Gumbel autoregressive models: An application using best practice life expectancy

Anthony Medford

amedford@sdu.dk

Longevity 16
Copenhagen
August 13, 2021

Aims

- To introduce Gumbel autoregressive models of order one – Gumbel AR(1).

Aims

- To introduce Gumbel autoregressive models of order one – Gumbel AR(1).
- Use Best Practice Life Expectancy (BPLE) to demonstrate model fitting.

What is Best Practice Life Expectancy?

- BPLE is the maximum life expectancy from among national populations during a given year, at a particular age.

What is Best Practice Life Expectancy?

- BPLE is the maximum life expectancy from among national populations during a given year, at a particular age.
- It is essentially a time series annual maxima.

Gaussian time series

- Gaussian models are the basic models for linear time series.
- Not uncommon to assume that the error process is itself Gaussian.
- Both the assumed innovations (error) and marginal distributions are often assumed to be Normally distributed.

Extreme (value) time series

- Where data records extreme events, Gaussian time series are often unsuitable.
- In time series approaches there is no discourse around the stationary marginal distribution or any attempt to set it out explicitly.

Gumbel AR(1) approach

- Exploits relationship between extreme value distributions and positive stable distributions.

Gumbel AR(1) approach

- Exploits relationship between extreme value distributions and positive stable distributions.
- Key step: adding a Gumbel distributed variable to a positive α-stable variable results also in a Gumbel distributed.

Gumbel AR(1) approach

- Exploits relationship between extreme value distributions and positive stable distributions.
- Key step: adding a Gumbel distributed variable to a positive α-stable variable results also in a Gumbel distributed.
- Analagous to familiar Gaussian AR model we get Gumbel AR model for time series.

Gumbel distribution

Distribution function:

$$F(x) = \exp\left\{-\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\} \tag{1}$$

Gumbel distribution

Distribution function:

$$F(x) = \exp\left\{-\exp\left[-(\frac{x-\mu}{\sigma})\right]\right\} \quad (1)$$

Characteristic function:

$$\phi_x(t) = E(\exp(itX)) = \Gamma(1 - i\sigma t)e^{i\mu t} \quad (2)$$

Gumbel distribution

Distribution function:

$$F(x) = \exp\left\{-\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\} \quad (1)$$

Characteristic function:

$$\phi_x(t) = E(\exp(itX)) = \Gamma(1 - i\sigma t)e^{i\mu t} \quad (2)$$

where $\Gamma(.)$ is the gamma integral and
$i = \sqrt{-1}$ is the complex number.

Positive α-stable distributions

A random variable S is α-stable if:

- There exist positive real numbers a and b such that $c_1 S_1 + c_2 S_2$ is equal in distribution to $aS + b$ (for all non-negative real numbers c_1 and c_2).

Positive α-stable distributions

A random variable S is α-stable if:

- There exist positive real numbers a and b such that $c_1 S_1 + c_2 S_2$ is equal in distribution to $aS + b$ (for all non-negative real numbers c_1 and c_2).
- S_1 and S_2 are independent, identically distributed (iid) copies of S.

The Gumbel / α-stable relationship

- If X is Gumbel distributed with parameters μ and σ and is independent of α-stable distributed S.

The Gumbel / α-stable relationship

- If X is Gumbel distributed with parameters μ and σ and is independent of α-stable distributed S.
- Then the sum $X + \sigma logS$ is also Gumbel distributed with parameters μ and σ/μ.

Basic setup I

Begin with the framework of the simple stationary AR(1) model:

$$X_t = \alpha X_{t-1} + \epsilon_t, \qquad 0 \leq \alpha \leq 1, = 1, 2, \ldots \tag{3}$$

Let $\epsilon_t = \alpha \log S_t$ where S is a positive stable random variable.

The marginal of $\{X_t\}$ to be Gumbel distributed.

Basic setup II

It can be shown that the mean and variance of the innovation rv are given by

$$E(\epsilon_t) = (1-\alpha)(\mu + \sigma\gamma) \qquad \text{and} \qquad Var(\epsilon_t) = \frac{(1-\alpha^2)(\pi^2\sigma^2)}{6} \tag{4}$$

where $\gamma \approx 0.5772$ is Euler's constant.

