Modeling Multi-population Longevity Risk with Mortality Dependence: A Lévy Subordinated Hierarchical Archimedean Copula Approach

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Outline

- Introduction
- Stochastic Mortality under a Multi-Country Setting
  - Lee-Carter Model
  - Copula Functions
- Applying LSHAC Model for Modeling Mortality Dependence across Countries
- Valuation Framework for Survivor Swaps
- Numerical Analysis of Survivor Swaps

Introduction

- Many researches focused on design & pricing of mortality linked securities

  - Survivor Swaps(Dawson, 2002; Blake, 2003; Lin and Cox, 2005; Dowd et al., 2006)

  - Longevity Bonds

Introduction

- Increase Hedge effectiveness
  - The underlying combined mortality index of a mortality linked security plays a important role
  - Cannot ignore basis risk in mortality linked securities

- Wang & Yang (2013) capture multi-country morality dynamics applying co-integration analysis

- Li and Hardy (2011) examine basis risk by four approaches

Introduction

- However, short-term catastrophe mortality shocks
  - Jump Effect

- Wang et al. (2011) demonstrate heavy-tailed distribution appear long-term mortality data

- Wang et al. (2014) find that time-varying Gumbel copula provide best goodness of fit for modeling mortality dependence.

Archimedean copulas

$$C(u_1, u_2, \ldots, u_d) = \psi[\psi^{-1}(u_1)+, \ldots, \psi^{-1}(u_d)],$$

is a d-dimensional Archimedean copula iff $\psi \in \mathcal{G}$ defined as

$$\left\{\psi : [0,\infty) \to [0,1] \,|\, \psi(0) = 1, \psi(\infty) = 0, (-1)^k \frac{d^k}{du^k}\psi(u) \geq 0, k \in \mathbb{N}\right\}$$

a set of *completely monotonic* (c.m.) functions.

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Drawbacks of AC

Exchangeable

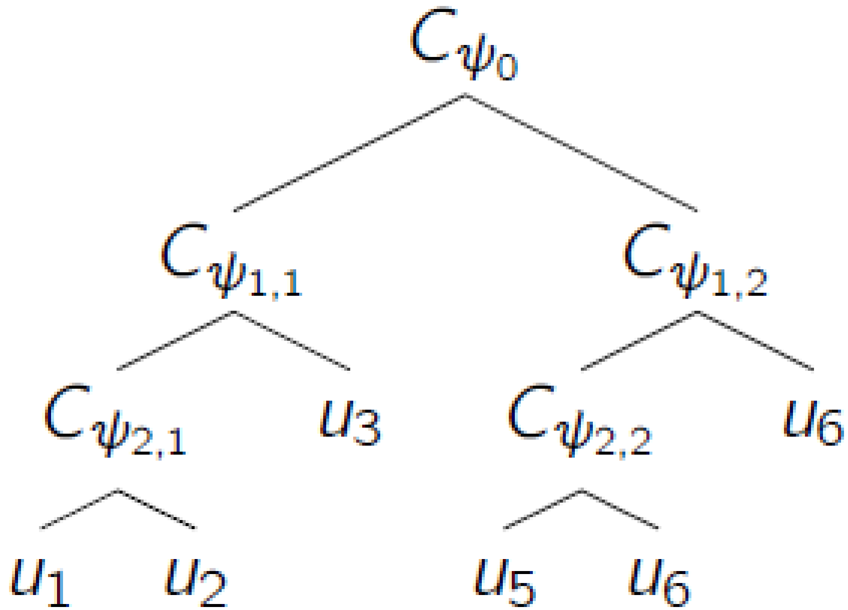
$$C(u_1,u_2)=\Pr ob(U_1\le u_1,U_2\le u_2)=\psi(\psi^{-1}(u_1)+\psi^{-1}(u_2))=C(u_2,u_1)$$

- invariant under permutation

=>not suitable for high dimensional asset model.

