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Optimal investment for a retirement plan with deferred annuities

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ABSTRACT

We construct an optimal investment portfolio model with deferred annuities for an individual investor saving in a retirement plan. The objective function consists of power utility in terms of consumption of all secured retirement income from the deferred annuity purchases, as well as bequest from remaining wealth invested in equity, bond, and cash funds. The asset universe is governed by a vector autoregressive model incorporating the Nelson–Siegel term structure and equity returns. We use multi-stage stochastic programming to solve the optimization problem numerically. Deferred annuity purchases are made continuously over the working lifetime of the investor, increasing particularly in the years before retirement. The investment strategy hedges price changes in deferred annuities, and bond holding and deferred annuity purchases increase when interest rates are high. Optimal investment and deferred annuity choices depend on realized and expected values of state variables. The optimal strategy is also compared with typical retirement plan strategies such as glide paths. Our results provide support for deferred annuities as a major source of retirement income.

1. Introduction

Constant-payment deferred annuities are a life-contingent income product that provides fixed lifetime income beginning at a pre-specified date, in most cases at retirement or at advanced age. Compared to nominal coupon-paying bonds, deferred annuities have the benefit that policyholders' mortality risk is pooled because they are sold by life insurers to large populations of policyholders. Premiums are determined by four factors: the deferment period, discount rates, mortality, and expenses. For retirees, deferred annuities reduce longevity risk in the distant future (Scott, 2008). For relatively young individuals who will work for a long period of time, they can be used to secure retirement income well before retirement (Horneff et al., 2010; Maurer et al., 2013). More broadly, social security and defined benefit pensions provide a type of deferred annuity, where governments and companies guarantee the pension benefits. Pension and annuity markets have started to afford individual investors greater freedom to use deferred annuities, but little research has been undertaken on the optimal purchase of such annuities before retirement. The U.S. Treasury allows target date funds to include deferred annuities among their assets in 401(k) plans (U.S. Treasury Department, 2014). Many retirement funds, however, apply a typical investment strategy, so-called glide paths, which reduce the proportion of equity and increase the proportion of bond in the investment portfolio as retirement approaches.

Immediate annuities, which pay a life-contingent benefit from the time the annuity contract is purchased, are more common than deferred annuities, and are the subject of considerably more research. Most studies focus on annuity purchases on or after retirement, i.e. in the decumulation phase. Koijen et al. (2011) present the optimal full-annuitization portfolio of nominal, inflation-indexed, and variable annuities at retirement, and the associated hedging strategy before retirement. In the hedging strategy, the optimal composition of nominal and inflation-linked bonds depends on an annuity strategy that will be used at retirement. Milevsky and Young (2007) derive analytically the optimal age for annuitization when wealth can only be fully annuitized just once. They also show that it may be optimal to buy immediate annuities gradually over time when this is permissible. Horneff et al. (2008, 2009) likewise find that buying an increasing amount of immediate annuities over the lifecycle enhances the individual's welfare.

Deferred annuities, by contrast to the above, have been less scrutinized. Huang et al. (2017) provide conditions for the optimal purchase of deferred annuities when the interest rate process is mean-reverting, but do not study the portfolio optimization problem. This is done in the simple environment of one risky asset and one risk-free asset by Horneff et al. (2010) and Maurer et al. (2013). They assume that the individual has an uncertain labour income stream, invests his disposable income, and is also allowed to buy deferred annuities at any time before retirement. The deferred annuities begin to pay lifetime benefits at a fixed retirement date. These studies show that deferred annuities have a crucial role in increasing welfare gains. The optimal strategy is

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to start to purchase deferred annuities early (from age 40) and to keep purchasing them over time up to about 80% of the final portfolio at retirement (Horneff et al., 2010; Maurer et al., 2013). The conclusion that deferred annuities should be purchased early is also supported by Konicz and Mulvey (2013) in a simpler setting where optimization is over annuity income assumed to be fully consumed during retirement.

This article provides further support for the early and regular purchase of deferred annuities prior to retirement. The framework that we consider is saving and investment in the accumulation phase within a retirement plan, which may be sponsored by an employer or provided by a life insurer. Individuals in the plan seek financial advice from pension planners and actuaries who focus on income rather than terminal wealth, as enjoined by Merton (2014). To this end, we solve numerically for the optimal allocation to stock, bond and cash, and for the optimal purchase of deferred annuities before retirement.

Our main contributions are as follows. First, we use a richer model of the financial market than Horneff et al. (2009, 2010), Maurer et al. (2013), Milevsky and Young (2007) and Konicz and Mulvey (2013), who consider only a constant risk-free rate and a risky asset with a single geometric Brownian motion. Koijen et al. (2011) argue that time-varying opportunities in equity return and a full term structure of interest rates are critical to long-term asset allocation. Diebold and Li (2006) find that the Nelson–Siegel model provides accurate long-term forecasts of yield curves. We use the framework of Konicz et al. (2016) with a vector autoregressive model combined with the Nelson–Siegel model which allows for time-varying equity risk premia and a full term structure of interest rates. This means that we quantify optimal asset allocation and deferred annuity purchase in a manner that can be used by actuaries and other risk managers to advise pension savers in the real world.