Parameter estimation: μ, σ

We use method of momemts for estimating μ, σ

$$\hat{\mu} = \bar{X} - \gamma \frac{\sqrt{6}}{\pi} s \qquad \text{and} \qquad \hat{\sigma} = \frac{\sqrt{6}}{\pi} s$$

(where $\bar{X} = \sum_{t=1}^{n} X_t/n$ and $s^2 = \sum_{t=1}^{n}(X_t - \bar{X})/n$.)

Parameter estimation: α

We estimate α using a Yule-Walker type approach,

$$\hat{\alpha} = \frac{\sum_{t=1}^{n-1}(X_t - \bar{X})(X_{t+1} - \bar{X})}{\sum_{t=1}^{n}(X_t - \bar{X})^2}.$$

Data

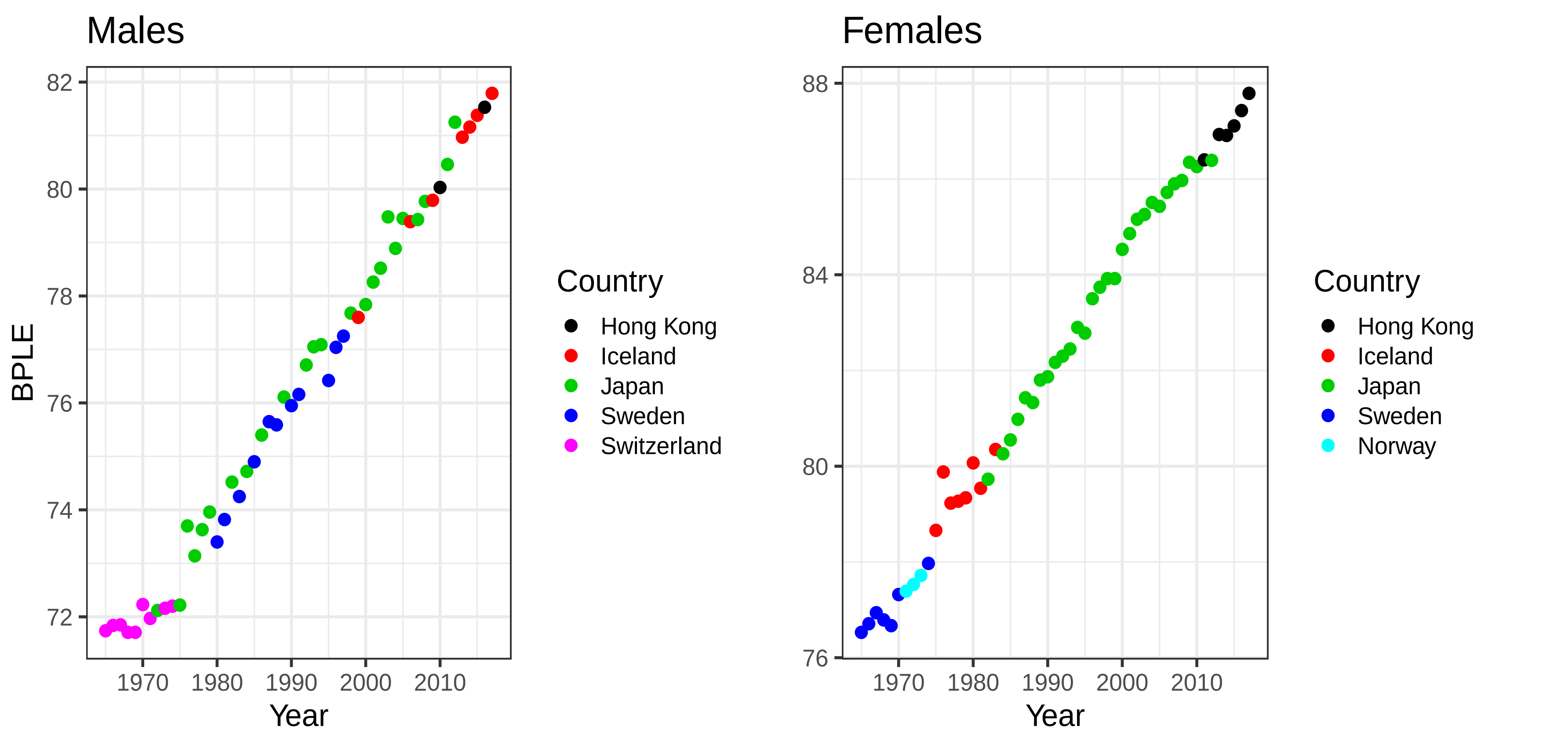

Figure: Best practice male and female life expectancies at birth, 1965 - 2017. Source: Human Mortality Database.

