Error analysis in simulation-based capital models: towards removal of proxy modelling errors from SCR estimates

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March 6, 2024

Outline

- Simulation-based capital models: the setting and simple recipes
- Practitioner viewpoints of proxy models
- Key ideas in the handling and removal of proxy errors
- Matching Adjustment portfolio example

The setting

The Solvency 2 capital requirement (SCR) is given by the amount of excess funds needed to cover the "1 year horizon, 99.5$^{\text{th}}$ percentile loss"

Internal models define a 1-year horizon loss distribution implicitly through

- Financial assumptions
- Stress calibrations
- Distribution assumptions on risk factors

Driving factors

Insurers and regulators have demanded increased

- Realism in risk and capital modelling, and
- Understanding of the interactions between risks (particularly in tail)

Reduction in costs & improvements in technology have allowed

- Development of sophisticated risk management systems
- Growth of complex businesses

Driving factors

However, the number of scenarios computed with the full model remains limited by

- complexity of base balance sheet calculations
- cost of computing capacity
- computational demand from other use cases

Therefore, faster to calculate alternatives to the full model are required, motivating the development of "proxy models"

Simulation-based capital models

Some basic mathematical notation

- Risk factor random variable R
- Risk factor realisation (of dimension m) $r \in \mathbb{R}^m$
- Loss function $x(r)$ (assume positive is gain)
- Loss distribution $x(R)$
- Distribution function $F_{x(R)}(s) = \mathbb{P}\left(x(R) \leq s\right)$
- Solvency capital requirement (SCR) $-F_{x(R)}^{-1}(0.005)$

Also, we use an index-bracket notation for sorted lists:

- Unsorted list $\{x_i\}_{i=1}^N$
- Sorted list $\{x_{(i)}\}_{i=1}^N$ (so that $x_{(i)} \leq x_{(i+1)}$)

Simulation-based capital models

Simulation-based capital models use Monte Carlo methods to estimate properties of the loss distribution such as the SCR

Basic Monte Carlo recipe

- Sample the risk factor distribution $\{r_i\}_{i=1}^{N}$
- Calculate loss scenarios $\{x(r_i)\}_{i=1}^{N}$
- Sort scenarios $x_i := x(r_i), \quad x_{(i)} \leq x_{(i+1)}$
- Estimate SCR $\xi = -x_{(k)}, \quad k = \lceil 0.005 \times N \rceil$

In its basic form, Monte Carlo may be impractical due to the number of scenarios required for convergence to blue$x(r)$ exceeding available computing capacity, an alternative approach is needed

Simulation-based capital models

In "proxy modelling" approximations of the full loss model which are faster to calculate are determined

These are used to create approximate loss samples that are feasible to calculate in large numbers

Modified Monte Carlo recipe

- Determine an approximate loss function ("proxy") $p(r) \approx x(r)$
- Sample the risk factor distribution $\{r_i\}_{i=1}^N$
- Calculate approximate losses $\{p(r_i)\}_{i=1}^N$
- Sort scenarios $p_i := p(r_i), \quad p_{(i)} \leq p_{(i+1)}$
- Estimate SCR $\xi_p = -p_{(k)}, \quad k = \lceil 0.005 \times N \rceil$

What do we primarily gain from the use of proxy functions?

Computational feasibility

What is introduced when we use proxy functions?

SCR uncertainty (approximations)
Communication requirements
Validation requirements

"We analysed proxy model errors for the purpose of developing an adjustment to final capital requirements, and found that the average error of the proxy model in a range of scenarios around the 99.5th percentile of the loss distribution is a surprisingly good estimate of the impact of the errors on capital requirements - even when the errors get quite big"

Proxy models: uncertain terms. The Actuary (2021)

"A proxy model will never replicate a heavy model exactly and so may produce large errors on particular individual scenarios. However, it can still produce sufficiently accurate quantile estimates as long as individual errors are free from systematic misstatements. Hursey et at. (2014) provided empirical evidence of this; however, the authors of this paper are not aware of a formal proof."

