Hedging of longevity-linked instruments

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Introduction

- Longevity risk affects both the public sector and private insurance companies
- Longevity-linked instruments, whose cash-flows are linked to mortality developments in a specified population, have been proposed to manage this risk
- Demand for longevity-linked instruments exists, but the supply side is less clear
- Supply of longevity-linked instruments might increase if their cash-flows could be hedged by trading in more liquid assets (cf. options markets)

Introduction

- Since the cash-flows of longevity-linked instruments seem to have much less to do with existing financial markets than simple stock options, longevity-linked cash flows cannot be perfectly hedged
- In other words, the markets are incomplete and the seller of such an instrument always retains some risk
- However, any connection between mortality and financial markets may help sellers of longevity bonds reduce the risk by approximate hedging of the bonds payouts

Overview

1. Modelling the law of a multivariate stochastic process consisting of mortality and asset returns, with particular emphasis on
   - Long-term development of mortality
   - Connections between mortality and asset returns
2. Utilizing this connection in the hedging of a mortality-linked cash flow using numerical optimization

Mortality model

- Let $E_{x,t}$ be the size of population aged $[x, x+1)$ (cohort) at the beginning of year t
- Denote by $D_{x,t}$ the number of deaths of people in this cohort
- Objective: model the values of $E_{x,t}$ over time $t = 0, 1, 2, \dots$ for a given set $X \subset \mathbb{N}$ of ages
- Assume the conditional distribution of $E_{x+1,t+1} = E_{x,t} - D_{x,t}$ given $E_{x,t}$ is binomial:

$$E_{x+1,t+1} \sim \mathrm{Bin}(E_{x,t}, p_{x,t})$$

where $p_{x,t}$ is the probability that an individual aged x and alive at the beginning of year t is still alive at the end of that year

Mortality model

- We reduce the dimensionality of $(p_{x,t})_{x \in X}$ by modelling the logistic probabilities by

$$\operatorname{logit} p_{x,t} := \ln\left(\frac{p_{x,t}}{1-p_{x,t}}\right) = \sum_{i=1}^{n} v_t^i \phi_i(x),$$

  where $\phi_i(x)$ are user-defined basis functions across cohorts, and v_t^i stochastic risk factors that vary over time
- In other words, $p_{x,t} = p_{v(t)}(x)$, where $v(t) = (v^1(t), \dots, v^n(t))$, and $p_v : X \to (0,1)$ is the parametric function defined for each $v \in \mathbb{R}^n$ by

$$p_v(x) = \frac{\exp\left(\sum_{i=1}^{n} v_i \phi_i(x)\right)}{1 + \exp(\sum_{i=1}^{n} v_i \phi_i(x))}$$

- Modelling the logit transforms instead of $p_{x,t}$ directly guarantees that $p_{x,t} \in (0,1)$.
- Historical values of v_t are constructed by maximum likelihood estimation, log-likelihood function is strictly concave

Mortality model

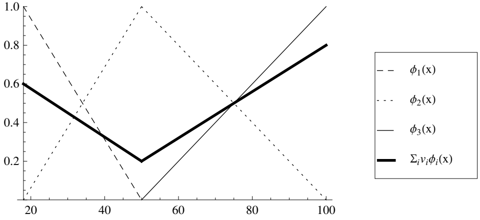
- For adult (18-100 years) mortality, we consider the model

$$\operatorname{logit} p_{x,t} = v_t^1 \phi_1(x) + v_t^2 \phi_2(x) + v_t^3 \phi_3(x),$$

- Basis functions are piecewise linear:

$$\phi_1(x) = \begin{cases} 1 - \frac{x-18}{32} & \text{for } x \leq 50 \\ 0 & \text{for } x > 50, \end{cases} \quad \phi_2(x) = \begin{cases} \frac{1}{32}(x-18) & \text{for } x \leq 50 \\ 2 - \frac{x}{50} & \text{for } x > 50, \end{cases}$$

$$\phi_3(x) = \begin{cases} 0 & \text{for } x \leq 50 \\ \frac{x}{50} - 1 & \text{for } x > 50. \end{cases}$$