Second, our model allows for realistic frictions and constraints. Koijen et al. (2011) obtain large short positions in bonds together with borrowing. This is unlikely to be achievable for individual investors, especially within their retirement plans. Short-selling is precluded in our model. Furthermore, transaction costs on stocks and bonds and charges on annuity purchases are included.

Third, our richer financial model comes at the cost of many more state variables than numerical dynamic programming can contend with. In possibly the most advanced numerical optimization model among the above, Koijen et al. (2011) employ simulation-enhanced dynamic programming but fail to incorporate realistic portfolio constraints and frictional costs. Instead, we use a key technique in operations research, multi-stage stochastic programming (MSP), as employed by Konicz et al. (2016). However, we consider the accumulation (pre-retirement) phase whereas they consider the decumulation (post-retirement) phase. We generate scenarios with a close fit to market data and we implement a step procedure which disallows arbitrage. Using the MSP method, Duarte et al. (2017) incorporate Solvency II regulatory constraints into an asset–liability management problem for a risk-averse pension plan operator. For other applications of MSP to individual retirement planning, see Consigli et al. (2012), Dempster and Medova (2011)

Fourth, we compute the welfare gains from having not just immediate annuities, but also deferred annuities. The welfare improvements from practical, deterministic strategies such as glide paths are also calculated, to aid actuaries and other financial planners.

We caution that our model, like all models, carries certain simplifying assumptions. We do not endogenize the optimal timing for annuitization. Annuities pay out only at a single (retirement) date, as in Maurer et al. (2013) and Horneff et al. (2010). We assume full annuitization at retirement, an empirically rare observation (the "annuity puzzle"), although not far off the 80% of total wealth that should be annuitized using deferred annuities according to the two abovementioned studies. We do not optimize over the full financial cycle, but only over the accumulation phase. Investment takes place within a retirement plan only; individuals are assumed to fully consume all retirement income and their expected utility from this consumption is maximized, similar to Koijen et al. (2011), Horneff et al. (2008) and Konicz and Mulvey (2013). We hope to address these simplifications in subsequent iterations of our model.

Our model shows that, as age increases, it is optimal to purchase more deferred annuities and reduce the portfolio allocation to risky financial assets in the retirement plan. The dynamic strategy is based on changes in state variables including asset returns, yield curves, and deferred annuity prices. The bond fund, in particular, has a critical role of hedging price changes in deferred annuities. An optimal strategy in some scenarios is to use the equity and bond funds to maximize investment returns and to wait for the best time to buy deferred annuities.

This article is organized as follows. We describe the portfolio optimization problem and define the price dynamics of available assets in Section 2. In Section 3, the time-varying and predictable market movements of equity returns and yield curves are defined using a vector autoregressive model with the Nelson–Siegel model. We solve the model by applying a multi-stage stochastic programming approach. Descriptions for the model formulation are given in Section 4. We investigate the numerical results of optimal investment and deferred annuity choice in Section 5. We also describe how the optimal solution changes as risk aversion, time preference, bequest motives, mortality, and contribution amounts vary. In Section 6, we discuss how the optimal strategy changes if deferred annuities are unavailable. Finally, we compare our optimal strategies with typical retirement-plan strategies, such as constant-mix, glide-path, and "100−age" strategies.

2. Investment for a retirement plan

2.1. Investment problem

We consider an individual who starts to contribute to a personal retirement plan at time 0, when he is δ years old, and who retires at time T. During the retirement planning period $[0, T)$, he contributes a fixed proportion ϕ of his labour income L_t (at time t) every year to the retirement plan. He can hold equity, bond and cash in the retirement fund, which is worth W_t at time t. Withdrawals from the fund are not allowed,[1] except to buy deferred annuities (DAs). DAs will pay out, if the individual is alive at retirement time T, every year from time T until he dies. Annuities are irreversible contracts, so the individual can buy, but not sell, annuities. Every unit of annuity that is bought pays out a secured income of £1 annually in retirement. If the individual dies before retirement, then the annuities do not pay out, but the wealth in his fund is bequeathed to his heirs. If he survives till retirement, then his accumulated wealth in the plan is fully annuitized by purchasing an immediate annuity.

During the retirement planning period $[0, T)$, the individual allocates his portfolio and purchases deferred annuities in order to maximize the expected utility of consumption of all income in retirement and of bequest before retirement. Note that we consider investment for a retirement plan only, and therefore we assume that the individual consumes fully his income from the retirement plan and that he can separate utility over consumption of retirement income from the utility over pre-retirement consumption.

1 Because of the tax privileges afforded to pension plans to encourage saving, withdrawals are in general restricted and attract a hefty tax liability.

