Unmasking Volatility, Discontinuous Continuity, VIX options via SPX options

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Joint Work with King Wang

Outline

- VIX from SPX remarks.

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- Stochastic Volatility Literature.

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- Continuous Time Models driven by Realized Variation.

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- Realized Quadratic Variations as Speed Drivers.

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- Markovian TFLP's.
- Continuous Time Models driven by Realized Variation.
- Realized Quadratic Variations as Speed Drivers.
- Some Stylized Results.

VIX and SPX

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- Following Jaber, Illand and Li (2023) the VIX_t is formally related to the process for S by

$$VIX_t = \lim_{h \downarrow 0} \sqrt{-\left(\frac{2}{h}\right) E^Q \left[\log(S_{t+h}/S_t) | \mathcal{F}_t\right]} \times 100.$$

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- Once the measure Q has been specified and calibrated, to SPX options for example, it may be employed to price options on the VIX.

Stochastic Volatility Literature

- In the context of continuous martingales living on Brownian filtrations with $\sigma = (\sigma_t, t > 0)$ being the instantaneous volatility for $\ln(S_t)$, it is well understood that

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- Options on the VIX then have a positive price only if the volatility is stochastic.
- There is an extensive literature on stochastic volatility models establishing consistency between the prices of VIX and SPX options.
- We cite a few examples.

Papanicolaou and Sircar(2014)

Fouque and Saporito(2018)

Guyon (2020)

Gatheral, Jusselin, and Rosenbaum(2020)

Blindness in Brownian Motion

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- For general Lévy processes one may enhance the degree of continuity by increasing the magnitude of the smaller moves relative to the larger ones.
- This feature is lost in the continuous limit.

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- They are infinitely divisible but have infinitely jumps in any interval most of which are arbitrarily small with no smallest jump among them.
- They may be considered to be good candidates for discontinuous approximations to continuity.
- Especially if one recognizes that both continuity and self-decomposability are a physical impossibility, both requiring infinite work.

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- Self-decomposability only requires a countable infinity of transactions associated with the infinite number of price jumps in any interval.
- They actually cannot exist.

CGMY Continuity Approximations

- The *CGMY* Lévy process has a unit time self-decomposable law with Lévy density on the positive side of the form

$$k(x; c, m, y) = c\frac{\exp(-mx)}{x^{1+y}}, \ x > 0.$$

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- For two sizes $a < (1+\lambda)a$ for $\lambda > 0$ the relative arrival rate of the smaller move to the larger move is given by the ratio

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- Speed, Scale and the Degree of Continuity are then represented by c, $b = 1/m$, and y.

Stochasticity of Speed, Scale or the Continuity Degree

- For 177 stocks over the period January 5, 2007 to June 30, 2023 or 4301 days estimates were obtained every 21 days for the speed, scale and continuity parameters c, b, and y separately for the positive and negative returns by matching observed tail probabilities to $CGMY$ tail probabilities.

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- In total there were 177×193 or a total of $34,611$ estimations. The $CGMY$ tail probabilities were obtained by Fourier inversion of the characteristic function.

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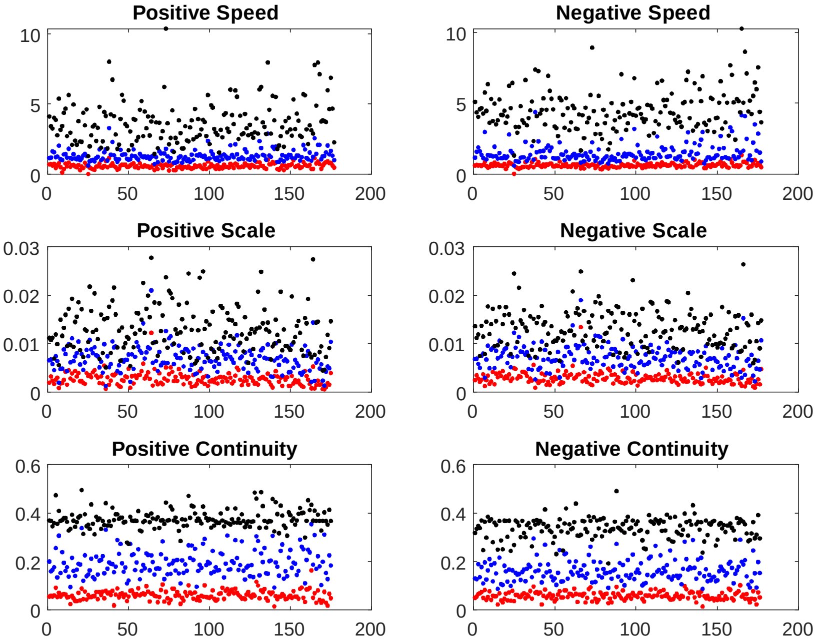
- The Figure presents the three quartile points across the 193 time points for the three parameters on both the positive and negative sides for the 177 equity assets.


