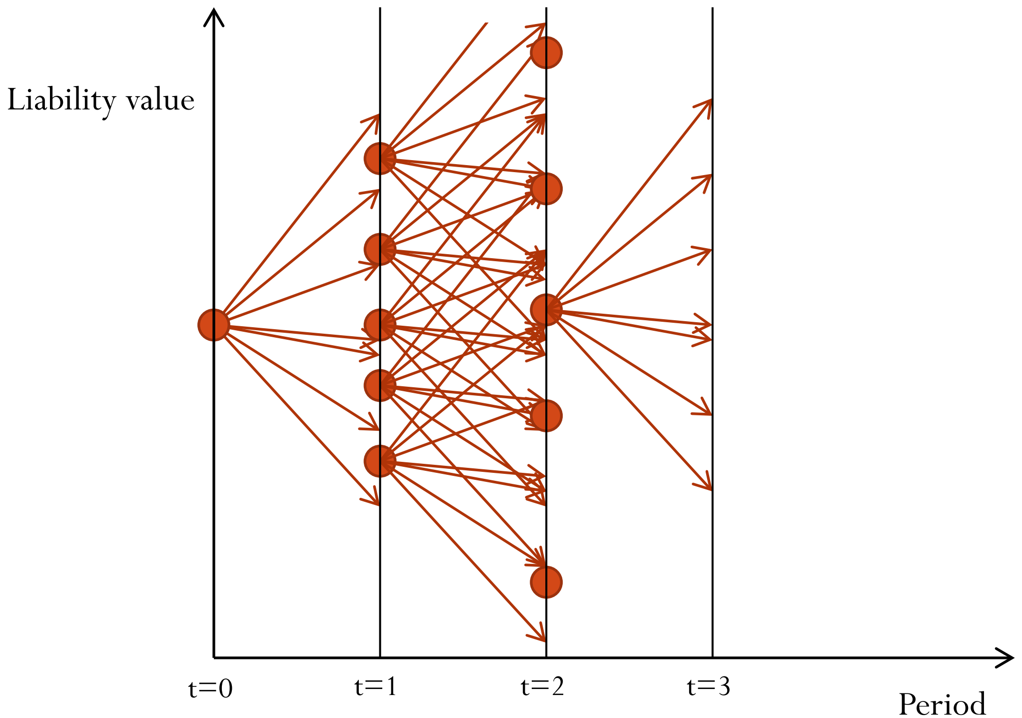
Addendum

- EIOPA (2014) suggests projecting deterministically to time t to avoid nested simulations
  - May distort estimation of VaR, especially in tails
- We propose alternative approach based on limited number of model points

Algorithm 1 Approximate estimation of the risk margin

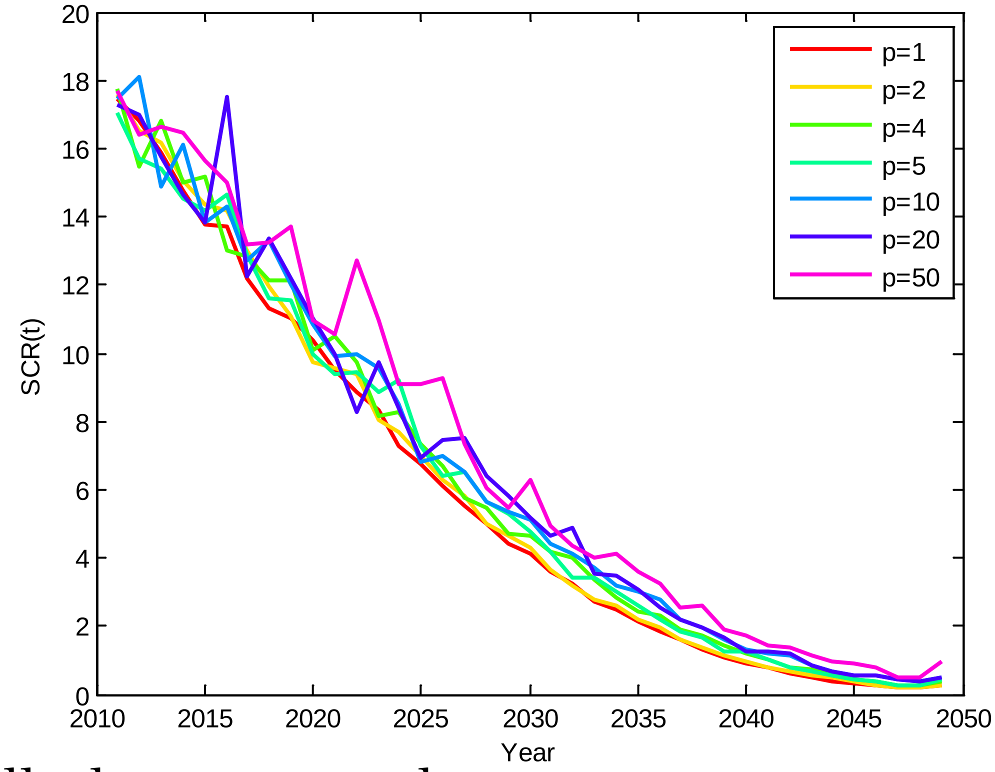
1: Perform N simulations to obtain empirical distribution of $\mathcal{L}(\tau+1)$ for estimation of $\mathrm{SCR}(\tau)$;
2: Select p sets of latent variables $\{\boldsymbol{\kappa}_{\tau+1}, \gamma_{\tau+1-x}\}$ corresponding to p model points in the distribution of $\mathcal{L}(\tau+1)$;
3: Perform N simulations for each model point to obtain p empirical distributions of $\mathcal{L}(\tau+2)|\mathcal{L}(\tau+1) = \mathcal{L}^{(i)}(\tau+1)$;
4: Calculate $\mathrm{SCR}^{(i)}(\tau+1)$ for each model point, and $\mathrm{SCR}(\tau+1) = \sum_{i-1}^{p} w_i \mathrm{SCR}^{(i)}(\tau+1)$ where w_i are a set of weights based on the relative probability of model point i;
5: Repeat steps 2 and 3 for each future year until the liabilities have run off;
6: Calculate the risk margin using $CoC \times \sum_{s=0}^{\infty} SCR(s)(1+r_s)^{-s}$

Addendum



- Fix p x N = 10,000
  - Trade off between high p (distribution at each time) and high N (robust estimate of 99.5% VaR)

Addendum



- Generally, low p means lower uncertainty in estimate, but biased SCR
- If p=10, SCR(0) = 5.4% and Risk Margin = 4.0% of best estimate of liability value.