Parameter estimates

	Male	Female
$\hat{\alpha}$	0.30 (.13)	0.49 (.12)
$\hat{\mu}$	70.5 (.05)	75.9 (.06)
$\hat{\sigma}$	0.27 (.04)	0.24 (.04)

Table: Estimated parameter values of fitted Gumbel AR(1) model with estimated asymptotic standard errors.

Order checking

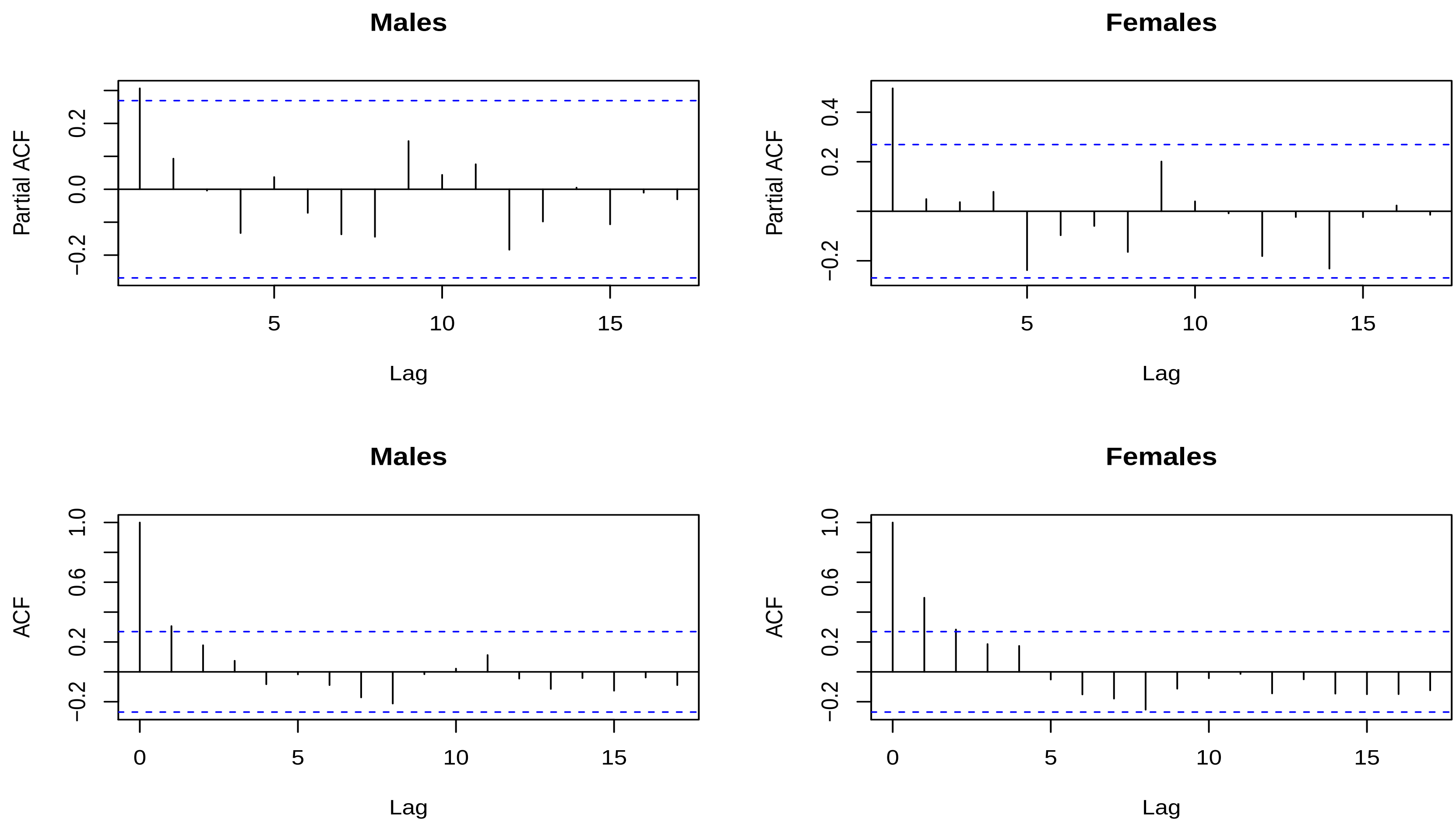

Figure: Order checking of fitted time. Top row: autocorrelation function. Bottom row: partial autocorrelation function.