- Interpretation: values of v_t^i points on the fitted $\operatorname{logit} p_{x,t}$ curve

Mortality model

- We consider female mortality dynamics of six large countries with relatively high life expectancies: Australia, Canada, France, Japan, UK, US
- Data consists of annual values of cohort sizes $E_{x,t}$ and numbers of deaths $D_{x,t}$ for each country, covering years 1950–2007 (Source: Human mortality database)

Statistical analysis: v^1

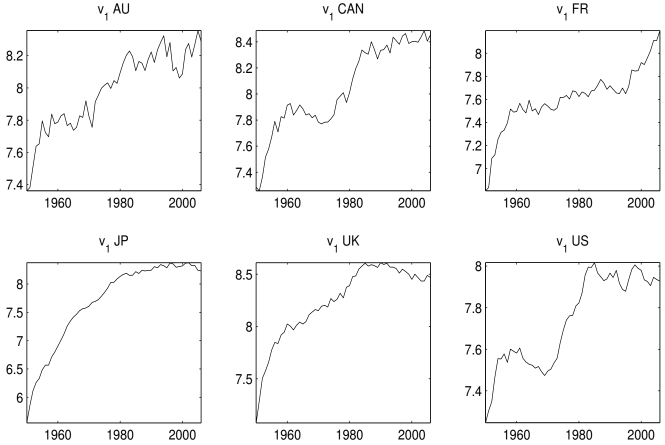


Figure: Historical values for risk factor v^1, females. Note the different scales.

Statistical analysis: v^1

- How long will mortality keep improving?
- Mortality risk factors often modelled as random walk with a negative drift, i.e. mortality will decline indefinitely
- Some experts predict that the decline in mortality will continue for the foreseeable future, while others suggest that human life expectancy might even decrease.
- In some sample countries, the historical values of v_t^1 seem to be stabilizing, a phenomenon already suggested by Wicksell
- In order to analyse this phenomenon, we fit the following regression into historical data:

$$\Delta v_t^1 = b + a v_{t-1}^1 + \varepsilon_t = -a\left(-\frac{b}{a} - v_{t-1}^1\right) + \varepsilon_t$$

- Statistical analysis provided support for this equation

Statistical analysis: v^2

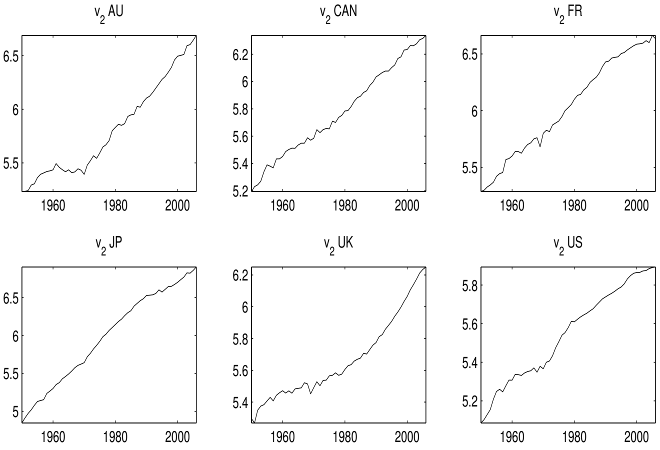


Figure: Historical values for risk factor v^2, females. Note the different scales.

Statistical analysis: v^2

- Rapid reduction in coronary disease amongst the middle-aged shows in risk factor v^2 reflecting the survival probability of 50-year-olds
- Can improvements in treatment and possible reductions in smoking outweigh the detrimental effects of obesity and other lifestyle-related factors?
- Historical values are not levelling out, but stabilizing behaviour similar to v^1 may be a future possibility for v^2
- To quantify the rate of improvement for v^2, we fit the regression