Positive Speed
Negative Speed
Positive Scale
Negative Scale
Positive Continuity
Negative Continuity
10
5
0
0.03
0.02
0.01
0.6
0.4
0.2
0 50 100 150 200

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- These observations motivate the choice of modeling the stochasticity in parameters of motion by stochastic Lévy speed with scale and the degree of continuity held fixed.
- Most of the paper employs the bilateral gamma model where the parameter y is zero.

Bilateral Gamma (BG) Details I

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- Let $\gamma_p = (\gamma_p(t), t > 0)$ and $\gamma_n = (\gamma_n(t), t > 0)$ be two standard gamma processes with unit mean and variance then for speed parameters c_p, c_n and scale parameters b_p, b_n for the positive and negative sides,

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- the bilateral gamma process $X_{BG} = (X_{BG}(t), t > 0)$ is defined by $X_{BG}(t) = b_p\gamma_p(c_pt) - b_n\gamma_n(c_nt)$.

Bilateral Gamma (BG) Details II

- The unit time chacteristic function of the bilateral gamma is given by

$$\phi_{BG}(u) = \left(\frac{1}{1 - iub_p}\right)^{c_p} \left(\frac{1}{1 + iub_n}\right)^{c_n}.$$

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- The process has a self decomposable limit law at unit time with Lévy density

$$k_{BG}(x) = \frac{c_n \exp\left(-|x|/b_n\right)}{|x|}\mathbf{1}_{x<0} + \frac{c_p \exp\left(-x/b_p\right)}{x}\mathbf{1}_{x>0}.$$

Variations and Quadratic Variations I

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- Specifically

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\begin{aligned}
V_L(t) &= \int_0^t \int_{-\infty}^{\infty} |x| \mu_L(dx, ds), \\
Q_L(t) &= \int_0^t \int_{-\infty}^{\infty} x^2 \mu_L(dx, ds).
\end{aligned}
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Variations and Quadratic Variations II

- The variation and quadratic variation processes, V_L, Q_L are themselves increasing Lévy processes in their own right and it is shown in, Madan and Wang (2023) for example, that their Laplace transforms are given by

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E\left[\exp\left(-\eta V_L(t)\right)\right] &= \exp\left(t\int_{-\infty}^{\infty}\left(e^{-\eta|x|}-1\right)k_L(x)dx\right), \\
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- Unlike Brownian motion where variation is infinite and quadratic variation is a constant. Here they are proper random variables.
- However, both realized variation and quadratic variation are increasing processes and not good candidates for state variables of a Markovian system.

Tempered Fractional Lévy Processes I

- For a tempering parameter λ and a fractional parameter δ define for Lévy innovations defined on the real line the processes $Y_L = (Y_L(t;\lambda,\delta), t \in \mathbb{R})$ and $Z_L = (Z_L(t;\lambda,\delta), t \in \mathbb{R})$ where

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- Separate versions for the positive and negative sides are obtained by partitioning the integrals over x for the positive and negative regions to obtain the processes $Y_{pL}(t;\lambda_p,\delta_p)$, $Y_{nL}(t;\lambda_n,\delta_n), Z_{pL}(t;\lambda_p,\delta_p), Z_{nL}(t;\lambda_n,\delta_n)$.

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- For $\delta = 0$ we have *OU* equations reflecting mean reversion as in a Heston stochastic volatility model.
- However, the uncertainty driving the *TFLP* processes are the paths of the original Lévy process and not some independent extraneous process of shocks.

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- The Laplace exponents for Y_L, and Z_L are then obtained as

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\psi_{Y_L}(\eta) &= \int_0^\infty \psi_V\left(e^{-\lambda s}s^\delta \eta\right) ds, \\
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- where ψ_V and ψ_Q are the Laplace exponents for variation and quadratic variation at unit time.