Marginal distribution checking: Gumbel vs Gaussian I

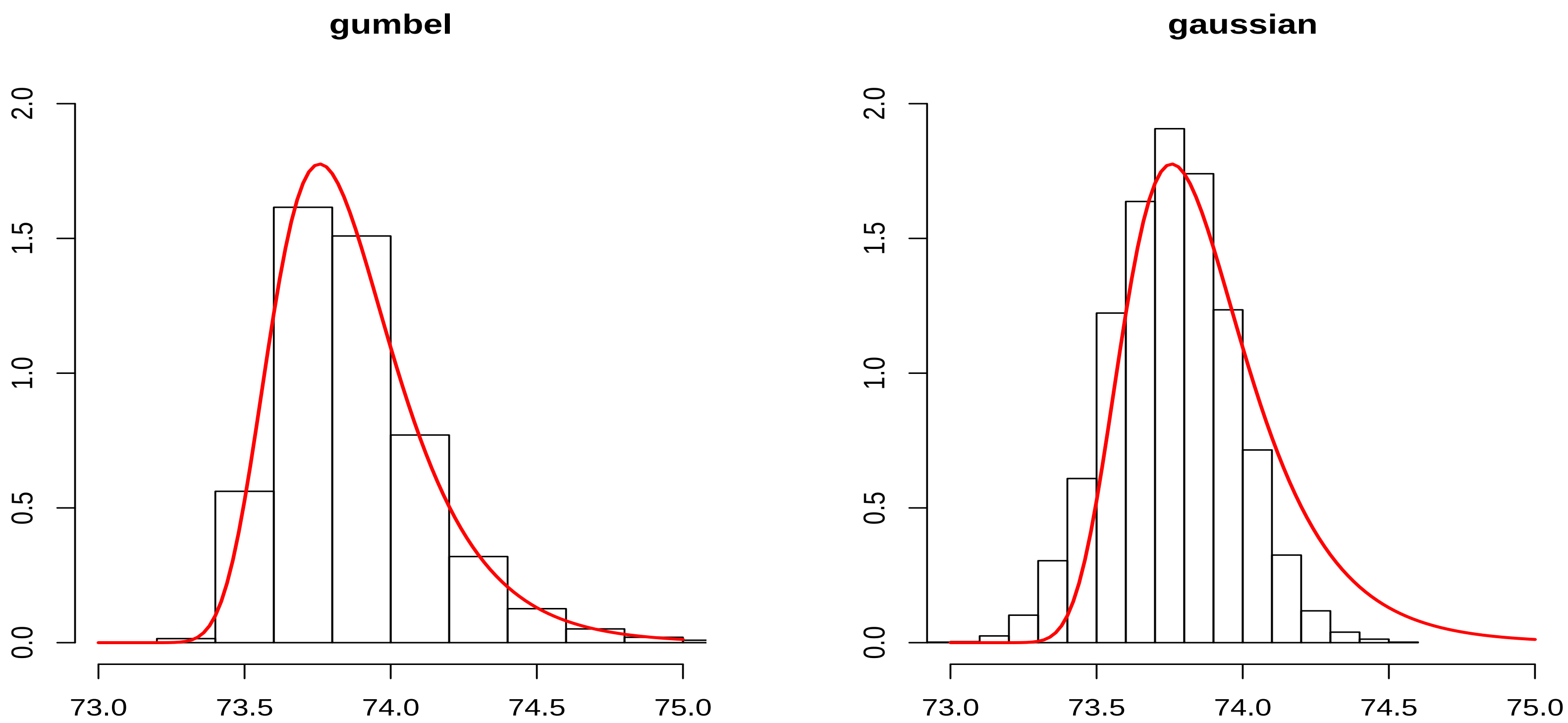

Figure: Conditional histograms of X_{t+1} for the Gumbel model and Gaussian model for males with superimposed extreme value distribution based the estimated parameters.