- Hierarchical Archimedean copula (HAC) has been proposed to "partially" overcome the disadvantage of Archimedean Copulas (AC).

3-level, six-dimensional HAC



"Copula of Copulas" Structure

$$C(u_1,u_2,u_3,u_4,u_5,u_6)$$

$$=C_{\psi_0}\left(\boxed{C_{\psi_{1,1}}\left(C_{\psi_{2,1}}(u_1,u_2),u_3\right)},C_{\psi_{1,2}}\left(C_{\psi_{2,2}}(u_4,u_5),u_6\right)\right)$$

3-level, six-dimensional HAC

$$C_{\psi_{1,1}}\left(C_{\psi_{2,1}}(u_1,u_2),u_3\right)$$

$$=\psi_{1,1}\left(\psi_{1,1}^{-1}\left(\psi_{2,1}\left(\psi_{2,1}^{-1}(u_1)+\psi_{2,1}^{-1}(u_2)\right)\right)+\psi_{1,1}^{-1}(u_3)\right)$$

$$=\psi_{1,1}\left(\psi_{1,1}^{-1}\circ\psi_{2,1}\left(\psi_{2,1}^{-1}(u_1)+\psi_{2,1}^{-1}(u_2)\right)+\psi_{1,1}^{-1}(u_3)\right)$$

$$\psi_0,\psi_{1,1},\psi_{1,2},\psi_{2,1},\psi_{2,2}\in\mathcal{G}, \qquad (3)$$

$$(\psi_0^{-1}\circ\psi_{1,i})' \quad \text{and} \quad (\psi_{1,i}^{-1}\circ\psi_{2,j})'\in\mathcal{G}, \qquad (4)$$

$$i=1,2, \quad j=1,2 \qquad (5)$$

Completely monotonic

- Condition (4), called **compatible condition,** yields a proper distribution function.

- However, it is hard to find suitable generators satisfying (4).

- Gumbel HAC satisfies Equation (4) and is easy to simulate. As a result, it is the most frequently used HAC in empirical study.

Levy Subordinated HAC

- Hering et al. (2010) first construct a two-level hierarchical model based on Lévy subordinators.

Theorem 2.1:

$(\psi_0^{-1} \circ \psi_1)'$ is c.m. $\Leftrightarrow \psi_0^{-1} \circ \psi_1 = \Psi$ is the Laplace exponent of a Lévy subordinator.

- Mai and Scherer (2012), introduced h-extendible copulas including a three-level LSHAC case.

Our Goals

- Dealing with the heavy-tailed distribution for long-term mortality data
- Introduce a LSHAC models with arbitrary levels to model mortality dependence (structure) across countries
- Propose a three-stage estimation procedure by additionally adding hierarchical clustering analysis to determine LSHAC structure.
- Empirically examine performance of LSHAC models for pricing survivor swaps

Morality model

- Multi-country mortality with dependence
  - Lee-Carter Model with jump diffusion (JD) & Generalized Hyperbolic (GH) innovation
- Lee Carter Model under multi-country

$$\ln m_{x,t}^{j} = a_x^{j} + b_x^{j} k_t^{j} + e_{x,t}^{j} \quad j = 1, \ldots, d$$