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dY &= \lambda(\alpha_0 - Y)dt, \\
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- The Markovian structure is useful in developing characteristic functions using well known procedures for exponential affine solutions.

Stochastic Lévy Speed: The Two Period Case I

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- Specifically the speed parameters are now on the two sides

$$c_{2p} + d_{2p}V_{pX}(t_1) + e_{2p}V_{nX}(t_1)$$
$$c_{2n} + d_{2n}V_{pX}(t_1) + e_{2n}V_{nX}(t_1)).$$

- The chacteristic function for the logarithm of the stock price at the second date $\ln(S_2)$ may be worked out in two ways.

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- The first assumes independence and takes the product of the marginal laws of X_1, $V_{pL}(t_1)$, $V_{nL}(t_1)$.

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- The first assumes independence and takes the product of the marginal laws of X_1, $V_{pL}(t_1)$, $V_{nL}(t_1)$.
- The second employs the joint law of the triple X_1, $V_{pL}(t_1)$, $V_{nL}(t_1)$.

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- where

$$\begin{aligned}
\alpha &= -2\left(c_{2p}\ln(1-b_{2p}) + c_{2n}\ln(1+b_{2n}) + b_{2p}c_{2p} - b_{2n}c_{2n}\right) \\
\beta &= -2(d_{2p}\ln(1-b_{2p}) + d_{2n}\ln(1+b_{2n}) + b_{2p}d_{2p} - b_{2n}d_{2n}) \\
\gamma &= -2(e_{2p}\ln(1-b_{2p}) + e_{2n}\ln(1+b_{2n}) + b_{2p}e_{2p} - b_{2n}e_{2n})
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- Many industry models employ such a value as an ad-hoc input.

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\begin{aligned}
g(\lambda) &= E\left[\exp\left(-\lambda\left(\beta A_{1p} + \gamma A_{1n}\right)\right)\right] \\
&= \left(\frac{1}{1+\beta\lambda b_{1p}}\right)^{c_{1p}t_1}\left(\frac{1}{1+\gamma\lambda b_{1n}}\right)^{c_{1n}t_1}
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$$

- Another magnitude of interest is the level of the forward VIX given by

$$
fwdvix = minvix + \int_{minvix}^{\infty} P(VIX_{t_1} > a)da.
$$

Continuous Time Case: Variation Driven TFLP for BG I

- Let L be BG with parameters b_p, c_p, b_n, and c_n and jump random measure $\mu_L(dx, dt)$.

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- Let $Y_p(t)$, $Y_n(t)$ be two continuous time speed drivers defined by

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\begin{aligned}
Y_p(t) &= \int_{-\infty}^{t}\int_{-\infty}^{\infty} e^{-\lambda_p(t-s)}(t-s)^{\delta_p} x^{+} \mu_L(dx, ds), \\
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- Let

$$
\begin{aligned}
\widetilde{Y}_p(t) &= \int_0^t Y_p(s)ds \\
\widetilde{Y}_n(t) &= \int_0^t Y_n(s)ds.
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Continuous Time Case: Variation Driven TFLP for BG II

- The logarithm of the characteristic function of $\ln(S(t))$ conditional on the paths of Y_p, Y_n is

$$\ln E[\exp(iu \ln S(t)) | Y_p, Y_n] = \alpha + \beta \widetilde{Y}_p(t) + \gamma \widetilde{Y}_n(t)$$

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$$\ln E\left[\exp(iu \ln S(t)) \middle| Y_p, Y_n\right] = \alpha + \beta \widetilde{Y}_p(t) + \gamma \widetilde{Y}_n(t)$$

- The coefficients are

$$\alpha = \left(\begin{array}{c} -c_p t \ln(1 - iub_p) + iuc_p t \ln(1 - b_p) \\ -c_n t \ln(1 + iub_n) + iuc_n t \ln(1 + b_n) \end{array} \right)$$

$$\beta = \left(\begin{array}{c} -d_p \ln(1 - iub_p) + iud_p \ln(1 - b_p) \\ -d_n \ln(1 + iub_n) + iud_n \ln(1 + b_n) \end{array} \right)$$

$$\gamma = \left(\begin{array}{c} -e_p \ln(1 - iub_p) + iue_p \ln(1 - b_p) \\ -e_n \ln(1 + iub_n) + iue_n \ln(1 + b_n) \end{array} \right)$$