Marginal distribution checking: Gumbel vs Gaussian II

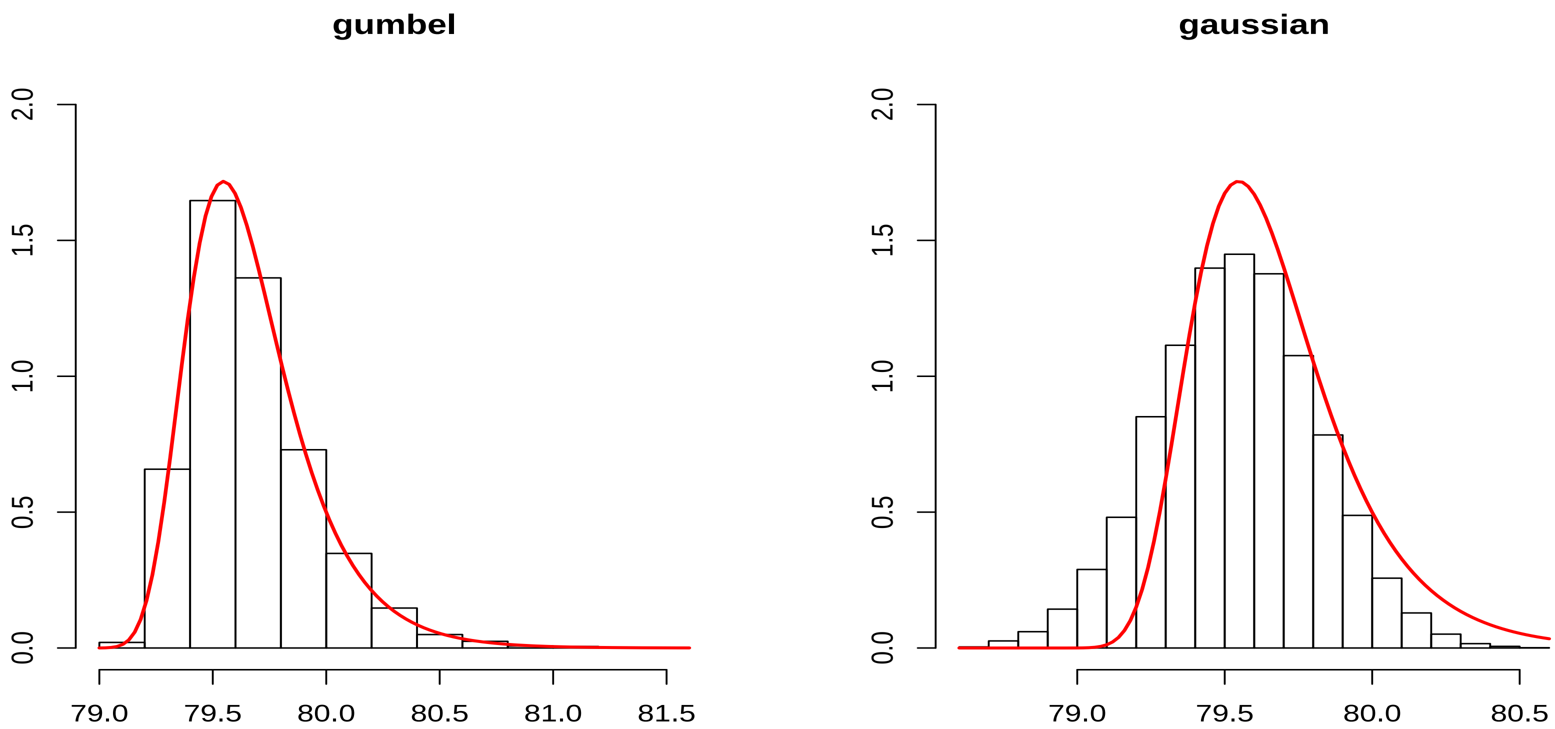

Figure: Conditional histograms of X_{t+1} for the Gumbel model and Gaussian model for females with superimposed extreme value distribution based on the estimated parameters.