$$\Delta v_t^2 = b + \varepsilon_t,$$

Statistical analysis: v^3 and GDP

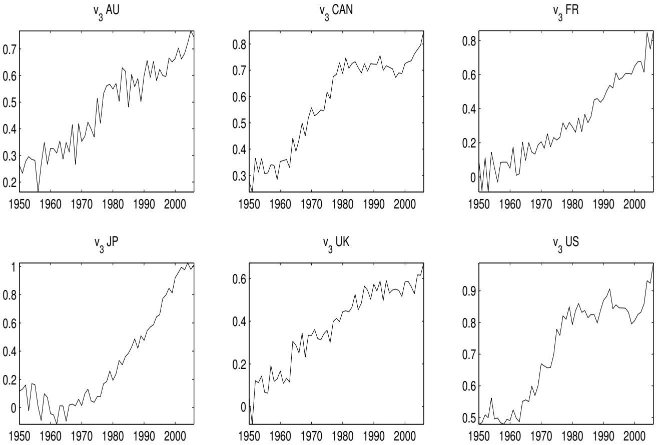


Figure: Historical values for risk factor v^3, females. Note the different scales.

Statistical analysis: v^3 and GDP

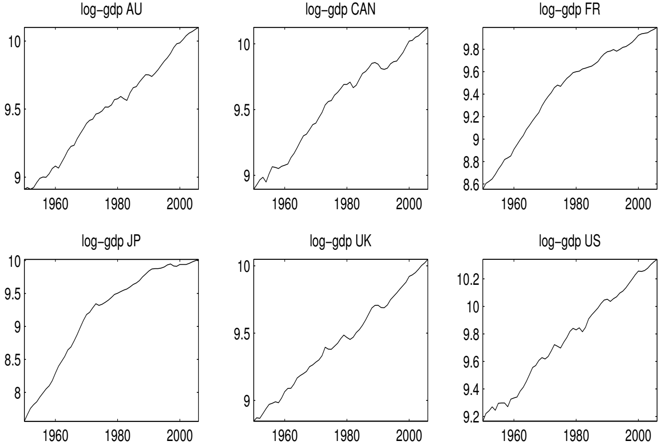


Figure: Historical values for logarithm of per capita GDP. Note the different scales.

Statistical analysis: v^3 and GDP

- Risk factor v^3 describes old-age mortality, with which the cash flows of mortality-linked instruments are often connected
- Long-term dependence between GDP and mortality has been observed in earlier works
- Similarities in the general shape of their plots support the observation that v^3 and log-per capita GDP may move together in the long run
- We analyse the dependence of v^3 on GDP with the regression

$$\Delta v_t^3 = b + a_1 v_{t-1}^3 + a_2 g_{t-1} + \varepsilon_t,$$

  where g_t is the log-per capita GDP
- Statistical analysis provided support for this equation

Statistical analysis: GDP and financial markets

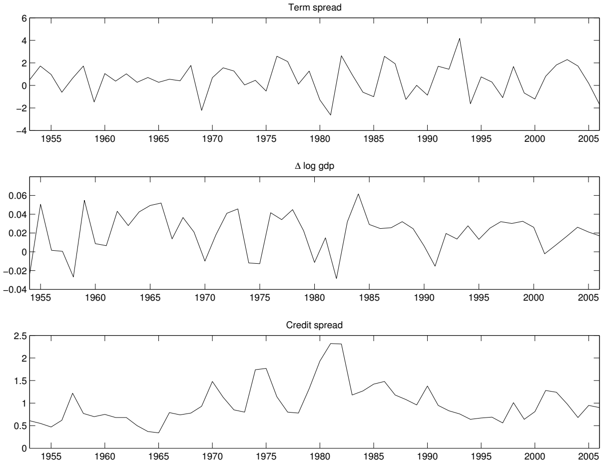


Figure: US term spread, differences of US GDP (logarithm) and credit spread.