Fractional Parameter

$\delta = 2$

- For $Y_p(t)$ with Markovian system $(Y_p, \alpha_{0p}, \alpha_{1p})$, the characteristic function of the $4 - tuple$ $(Y_p(t), \alpha_{0p}(t), \alpha_{1p}(t), \widetilde{Y}_p(t))$ is given by the exponential affine solution

$$\begin{aligned} & \phi\left(u_1(s), u_2(s), u_3(s), u_4(s)\right) \\ = \; & E_s\left(\exp(iu_1 Y_p(t), iu_2\alpha_0(t), iu_3\alpha_1(t), iu_4\widetilde{Y}_p(t))\right) \\ = \; & \exp(\theta_0(s) + \theta_1(s)Y_p(s) + \theta_2(s)\alpha_0(s) + \theta_3(s)\alpha_1(s) + \theta_4(s)\widetilde{Y}_p(s) \end{aligned}$$

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- The resulting differential equations are with $\theta_4(s) = iu_4$.

$$\begin{aligned} \theta_{0s} - \ln\left(1 - \theta_3(s)\right)) &= 0 \\ \theta_{1s} - \lambda_p\theta_1(s) &= 0 \\ \theta_{2s} + \lambda_p\theta_1(s) - \lambda_p\theta_2(s) &= 0 \\ \theta_{3s} + \lambda_p\theta_2(s) - \lambda_p\theta_3(s) &= 0 \end{aligned}$$

Characteristic Function Solution

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- $$
\begin{aligned}
\theta_4(0,\mathbf{u}) &= iu_4 \\
\theta_1(0,\mathbf{u}) &= \left[iu_1 + \frac{iu_4(e^{\lambda_p t}-1)}{\lambda_p} \right] e^{-\lambda_p t} \\
\theta_2(0,\mathbf{u}) &= \left(iu_2 + \lambda_p \left[iu_1 + \frac{iu_4}{\lambda_p}\left(e^{\lambda_p t}-1\right) \right] t \right) e^{-\lambda_p t} \\
\theta_3(0,\mathbf{u}) &= \left(\begin{array}{c} iu_3 + \lambda_p iu_2 t + \\ \lambda_p^2 \left[\begin{array}{c} iu_1 + \\ \frac{iu_4}{\lambda_p}\left(e^{\lambda_p t}-1\right) \end{array} \right] t^2/2 \end{array} \right) e^{-\lambda_p t} \\
\theta_0(0,\mathbf{u}) &= -\int_0^t \ln \left(\begin{array}{c} 1 - e^{-\lambda_p v} \times \\ \left[\begin{array}{c} iu_3 + \lambda_p iu_2 v + \\ \lambda_p^2 \left[\begin{array}{c} iu_1 + \\ \frac{iu_4}{\lambda_p}\left(e^{\lambda_p t}-1\right) \end{array} \right] v^2/2 \end{array} \right] \end{array} \right) dv
\end{aligned}
$$

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- $$v(\eta) = \Gamma(-y_p)\left((M+\eta)^{y_p} - M^{y_p}\right).$$

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$$I(\nu, a, \lambda) = \int_0^\infty x^{\nu-1} e^{-ax-\lambda x^2} dx = (2\lambda)^{-\nu/2}\Gamma(\nu) h_{-\nu}\left(\frac{a}{\sqrt{2\lambda}}\right)$$

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- for $\mathrm{Re}(\nu) > 0$, where the functions $h_\nu(z)$ are the Hermite functions.
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-

$$\begin{aligned} w(\eta) &= -\left[\frac{2\eta_1}{y_p} + \frac{M^2}{y_p(1-y_p)}\right] I(2-y_p, M, \eta) \\ &\quad - \frac{2\eta M}{y_p(1-y_p)} I(3-y_p, M, \eta) \\ &\quad + \frac{M^{y_p}}{y_p(1-y_p)}\Gamma(2-y_p). \end{aligned}$$

Two Period Results I

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- There were thus five calibrations for each of the 209 days.