Androschuck et al (2017). Simulation-based capital models. British Actuarial Journal, 22(2), 257-335

Using the full model

For the Monte Carlo recipe for SCR, we wish to know the k^{th} largest loss scenario $x_{(k)}$ associated with the "critical scenario" r_{i_k} satisfying $x_{(k)} := x(r_{i_k})$. Unfortunately, this cannot be directly computed

An approach: use the proxy model's critical scenario

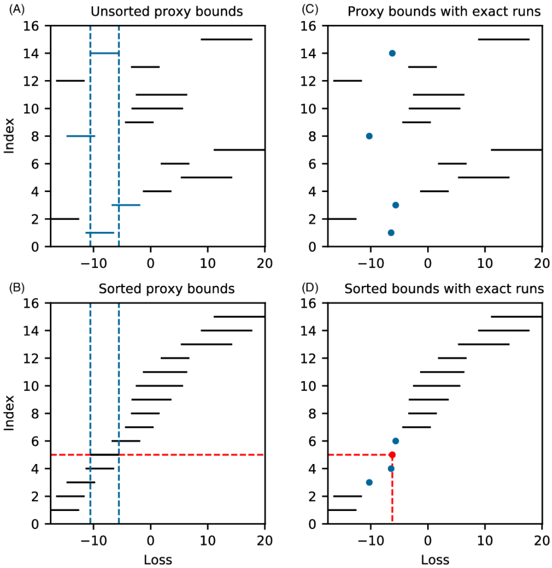
Suppose $p_{(k)} = p(r_{j_k})$ for some r_{j_k} and propose $-x(r_{j_k})$ as an SCR estimate. Here we hope $x(r_{j_k}) \approx x(r_{i_k})$ however we don't know the error $|x(r_{j_k}) - x(r_{i_k})|$ since ranks are not preserved between models

We show that under certain mathematical and computational assumptions the SCR can be recovered without approximation error

Key ideas

- In a Monte Carlo setting, consider proxy functions that are upper and lower bounds on the full model
- Bounds are preserved under sorting giving computable SCR bounds at the (unknown) critical loss scenario
- Consideration of the ordering of possible loss scenarios identifies a subset of scenarios containing the critical scenario
- Full computation of the scenario within the subset (if feasible) gives the SCR without approximation error

Removal of proxy errors by example



Key ideas

Bounding the loss function

- Determine bounding proxy functions $l(r) \leq x(r) \leq u(r)$
- Sample the risk factors $\{r_i\}_{i=1}^N$
- Calculate proxy values $\{l(r_i)\}_{i=1}^N$, $\{u(r_i)\}_{i=1}^N$ ($\{x(r_i)\}_{i=1}^N$ is infeasible)
- Write $l_i := l(r_i)$, $x_i := x(r_i)$, $u_i := u(r_i)$ and observe $l_i \leq x_i \leq u_i$

Componentwise inequality preserved under sorting

If $l_i \leq x_i \leq u_i$ then the (independently) sorted lists satisfy $l_{(i)} \leq x_{(i)} \leq u_{(i)}$

Observe:

- The (unknown) SCR value $x_{(k)}$ is bounded by the k^{th} sorted proxy bounds $l_{(i)}$ and $u_{(i)}$

Key ideas

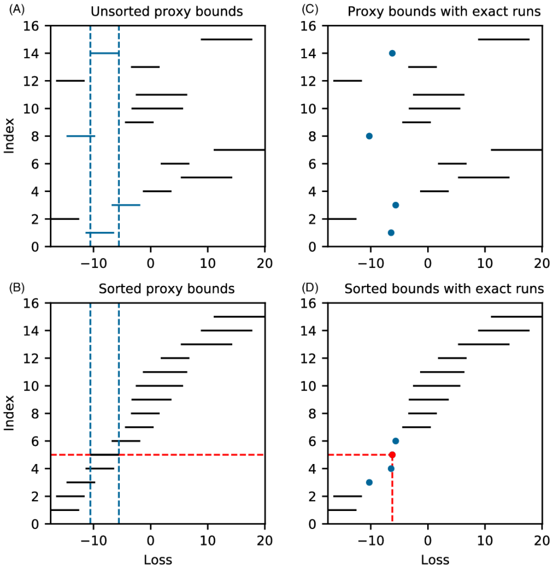
Identifying the set of possible critical scenarios