Statistical analysis: GDP and financial markets

- GDP is linked with financial markets
  - Yield curve/term spread
  - Credit spread
  - Stock prices
- Combining earlier results on the relation of GDP to term spread s_t^T and credit spread s_t^C, we analyse these connections with the regression

$$\Delta g_t = b + a_1 s_{t-1}^T + a_2 s_{t-1}^C + \varepsilon_t,$$

- In the above, s_t^T is the difference between US treasury log-rates of 5 and 1 years, and s_t^C is the log-difference between Moody's corporate bond log-yields with ratings BAA and AAA (Source: FRED: Federal reserve economic data)
- Statistical analysis provided support for this equation

Modelling the risk factors

- We propose to model mortality and financial markets with the following system of equations:

$$
\begin{aligned}
\Delta v_t^1 &= a^{11} v_{t-1}^1 + b^1 + \varepsilon_t^1 \\
\Delta v_t^2 &= b^2 + \varepsilon_t^2 \\
\Delta v_t^3 &= a^{33} v_{t-1}^3 + a^{34} g_{t-1} + b^3 + \varepsilon_t^3 \\
\Delta g_t &= a^{45} s_{t-1}^T + a^{46} s_{t-1}^C + b^4 + \varepsilon_t^4 \\
\Delta s_t^T &= a^{55} s_{t-1}^T + b_5 + \varepsilon_t^5 \\
\Delta s_t^C &= a^{66} s_{t-1}^C + b_6 + \varepsilon_t^6
\end{aligned}
$$

- This is a linear stochastic difference equation

$$\Delta x_t = A x_{t-1} + b + \varepsilon_t$$

for $x = [v_t^1, v_t^2, v_t^3, g_t, s_t^T, s_t^C]$

- The random vector ε_t follows a Gaussian distribution

Modelling the risk factors

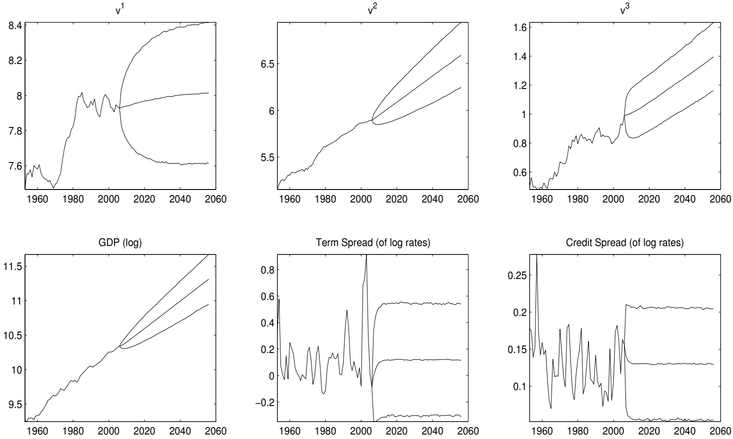


Figure: Medians and 95% confidence intervals for MC simulations (N=10000), US females.

Modelling the risk factors

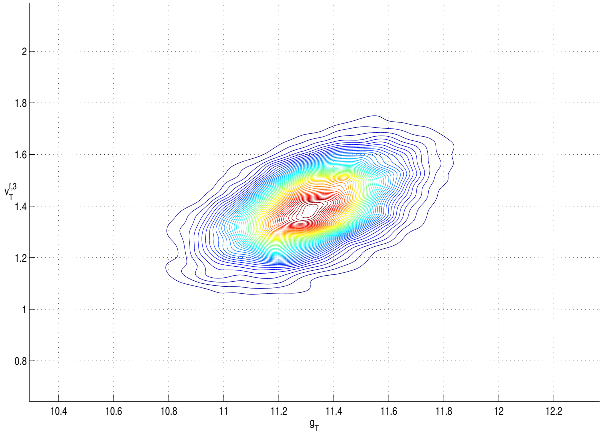


Figure: Kernel density estimate for the joint probability density function of v^3_{2056} and g_{2056}

Hedging longevity-linked instruments

- We observed connections between old-age longevity and interest rate spreads (via GDP)
- When term spread is high, old-age longevity improves faster
- As a result, investment returns on long-maturity bond tend to be relatively high when old-age longevity improves fast
- When credit spread is high, old-age longevity improves more slowly
- Corporate bond returns tend to be high when old-age mortality improves slowly