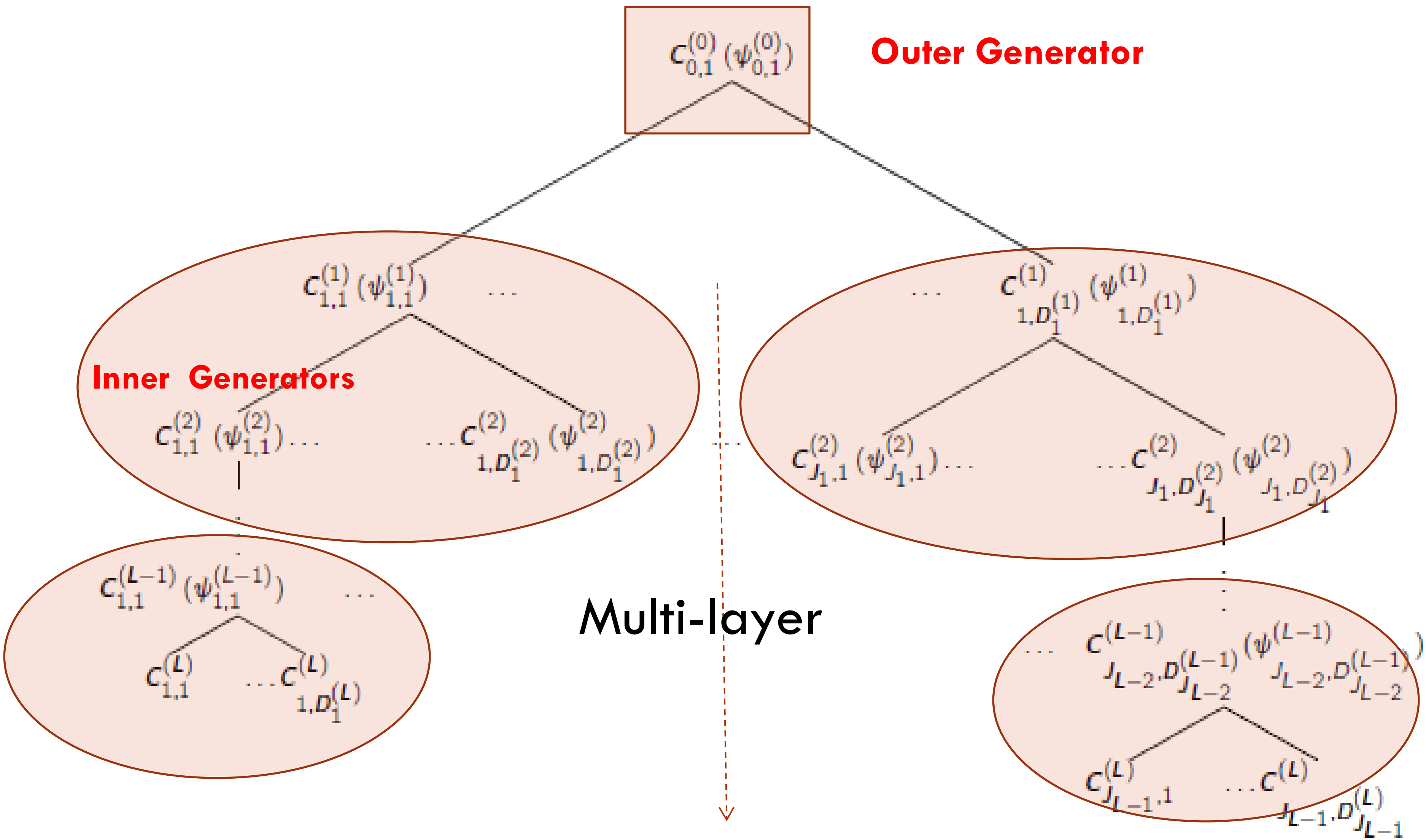
$$\Delta k_t^{j} = \gamma^{j} + \alpha_{t-1}^{j} k_{t-1}^{j} + \varepsilon_t^{j} + \beta_{t-1}^{j} \varepsilon_{t-1}^{j} \quad j = 1, \ldots, d$$

Generalized Hyperbolic (GH)

- Morality model
  - Lee-Carter with jump diffusion (JD) & Generalized Hyperbolic (GH) innovation

- GH distribution nests many well-known highly flexible distributions
  - ✓ Normal
  - ✓ Student-*t*
  - ✓ the hyperbolic distribution of Barndorff-Nielsen and Blæsild (1981)
  - ✓ the Variance Gamma (VG) distributions of Madan and Seneta (1987, 1990)
  - ✓ Normal Inverse Gaussian
  - ✓ the GH skewed *t* distribution of Prause (1999).