- All loss scenarios satisfy $x_i \in [l_i, u_i]$
- The critical scenario satisfies $x_{(k)} \in [l_{(k)}, u_{(k)}]$
- Any scenario i (associated with r_i) cannot be the k^{th} critical scenario if $[l_i, u_i] \cap [l_{(k)}, u_{(k)}] \neq \emptyset$.
- The set $\mathcal{A}_k := \{i : [l_i, u_i] \cap [l_{(k)}, u_{(k)}] \neq \emptyset, 1 \leq i \leq N\}$ contains the index i_k of the critical scenario r_{i_k} satisfying $x_{(k)} = x(r_{i_k})$

Observe:

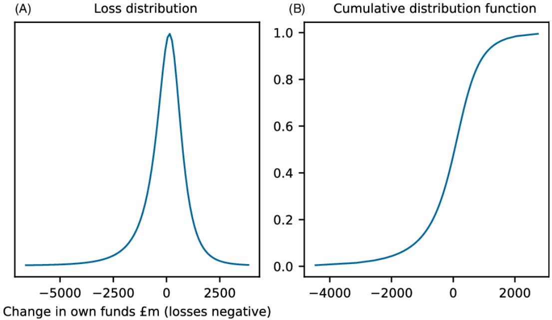
- Full calculation (if feasible) over all scenarios in $\mathcal{A}_k$ yields the critical scenario without approximation error.
- The feasibility of of calculating all scenarios in $\mathcal{A}_k$ is a function of the tightness of the proxy error bounds

Removal of proxy errors by example

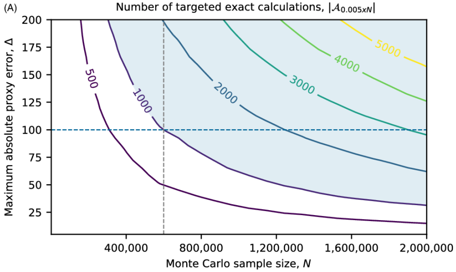


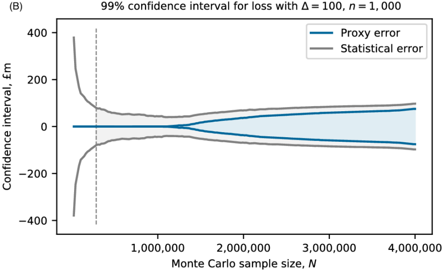
Numerical example

A family of distributions was selected that allow for skew and fat tails
Parameters chosen so that the loss distribution appears realistic



Numerical example





Key points

The methods presented may

- Help estimate SCR uncertainty by providing error bounds
- Simplify SCR communication
- Lead to the removal of proxy errors in the SCR

Possible weaknesses of the methods presented

- SCR bounds may be too large for use
- Computations required to remove proxy errors may be infeasible
- Require validated proxies that are bounds to the full model

How can we derive proxy functions that are bounds on the full model?

Matching Adjustment example

Key ideas

- Under Solvency 2, and for approved firms, eligible liabilities may be valued through a discount rate derived from the yield on a portfolio of "matching" assets. For simplicity, we take the liability valuation to be the matching portfolio value
- An eligible portfolio may be formed through an optimisation of excess funds where eligible portfolios are identified through optimisation constraints
- Schematically the computation involves gathering asset valuations, and asset and liability cashflows, denoted $\pi(r)$, and then performing a minimum cost optimisation $\mathcal{X}$:

$$\text{Full model:} \quad r \xrightarrow{\text{Heavy}} \pi(r) \xrightarrow{\text{Light}} \mathcal{X}(\pi(r))$$

Matching example

Kocherlakota et al. (1988)

- Suppose the cash balance (positive/negative) accrues at different (reinvestment/borrowing) rates with discount curves $DF_{\pm}(0, t_j)$
- Asset i has value v_i and cashflows $c_{i,j}$ at time t_j, liability cashflows denoted l_j at t_j
- Matching portfolio weights $0 \leq \alpha_i \leq 1$
- Denote balance at time t_j by $b_j^+ - b_j^-$ where $b_j^+, b_j^- \geq 0$
- Cash accrual constraint

$$b_j^+ - b_j^- = b_{j-1}^+ \frac{DF_+(0, t_{j-1})}{DF_+(0, t_j)} - b_{j-1}^- \frac{DF_-(0, t_{j-1})}{DF_-(0, t_j)} + \sum_i \alpha_i c_{i,j} - l_j$$

- Terminal balance constraint $b_n^- = 0$
- Choose α_i to minimise liability valuation $\mathcal{L} = \sum_i v_i \alpha_i$