Two Period Results II

Table 1
Quartiles for the Number of Options

	First Maturity			Second Maturity		
Month	Q1	Q2	Q3	Q1	Q2	Q3
1	154	173	205	157	181	280
2	165	184	313	122	210	322
3	121	189	292	47	80	216
4	55	113	226	46	59	187
5	47	97	199	49	69	122

Two Period Results III

-

Table 2

Quartiles for the Average Percentage Error

	First Maturity			Second Maturity		
Month	Q1	Q2	Q3	Q1	Q2	Q3
1	0.0236	0.0392	0.0677	0.0128	0.0195	0.0342
2	0.0143	0.0224	0.0405	0.0092	0.0135	0.0223
3	0.0107	0.0149	0.0235	0.0078	0.0109	0.0186
4	0.0079	0.0119	0.0199	0.0057	0.0088	0.0132
5	0.0056	0.0090	0.0145	0.0045	0.0069	0.0112

MinVIX and FwdVIX

Table 3

Quartiles for the Minimum and forward VIX

	Minimum VIX			Forward VIX		
Month	Q1	Q2	Q3	Q1	Q2	Q3
1	0.1272	0.1820	0.2394	0.2000	0.2511	0.3222
2	0.1423	0.2101	0.2747	0.2561	0.3044	0.3961
3	0.1236	0.2010	0.2869	0.2526	0.3151	0.4111
4	0.1432	0.2293	0.3522	0.2639	0.3566	0.5195
5	0.1576	0.2667	0.3981	0.3118	0.4497	0.5981

Parameter Values

- Table 4
  Two Period Model Parameter
  Quartiles at the First Maturity

Parameter	Q1	Q2	Q3
b_{1p}	0.0020	0.0032	0.0047
c_{1p}	242.36	377.26	686.04
b_{1n}	0.0527	0.0680	0.0882
c_{1n}	3.2832	6.3789	10.801
b_{2p}	0.0023	0.0040	0.0085
c_{2p}	119.61	486.94	1138.8
d_{2p}	6.9869	30.437	101.60
e_{2p}	4.4217	13.532	31.784
b_{2n}	0.0859	0.1030	0.1306
c_{2n}	0.5929	2.4577	4.2128
d_{2n}	9.4884	23.776	33.904
e_{2n}	3.9607	13.598	27.622

Stylized VIX Match

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Stylized VIX Match

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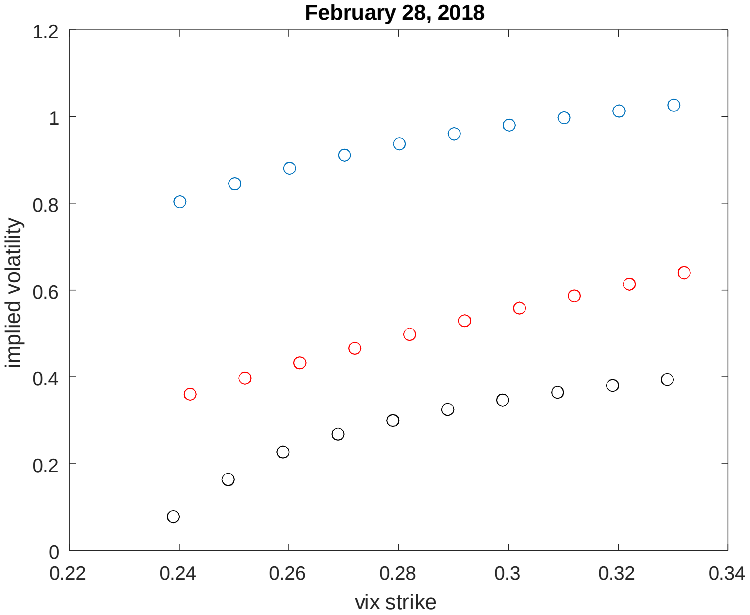
February 28, 2018
1.2
1
0.8
0.6
0.4
0.2
0
implied volatility
0.22
0.24
0.26
0.28
0.3
0.32
0.34
vix strike

February 28, 2018

- The parameter values associated with Figure 33 are as provided in Table 5.

February 28, 2018

- The parameter values associated with Figure 33 are as provided in Table 5.
- We observe substantial values for d_{2p} and e_{2n} that one anticipates drive the high levels for the *VIX* implied volatilities at the one month maturity.