Model diagnostics I

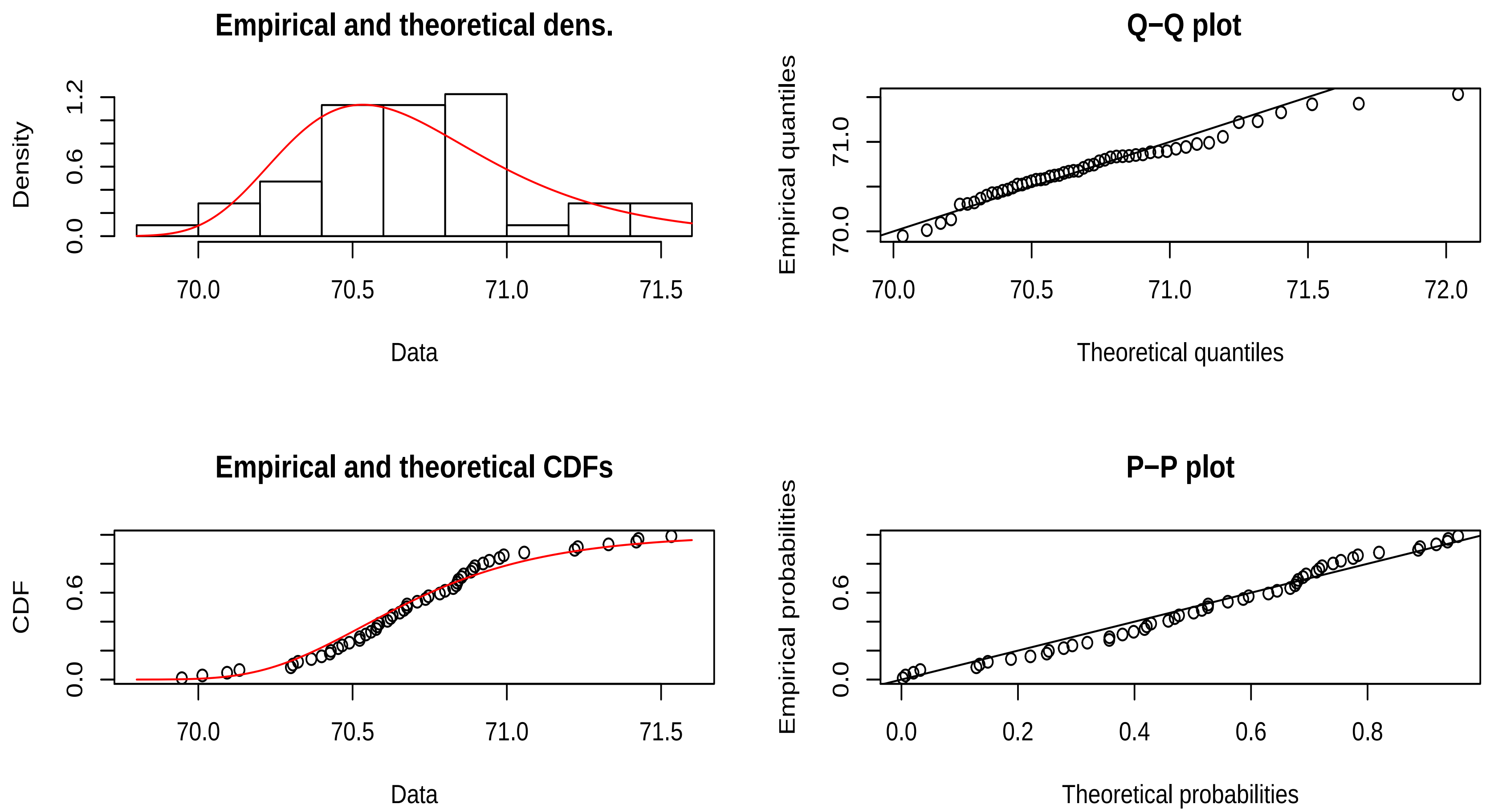

Figure: Gumbel fit diagnostic plots for males

Model diagnostics II

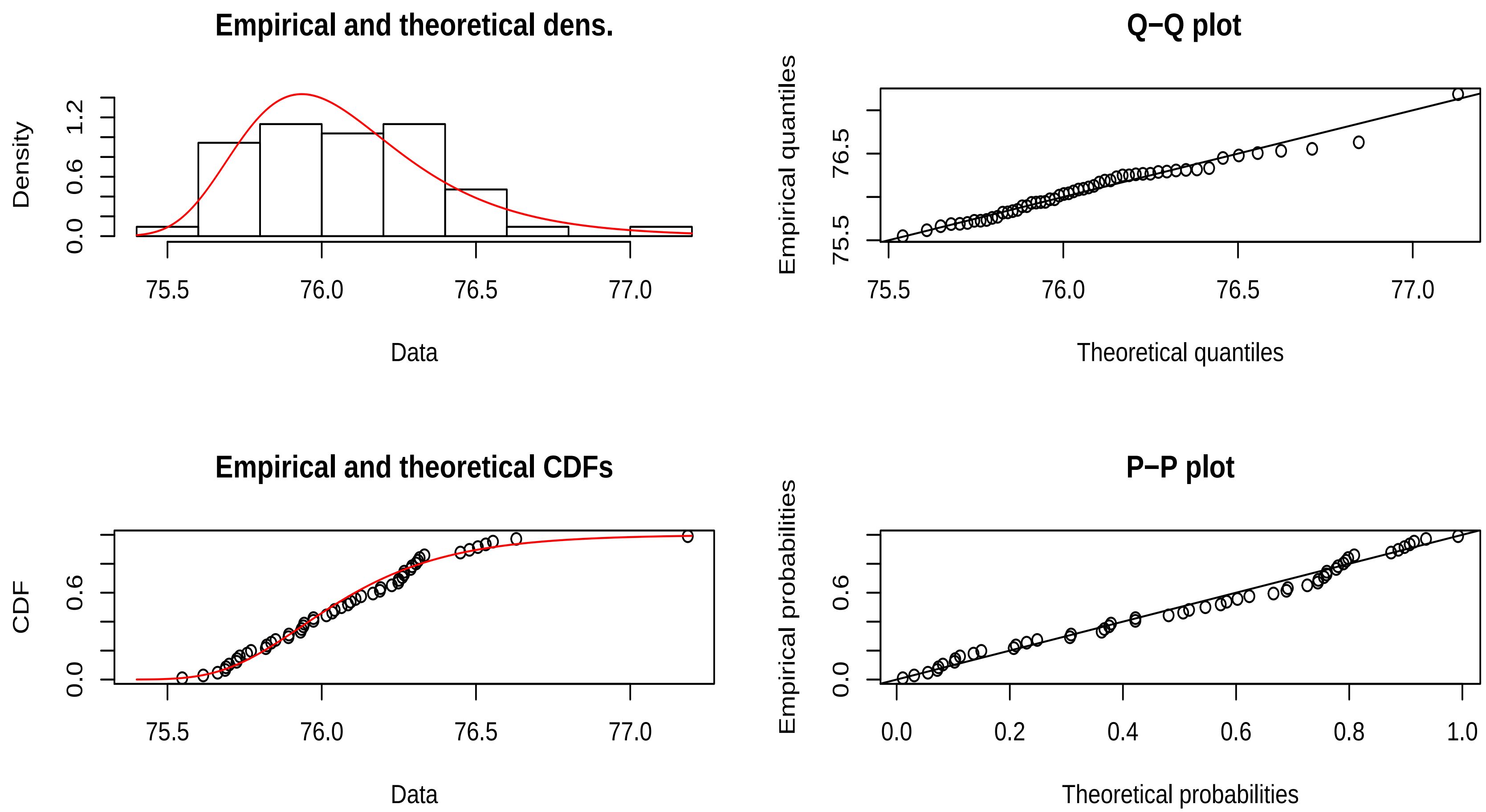

Figure: Gumbel fit diagnostic plots for females

Recap

- Have presented a method that fits extreme value time series data better than the conventional gaussian approach .

Recap

- Have presented a method that fits extreme value time series data better than the conventional gaussian approach .
- It is an alternative way to model short-term dependencies among maxima coming from light-tailed distributions.

Recap

- Have presented a method that fits extreme value time series data better than the conventional gaussian approach .
- It is an alternative way to model short-term dependencies among maxima coming from light-tailed distributions.
- We remain in the familiar ARIMA model frame work but with some added complexity to reflect the extreme value marginal distributions from the innovations.

Future work

- To make the model more flexible.

Future work

- To make the model more flexible.
  - Accommodate the general class of ARIMA time series, not just AR(1) models

Future work

- To make the model more flexible.
  - Accommodate the general class of ARIMA time series, not just AR(1) models
  - Fit the other extreme value distributions, not just the Gumbel

Future work

- To make the model more flexible.
  - Accommodate the general class of ARIMA time series, not just AR(1) models
  - Fit the other extreme value distributions, not just the Gumbel
- Explore forecasting using these types of models.

Future work

- To make the model more flexible.
  - Accommodate the general class of ARIMA time series, not just AR(1) models
  - Fit the other extreme value distributions, not just the Gumbel
- Explore forecasting using these types of models.
- Can this these models improve high age mortality modelling and forecasting?