Figure : General Framework of LS-HAC Model



Generators of LS-HAC

=> For Calibration Purpose

Corollary 2.1. *For $1 \le l \le L$, in level l, the j_l-th copula generator in position s_{l-1}: $\psi^{(l)}_{s_{l-1},j_l}$, can be expressed as:*

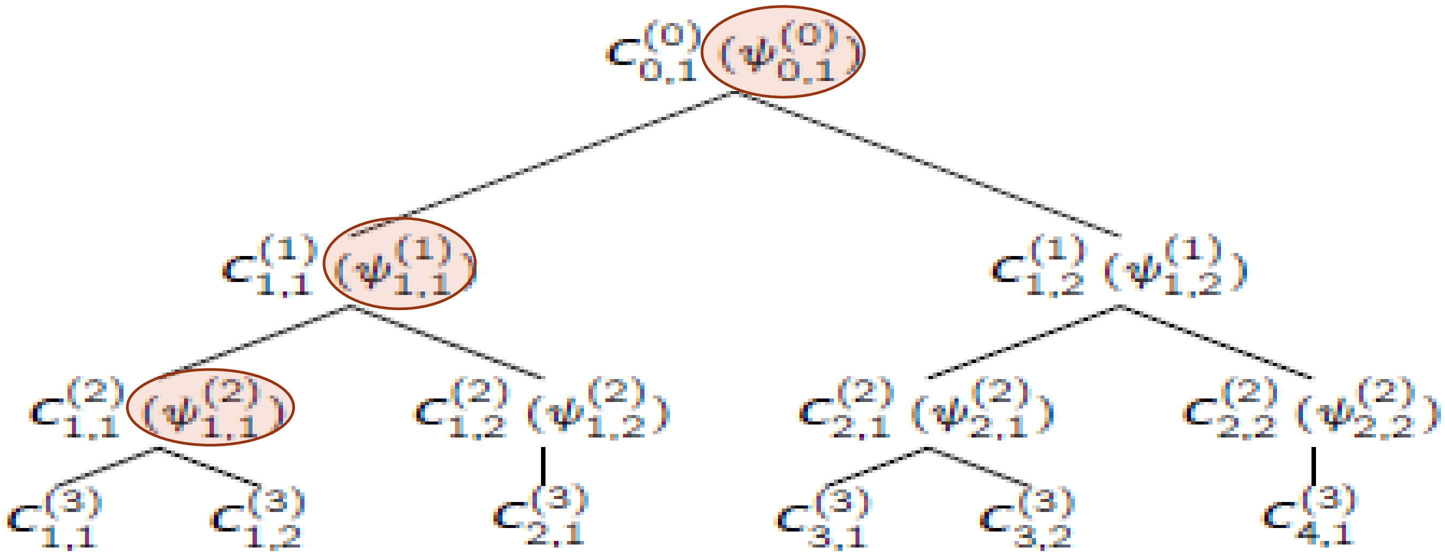
Outer generator **Lévy subordinators**

Inner generator →

$$\psi^{(l)}_{s_{l-1},j_l}(u) = \psi^{(0)}_{0,1} \bigodot_{i=1}^{l} \tilde{\Psi}^{(i)}_{s_{l-1},j_l}(u), \tag{10}$$

where $\bigodot_{k=1}^{n} f_k := f_1 \circ \ldots \circ f_n$, and $\psi^{(l)}_{s_{l-1},j_l}$ is c.m..

A three-layer Six-Dimensional LS-HAC



$$\psi_{1,1}^{(1)} = \psi_{0,1}^{(0)} \circ \widetilde{\Psi}_{1,1}^{(1)},$$

$$\psi_{1,1}^{(2)} = \psi_{1,1}^{(1)} \circ \widetilde{\Psi}_{1,1}^{(2)} = \psi_{0,1}^{(0)} \circ \widetilde{\Psi}_{1,1}^{(1)} \circ \widetilde{\Psi}_{1,1}^{(2)}$$

In the previous expressions, $\psi_{i,j}^{(l)}$ is AC generators and $\widetilde{\Psi}_{s,t}^{(l)}$ is Laplace exponent of Lévy subordinators.