Matching example

Linear matching optimisation restated into standard "A, b, c" matrix form

Linear optimisation

- Asset value $\mathfrak{A}(\pi(r)) := c(r)^T I$ (sum of asset values)
- Liability value $\mathcal{L}(\pi(r)) = \inf\{c(r)^T\alpha | A(r)\alpha \geq b(r), \alpha \geq 0\}$
- Loss function defined as change in assets less liabilities

$$\mathcal{X}(\pi(x(r)) := \mathfrak{A}(\pi(r)) - \mathcal{L}((\pi(r)) - (\mathcal{A}_0 - \mathcal{L}_0)$$

where "data", such as cashflow values, is gathered into $\pi(r) := [A(r), b(r), c(r)]$, $A(r)$ is a constraint matrix, $b(r)$ and $c(r)$ are vectors

Matching Adjustment (example)

A proxy for the loss function can be formed through data-proxies

$$\text{Proxy model:} \quad r \xrightarrow{\text{Light}} \pi^*(r) \xrightarrow{\text{Light}} \mathcal{X}(\pi^*(r))$$

Under suitable differentiability, an approximating tangent plane can be used to approximately bound the error

Approximate error bounds

$$\begin{aligned}
|\mathcal{X}(\pi(r)) - \mathcal{X}(\pi^*(r))| \lessapprox \\
& \sum_{i,j} \left|\frac{\partial \mathcal{X}}{\partial A^*_{i,j}}(\pi^*(r))\right| |A_{i,j} - A^*_{i,j}| \\
& + \sum_{j} \left|\frac{\partial \mathcal{X}}{\partial b^*_{j}}(\pi^*(r))\right| |b_j - b^*_j| + \sum_{i} \left|\frac{\partial \mathcal{X}}{\partial c^*_{i}}(\pi^*(r))\right| |c_i - c^*_i|
\end{aligned}$$

Key ideas

The loss function is formed from an optimal-value function of a linear optimisation problem

Primal and dual problems

- Primal value $\mathcal{L}(\pi(r)) = \inf\{c(r)^T\alpha | A(r)\alpha \geq b(r), \alpha \geq 0\}$
- Dual value $\mathcal{D}(\pi(r)) = \sup\{b(r)^T\lambda | A(r)^T\lambda = c(r), \lambda \geq 0\}$

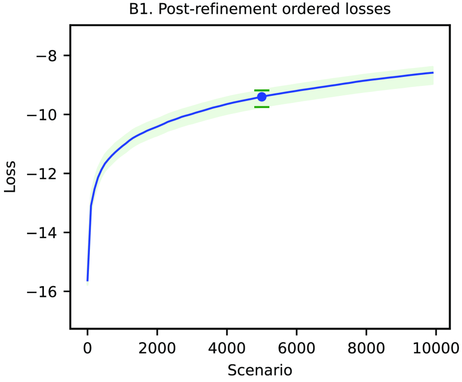
Derivatives

If the optimal-value function $\mathcal{L}$ is differentiable wrt to its data then

- $\frac{\partial \mathcal{L}}{\partial A_{i,j}} = -\alpha_i(r)\lambda_j(r), \frac{\partial \mathcal{L}}{\partial b_j} = \lambda_j(r), \frac{\partial \mathcal{L}}{\partial c_i} = \alpha_i(r)$

Stylized examples

- Green region shows approximate error bounds shown against actual
- Approximate error bounds give an attribution and refinement mechanism
- Removal of proxy errors at the capital scenario k is dependent on size of bounds and the subset $\mathcal{A}_k$



In conclusion

The framework provides practitioners mechanisms for the elimination of proxy errors from SCR estimates through

- Bounding impact of proxy errors on SCR estimates
- Understanding the computational cost of eliminating proxy errors from SCR estimates
- Use of error attribution to refine proxies to make elimination feasible

In conclusion, error analysis has an important role in removing proxy modelling as a potential single point of failure of internal models

References

- Crispin, D. & Kinsley, S. (2022). Eliminating proxy errors from capital estimates by targeted exact computation. Annals of Actuarial Science, 1-24, doi:10.1017/S1748499522000161
- Crispin DJ. Error propagation and attribution in simulation-based capital models. Annals of Actuarial Science. Published online 2023:1-29. doi:10.1017/S1748499523000210