Table 5

Parameters for February 28, 2018

b_{1p}	0.0049	b_{2p}	0.0016	b_{2n}	0.1544
c_{1p}	346.43	c_{2p}	4145.5	c_{2n}	1.0745
b_{1n}	0.0811	d_{2p}	263.34	d_{2n}	0.5133
c_{1n}	4.1799	e_{2p}	69.950	e_{2n}	43.212

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- The calibration was conducted on six maturities simultaneously.
- The quartiles for the average percentage error over the six maturities across the 100 days were 0.0365, 0.0491, and 0.0663.
- The quartiles for the number of options in the calibrations were 466, 743, and 1100.

Estimated Parameter Quartiles I

- Table 7 presents quartiles for the 10 estimated motion parameters.

Table 7
Bilateral Gamma Stochastic Variation
Parameter Quartiles

Parameter	Q1	Q2	Q3
b_p	0.0032	0.0070	0.0107
c_p	0.0209	0.2379	2.1727
b_n	0.0851	0.1634	0.1958
c_n	0.0265	0.3232	1.3462
d_p	0.2657	3.5566	12.660
d_n	0.0000	0.0001	0.1361
e_p	0.0034	0.1391	2.8386
e_n	0.0080	0.0796	0.4134
λ_p	0.2182	1.9812	6.4974
λ_n	0.9400	2.9709	5.9415

Estimated Parameter Quartiles II

- Table 7 presents quartiles for the 8 estimated initial level parameters.

Table 7
Bilateral Gamma Stochastic Variation
Parameter Quartiles

Parameter	Q1	Q2	Q3
Y_{0p}	1.0739	7.0695	22.888
α_{0p}	0.1038	0.8192	4.0457
α_{1p}	0.0036	0.0965	0.4672
$\widetilde{Y}_{0p}$	0.0389	0.4383	2.4920
Y_{0n}	0.2145	1.0989	4.1500
α_{0n}	1.0198	2.6353	7.0929
α_{1n}	0.0119	0.2538	1.8444
$\widetilde{Y}_{0n}$	0.0001	0.0088	0.4463

Instantaneous VIX Implied Vols

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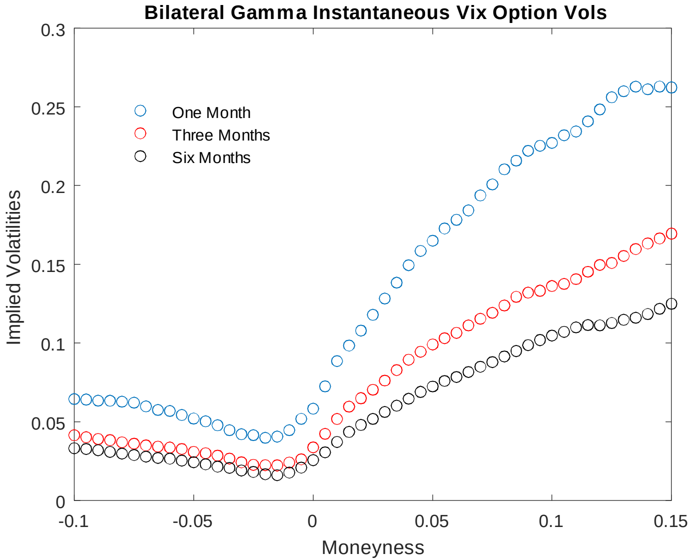
- They follow the pattern observed in traded *VIX* options.


Bilateral Gamma Instantaneous Vix Option Vols
One Month
Three Months
Six Months
Implied Volatilities
Moneyness
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.1
-0.05
0
0.05
0.1
0.15

Conclusion I

- It is argued that volatility has a number of dimensions that become visible in discontinuous approximations of continuity and that are lost in the continuous limit.

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- They include the speed and scale of movements along with the degree of proximity to continuity.
- The coarsest property is speed with scale and continuity proximity being somewhat finer considerations.
- Models with stochastic speed are formulated where the current speed looking forward is adapted to or driven by levels of realized speed looking backwards.

Conclusion II

- With a view towards building a Markovian dynamics the state variables are taken to be Tempered Fractional Lévy Processes with integral fractional parameter, driven by the variation or quadratic variation of an underlying Lévy process.

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- A simple two period model is seen to provide VIX option prices matching stylized features of the VIX option market.
- A continuous time model based on bilateral gamma and *CGMY* levels of realized variations delivers prices for options on the future level of the instantaneous VIX.