Generators of LSHAC

In this paper, we consider

- AC generators
  - Clayton (CL)
  - Gumbel (GM)
  - Inverse Gaussian (IG)

Family	$\psi(u)$	$C(u_1,\ldots,u_d)$	Parameter
CL	$\psi_{CL}(u)=(1+u)^{-\frac{1}{\theta}}$	$\big(1+\sum_{i=1}^{d}(u_i^{-\theta}-1)\big)^{-\frac{1}{\theta}}$	$\theta>0$
GM	$\psi_{GM}(u)=\exp\big(-x^{\frac{1}{\theta}}\big)$	$\exp\big(-\sum_{i=1}^{d}(-\log u_i)\big)^{\frac{1}{\theta}}$	$\theta\geq 1$
IG	$\psi_{IG}(u)=\exp\big(\frac{1}{\theta}(1-\sqrt{1+2\theta^2 x})\big)$	$\exp\big(\frac{1}{\theta}(1-\sqrt{\sum_{i=1}^{d}(1-\theta\log u_i)^2+(1-d)})\big)$	$\theta>0$

Note: CL: Clayton family, GM: Gumbel family, IG: Inverse Gaussian family

Lévy Subordinators of LSHAC

- Lévy Subordinators
  - Gamma (G)
  - Gumbel or Stable (GM)
  - Inverse Gaussian (IG)

Subordinator	$\Psi(u)$	Parameters
G	$\Psi_G = a\log(1 + \frac{u}{b})$	$a > 0, b > 0$
GM	$\Psi_{GM} = u^a$	$0 < a < 1$
IG	$\Psi_{IG} = a\sqrt{2u + b^2} - ab$	$a > 0, b > 0$

Note: G: Gamma process, GM: Stable process, IG: the Inverse Gaussian process

LSHAC: Gumbel HAC as special case

- *Note that Gumbel HAC is a special case of LSHAC when outer generator and subordinator are both Gumbel type.*

Corollary 2.2. *In the structure of an Gumbel-HAC, the l-th level copula generator $\psi^{(l)}(u)$ can be expressed as $(l \geq 1)$:*

$$\psi^{(l)}(u) = \psi^{(0)} \bigodot_{k=1}^{l-1} \tilde{\Psi}^{(k)}(u) = \exp\Big(- u^{\prod_{k=1}^{l-1} \frac{1}{\theta_k}} \Big).$$

From the parameterization in Table 1 and Table 2, $\psi^{(0)}$ represents a GM generator with $\theta = \theta_0, \theta_0 > 1$, where $\tilde{\Psi}^{(k)}$ denotes the k-th GM subordinator with $a = 1/\theta_k, \theta_k > 1$.

LSHAC Estimation

$$L(\boldsymbol{\theta}) = \sum_{j=1}^{d} \sum_{t=1}^{T} \log f_j \left(x_{t,j} | \mathcal{F}_{t-1}; \boldsymbol{\theta}^M \right) + \sum_{t=1}^{T} \log \left(c \left(F_1(x_{t,1}), \ldots, F_d(x_{t,d}) | \mathcal{F}_{t-1}; \boldsymbol{\theta}^M, \boldsymbol{\theta}^C, \mathcal{S} \right) \right)$$

- **Parameter sets**

$$\boldsymbol{\theta} = (\boldsymbol{\theta}^M, \boldsymbol{\theta}^C, \mathcal{S})$$