Hedging longevity-linked instruments

- Traditional actuarial view on valuation of cash flows: discounting expected cash flows
- Alternative approach: determining the minimal capital required to cover the claims at an acceptable level of risk

The value of liabilities depends essentially on

- **Probability distribution**: description of future development of claims and investment returns, both involving significant uncertainties
- **Risk preferences**: the level of risk at which assets should cover liabilities
- **Hedging strategy**: investment strategy for the given capital

Hedging longevity-linked instruments

- Denote $S_{x,t} \in [0,1]$ the proportion of survivors in cohort $x \in X \subset \mathbb{N}$ at times $t = 0, 1, \dots, T$ (*Survivor index*)
- (Annuity) longevity bond: coupon payments proportional to $S_{x+t,t}$ at times $t = 1, 2 \dots, T$ in exchange for an initial payment V_0
- Survivor swap: exchange of a cash flow proportional to $S_{x+t,t}$ for a fixed cash flow proportional to $\bar{S}_t$ at times $t = 1, 2, \dots, T$
- Insurance claims/pension fund management: cash flow c_t that depends on $S_{x+t,t}$ as well as consumer price and pension indices
- Other variants (e.g. zero-coupon bond with terminal payment $S_{x,T}$)

Hedging longevity-linked instruments

We use the following market model:

- A finite set J of liquid assets
- Return of asset j over period $[t-1, t]$ is denoted by $R_{t,j}$,
- The amount of wealth invested in asset j at time is t $h_{t,j}$
- $S_t = (S_{x,t})_{x \in X}$, $R_t = (R_{t,j})_{j \in J}$ and $h_t = (h_{t,j})_{j \in J}$ are the vectors of survivor indices, returns and investments, respectively
- $(S_t)_{t=0}^T$, $(R_t)_{t=0}^T$, $(h_t)_{t=0}^T$ are adapted stochastic processes on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t=1}^T, \mathbb{P})$
- $\mathbb{P}$ reflects the investor's views on the future development of the stochastic factors

Hedging longevity-linked instruments

Least initial capital required for covering a longevity bond is

$$\begin{aligned} &\text{minimize} && \sum_{j\in J} h_{0,j} \quad \text{over } h \in \mathcal{N} \\ &\text{subject to} && \sum_{j\in J} h_{t,j} \le \sum_{j\in J} R_{t,j} h_{t-1,j} - S_{x+t,t} \quad t = 1,\dots,T \\ & && h_{t,j} \in \mathcal{D}_t,\ t = 1,\dots,T \\ & && \rho(\sum_{j\in J} h_{T,j}) \le 0. \end{aligned} \tag{1}$$

- $\mathcal{N}$ denotes the $\mathbb{R}^J$ -valued investment strategies, adapted to the filtration $(\mathcal{F})_{t=1}^T$
- $\mathcal{D}_t(\omega) \in \mathbb{R}^J$ is the set of feasible investment strategies at time t and state ω
- $\rho : L^0(\Omega, \mathcal{F}_T, \mathbb{P}) \to \mathbb{R}$ is a convex risk measure; measuring risk/loss/disutility

Hedging longevity-linked instruments

The asset-liability management problem of an agent with initial capital $\bar{w}$ and liabilities $\bar{c} = \bar{c}_t$ is

$$
\begin{aligned}
\text{minimize} \quad & \rho(\sum_{j\in J} h_{T,j}) \quad \text{over } h \in \mathcal{N} \\
\text{subject to} \quad & \sum_{j\in J} h_{0,j} \le \bar{w} \\
& \sum_{j\in J} h_{t,j} \le \sum_{j\in J} R_{t,j} h_{t-1,j} - \bar{c}_t \quad t = 1,\dots,T \\
& h_{t,j} \in \mathcal{D}_t, t = 1,\dots,T
\end{aligned} \tag{2}
$$

We denote the optimum value of this problem by $\varphi(\bar{w}, \bar{c})$. The minimum price for which the agent is willing to sell a longevity bond is $\pi(\bar{w}, \bar{c}; S) := \inf\{w | \varphi(\bar{w} + w, \bar{c} + S) \le \varphi(\bar{w}, \bar{c})\}$ (*indifference pricing*)