Marginal Parameters

Copula Parameters

Hierarchica
l Structure

LSHAC Estimation

- **Three-Stage Estimation**
  - Stage 1: Estimating Margins

$$\hat{\Theta}_j^M = \arg\max_{\Theta_j^M} \sum_{t=1}^{T} \ln f_j\left(x_{t,j} \mid \mathfrak{I}_{t-1}; \Theta_j^M\right), \quad \hat{\Theta}_M = \left\{\hat{\Theta}_1^M, \ldots, \hat{\Theta}_d^M\right\}$$

  - Stage 2: Determining Hierarchical Structure

    => ***Hierarchical Clustering Analysis***

  - Stage 3: Estimating LSHAC parameters

$$\hat{\Theta}_C = \arg\max_{\Theta_C} \sum_{t=1}^{T} \log\left(c\left(F_1(x_{t,1}), \ldots, F_N(x_{t,d}) \middle| \mathfrak{I}_{t-1}; \hat{\Theta}_M; \hat{S}; \Theta_C\right)\right)$$

Step1: Estimate the parameters of LC Model

- After obtaining the marginal parameters, we use **probability transform** based on the best goodness-of-fit residual distribution to obtain the inputs of copulas.

$$u_{t,j} = F_{Z_j}(z_{t,j}), \quad j = 1, \ldots, d$$

Stage 2: Hierarchical Clustering Analysis

- Despite its importance, few papers discuss the grouping method of the HAC or LSHAC models.

- Okhrin et. al. (2013) introduce an estimation procedure to determine optimal structure of HAC by evaluating all possible structures.
  - Sequentially determine the copula parameters for each level.

- We employ **hierarchical clustering method** to determine the structure of LSHAC.

Dissimilarity (Proximity) Matrix

Using six dimension data as an example:

$$
\begin{array}{c|cccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
\hline
 & d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & x_1 \\
 & d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} & d_{2,5} & d_{2,6} & x_2 \\
 & d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4} & d_{3,5} & d_{3,6} & x_3 \\
 & d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} & d_{4,5} & d_{4,6} & x_4 \\
 & d_{5,1} & d_{5,2} & d_{5,3} & d_{5,4} & d_{5,5} & d_{5,6} & x_5 \\
 & d_{6,1} & d_{6,2} & d_{6,3} & d_{6,4} & d_{6,5} & d_{6,6} & x_6
\end{array}
$$

- $d_{i,j}$ represents dissimilarity of i and j .
- $d_{i,i}=0$ and $d_{i,j} = d_{j,i}$ (symmetric)
- Large value of $d_{i,j}$ represents high degree of dissimilarity

Euclidean V.S. Dependence Metric :

- Euclidean metric: $d_{i,j}^{\text{Euclid}} = \sqrt{\sum_k (x_{k,i} - x_{k,j})^2}$

- Dependence metric $d_{i,j}^{\text{Corr}} = 1 - \Re_{i,j}$

where $\Re_{i,j}$ is an **association measure**, such as *Spearman's rho* and *Kendall's tau*.

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A Trade-off measure

- In literature, a sequential procedure according to Kendall's tau is used to determine hierarchical structure of classical Gumbel HAC.

- It ignores ***dissimilarity*** from "Euclidean Distance" .

- In this paper, we propose a new measure by dividing Euclidean metric by dependence metric

$$\frac{d_{i,j}^{\text{Euclid}}}{1+\tau_{i,j}}$$

Empirical Results

LSHAC Model for Modeling Mortality Dependence across Countries

Goodness of Fits-Kt

Denmark_kt

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-167.21	170.21	174.11	5	5	5
T	-158.55	162.55	167.74	4	1	1
JD	-157.53	163.53	171.31	1	4	4
VG	-158.40	163.40	169.89	3	3	3
NIG	-158.07	163.07	169.56	2	2	2

Finland_kt

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-168.22	171.22	175.11	5	5	5
T	-165.07	169.07	174.26	4	4	3
JD	-158.67	164.67	172.45	1	1	2
VG	-159.68	164.68	171.17	2	2	1
NIG	-162.79	167.79	174.28	3	3	4

Goodness of Fits-Kt

France_kt

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-176.74	179.74	183.63	5	5	5
T	-170.18	174.18	179.37	4	3	1
JD	-168.43	174.43	182.21	3	4	4
VG	-168.02	173.02	179.51	2	2	3
NIG	-167.95	172.95	179.43	1	1	2

Iceland_kt

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-197.98	201.98	207.17	5	1	*1*
T	-197.20	202.20	208.68	2	2	2
JD	-196.61	203.61	212.69	1	5	5
VG	-197.31	203.31	211.10	4	4	4
NIG	-197.21	203.21	210.99	3	3	3

Goodness of Fits-Kt

Italy_kt

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-182.41	185.41	189.30	5	5	2
T	-179.85	183.85	189.04	4	1	1
JD	-179.10	185.10	192.88	1	4	5
VG	-179.46	184.46	190.94	2	2	3
NIG	-179.52	184.52	191.01	3	3	4

Netherlands_kt

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-191.48	194.48	198.37	5	5	5
T	-178.93	182.93	188.12	4	3	3
JD	-177.26	183.26	191.05	3	4	4
VG	-176.03	181.03	187.52	1	1	1
NIG	-176.39	181.39	187.87	2	2	2

Goodness of Fits-Kt

Norway_kt

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-133.92	136.92	140.81	5	1	*1*
T	-133.91	137.91	143.10	4	2	2
JD	-133.76	139.76	147.55	1	5	5
VG	-133.78	138.78	145.26	2	3	3
NIG	-133.79	138.79	145.28	3	4	4

Sweden_kt

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-147.57	150.57	154.46	4	4	4
T	-171.60	175.60	180.79	5	5	5
JD	-138.17	144.17	151.95	2	3	3
VG	-137.27	142.27	148.76	1	1	1
NIG	-138.23	143.23	149.72	3	2	2

Goodness of Fits-Kt

Switzerland_kt

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-137.02	140.02	143.91	5	1	*1*
T	-137.02	141.02	146.21	4	4	2
JD	-135.62	141.62	149.41	1	5	5
VG	-135.81	140.81	147.30	2	2	3
NIG	-135.98	140.98	147.47	3	3	4

UK_England_and_Wales_kt

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-160.60	163.60	167.49	5	5	2
T	-158.00	162.00	167.19	4	2	1
JD	-156.34	162.34	170.13	1	4	5
VG	-157.01	162.01	168.49	3	3	4
NIG	-156.98	161.98	168.47	2	1	3

Goodness of Fits-Copula Model

Copula Model				Parameters	LLF	AIC	BIC
Elliptical Copulas							
Gaussian				45	220.17	-175.17	-116.78
Student's T				46	242.64	-196.64	**-136.95**
AC (Gumbel)				1	99.11	-98.11	-96.81
AC (Clayton)				1	87.17	-86.17	-84.87
AC (IG)				1	109.12	-108.12	-106.82
LSHAC							
Level0	Level 1	Level 2	Level 3				
GM	GM	GM	GM	8	156.04	-148.04	-137.66
GM	GM	IG	GM	11	165.39	-154.39	-140.12
GM	IG	GM	GM	10	160.76	-150.76	-137.78
CL	GM	GM	GM	8	158.41	-150.41	-140.03
CL	GM	IG	GM	11	165.61	-154.61	-140.34
IG	**GM**	**GM**	**GM**	**8**	**164.87**	**-156.87**	**-146.49**
IG	GM	GM	IG	10	161.66	-151.66	-138.69
IG	GM	IG	GM	11	171.42	-160.42	-146.15
IG	GM	IG	IG	13	167.82	-154.82	-137.95
IG	IG	GM	GM	10	163.45	-153.45	-140.47

Valuation for Survivor Swaps

- Survivor swaps
  - Agreement the periodic exchange of a series of preset payment for mortality-dependent payments
  - Value of notional principal multiplied by a fixed rate receives the value of the notional principal multiplied by the unexpected shock in survival probability (floating-ratepayer)
  - Transfer the unexpected shock in mortality improvement