Hedging longevity-linked instruments

- We choose ρ to be the entropic risk measure whose acceptance set is $\mathcal{A} = \{X \in L^0 \mid \rho(X) \leq 0)\} = \{X \in L^0 \mid Eu(X) \geq u(0)\}$, where u is the exponential utility function
- We apply a numerical optimization procedure for constructing hedging strategies from a given set of parametric dynamic investment strategies (basis strategies)
- Computations are based on approximating the probability distribution of the risk factors by a sample of scenarios
- Numerical methods of convex optimization are applied to find an optimal diversification amongst basis strategies
- Effectiveness of the hedge depends on how well returns on the basis strategies conform to the liabilities

Hedging longevity-linked instruments

Well-known basis strategies

- Buy and hold (B&H): initial asset allocation is held over time
- Fixed proportions (FP): asset allocation is rebalanced in each period so that the weight of an asset in the total portfolio stays fixed
- Target date fund (TDF): proportion invested in risky assets decreases in time

Hedging longevity-linked instruments

Liability-dependent basis strategies

- Spread strategies: proportion invested in long-maturity bonds/corporate bonds at time t depends on term spread s_t^T / credit spread s_t^C
- Survival index strategies: proportions invested in long-maturity/corporate bonds depend on survival index S_t
- Survival index-wealth strategies: proportions of long-maturity bonds depends on on S_t/w_t
- GDP strategies: proportion of long-maturity bonds depends on on log-GDP g_t

Hedging longevity-linked instruments

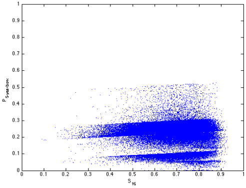
- What is the least initial capital required for a longevity bond? How do the observed connections between mortality and asset returns affect the capital requirement and hedging strategy?
- The set J of assets consists of
  1. US Treasury bills (1-year)
  2. US Treasury bonds (5-year)
  3. US corporate bonds (5-year)
  4. US equity (S&P total return index)
- Liabilities depend on the survival index $S_{x,t}$ of US females aged 65 at the beginning of year 2008
- Parameter of utility function $\rho = 1.1$
- T=30, N=500000

Table: Weights in optimal hedging strategy, well-known strategies.

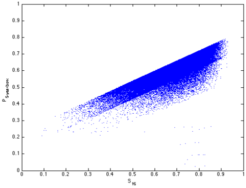
Capital requirement $w_0 = 14.2$	
Weight	Type
0.108	FP
0.048	FP
0.006	FP
0.839	B&H Treasury bonds

Table: Weights in optimal hedging strategy, liability-dependent and well-known strategies.

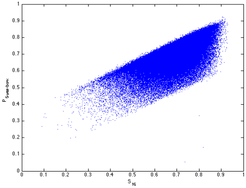
Capital requirement $w_0 = 14.0$	
Weight (%)	Type
0.246	Term spread (long vs. short treasury bond)
0.016	Term spread (long treasury bond vs. equity)
0.575	Survival index (long treasury bond vs. equity)
0.060	Survival index/wealth ratio (long treasury bond vs. equity))
0.036	Term spread (long vs. short treasury bond))
0.005	Credit spread (corporate bond vs. equity)
0.062	Credit spread (corporate bond vs. equity)



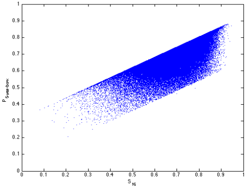
(a) $\rho = 0.1$



(b) $\rho = 0.3$



(c) $\rho = 0.7$



(d) $\rho = 1.1$

Figure: Proportion of wealth invested in long-maturity bonds as a function of the survival index for different risk levels, $t = 15$.

Conclusions

- We present a simple stochastic model of mortality and financial markets, describing their long-term development and mutual connections
- Using this model, we apply numerical methods to construct good hedging strategies for longevity-linked cash flows, given user's risk preferences
- Adjusting investment strategies to liabilities may allow for reduced capital requirements and hedging costs for mortality-linked instruments.

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