Valuation Framework for Survivor Swaps

- Risk neutral of survivor probability
  - *N*-year survival probability

$$ {}_n p_{x_0}^{j} = p_{x_0}^{j}(t_0, n) = \prod_{j=0}^{n-1} p_{x_0}^{j}(t_0 + j) = \exp\left(-A_n^j\right) \quad j = 1, \ldots, 2N $$

  - The ${}_t p_{x_0}^{j}$ under P measure

$$ F_t^j(x) = Porb_P\left({}_t p_{x_0}^{j} \leq x\right) $$

Valuation Framework for Survivor Swaps

- Risk neutral of survivor probability
  - ${}_t p_{x_0}^j$ P measure to Wang measure Q

$$\tilde{F}_t^j(x) = \Phi\left(\Phi^{-1}\left(F_t^j(x)\right) + \lambda_W\right)$$

  - Denuit et al. (2007) shows

$$E_Q\left[{}_t p_{x_0}^j\right] = \int_0^1 \left(1 - \Phi\left(\Phi^{-1}\left(F_t^j(y)\right) + \lambda_W\right)\right) dy$$

Valuation Framework for Survivor Swaps

- Fair value of Survivor Index Swap

$$LS(t_0)=\sum_{t=1}^{T}\left(B(0,t)\sum_{j=1}^{2N}Q_j(0)L_jE_Q\left[{}_t p_{x_0}^j\right]\right)-(1+\pi)\sum_{t=1}^{T}\left(B(0,t)\sum_{j=1}^{2N}Q_j(0)L_jH_t^j\right)$$

- Fair swap premium

$$\pi=\frac{\sum_{t=1}^{T}\left(B(0,t)\sum_{j=1}^{2N}Q_j(0)L_jE_Q\left[{}_t p_{x_0}^j\right]\right)}{\sum_{t=1}^{T}\left(B(0,t)\sum_{j=1}^{2N}Q_j(0)L_jH_t^j\right)}-1$$

Valuation Framework for Survivor Swaps

- From the standpoint of pay-floating, the unexpected loss

$$L(t) = \left((S_{x_0}(0,t) - (1+\pi)H(t) \right)$$

- The present value of total unexpected loss

$$PVL = \sum_{t=1}^{T} B(0,t) L(t)$$

Fair Swap Rate(b.p.)

Yield Rates	Model	λ = -0.1	λ = -0.15	λ = -0.2
Original	**Time-varying LSHAC Levy**	**137.36**	**150.39**	**163.33**
	No dependence+Normal	**135.47**	**148.51**	**161.47**
up 2%	**Time-varying LSHAC Levy**	**107.61**	**118.69**	**129.69**
	No dependence+Normal	**106.08**	**117.15**	**128.17**
up 4%	**Time-varying LSHAC Levy**	**83.23**	**92.64**	**101.97**
	No dependence+Normal	**81.99**	**91.39**	**100.74**

Analysis of Longevity Risk

Time to Maturity	Model	VaR95	VaR99	CTE95	CTE99
15	Time-varying LSHAC Levy	0.0716	0.1036	0.0912	0.1198
	No dependence+Normal	0.0578	0.0824	0.0728	0.0937
20	Time-varying LSHAC Levy	0.1244	0.1790	0.1584	0.2078
	No dependence+Normal	0.1059	0.1508	0.1333	0.1718
25	Time-varying LSHAC Levy	0.1764	0.2552	0.2253	0.2971
	No dependence+Normal	0.1538	0.2212	0.1952	0.2533

Contribution

- Employs LSHAC introduce mortality dependence

- Propose a three-stage estimation procedure by additionally adding hierarchical clustering analysis to determine LSHAC structure

- Value at Risk (VaR) & Conditional Tail Expectation (CTE) of survivor swaps

Thank You