The individual investor has a power utility function $u(t,x) = e^{-\rho t}x^{1-\gamma}/(1-\gamma)$ in terms of cash flow or wealth x at time t. The individual therefore has a constant relative risk aversion (CRRA) coefficient γ. As γ tends to one, the utility function becomes logarithmic. We assume that all retirement income is used for consumption, so the utility function is defined in terms of income from annuities. The time preference coefficient $0 \le \rho \le 1$ reflects the individual's preference for early income over late income. The utility function is also defined with regard to the bequest amount W_t before retirement. A bequest parameter κ captures the importance of bequest relative to retirement income.

In the following, standard actuarial notation is used for survival and death probabilities. The probability that a person aged δ years survives until age $\delta+t$ is denoted by ${}_tp_\delta$. The probability that a $(\delta+t)$-year old person dies over the following Δt years is denoted by ${}_{\Delta t}q_{\delta+t}$, abbreviated to $q_{\delta+t}$ when $\Delta t = 1$. For practical purposes, we also assume that a person cannot live beyond age ω, which is the maximum age in an actuarial life table, so the individual investor dies before or at time $\tau = \omega - \delta$, since he is aged δ at time 0.

Let the total number of units of deferred annuities purchased by time t be $X_{A,t}$, where the subscript A stands for annuities. Since each unit of annuity provides £1 annually in retirement, the secured retirement income by time t is $X_{A,t}$. If the annuity price is $S_{A,t}$, then the investor pays $S_{A,t}(X_{A,t} - X_{A,t-1})$ to buy annuities at time $t \in [0,T]$ (with $X_{A,-1} = 0$).

We also assume that the investor buys and sells units or shares in an equity fund, a cash fund and a bond fund, denoted by E, C and B respectively. Let $X_{E,t}$ be the number of units of the equity fund held in the retirement plan at time t, and $S_{E,t}$ be the price of equity units at time t. A corresponding notation holds for the cash and bond funds. At time t, the individual decides how much to hold in equity, cash and bond, and how many annuity units to buy. The decision variable for the individual at time $t \in [0,T)$ is therefore $X_t = [X_{E,t},\, X_{C,t},\, X_{B,t},\, X_{A,t}]'$.

The objective function, budget constraints, and variable constraints for the retirement planning problem are given by the equations below:

$$\max_{\{X_t,t\in[0,T)\}} \quad \mathbb{E}_0\left[\sum_{t\in[T,\tau)} {}_tp_\delta\; u(t,X_{A,T}) + \sum_{t\in[0,T)} {}_tp_\delta \cdot q_{\delta+t}\cdot \kappa^\gamma u(t+1,W_{t+1})\right], \tag{1a}$$

$$\text{s.t.} \quad W_{t+1} = W_t + \phi\cdot L_t - S_{A,t}(X_{A,t}-X_{A,t-1}) + \sum_{i\in\{E,C,B\}} (S_{i,t+1}-S_{i,t})X_{i,t} \quad \text{for } t\in[0,T), \tag{1b}$$

$$X_{i,t} \ge 0 \quad \text{for } i\in\{E,C,B,A\} \text{ and } t\in[0,T), \tag{1c}$$

$$X_{A,t+1} \ge X_{A,t} \quad \text{for } t\in[0,T), \tag{1d}$$

$$X_{i,T} = 0 \quad \text{for } i\in\{E,C,B\}, \tag{1e}$$

$$X_{A,T} = X_{A,T-1} + W_T/S_{A,T}, \tag{1f}$$

$$W_t \ge 0 \quad \text{for } t\in[0,T], \tag{1g}$$

$$W_0 = w_0 \quad \text{w.p. } 1. \tag{1h}$$

In Eq. (1a), the decision variables over which expected utility is maximized are the portfolio and annuity purchase decisions over the planning horizon $[0,T)$. Since the retirement income which is secured through deferred annuity purchase by retirement time T is $X_{A,T}$, the utility of secured income at time $t \in [T,\tau)$ during retirement is $u(t,X_{A,T}) = e^{-\rho t}(X_{A,T})^{1-\gamma}/(1-\gamma)$. If the individual dies during period $[t,t+1)$, then wealth W_{t+1} constitutes a bequest, so the utility of bequest is then $\kappa^\gamma u(t,W_{t+1}) = \kappa^\gamma e^{-\rho t}(W_{t+1})^{1-\gamma}/(1-\gamma)$.

The budget constraint, in Eq. (1b), shows the dynamics of wealth W_t in the retirement plan. Wealth is increased by a contribution which is a fixed proportion ϕ of labour income L_t, as well as by increases in the price of equity, cash and bond funds, $(S_{i,t+1} - S_{i,t})$ for $i = \{E,C,B\}$. Wealth in the retirement plan is reduced if there is a withdrawal of $S_{A,t}(X_{A,t} - X_{A,t-1})$ to buy deferred annuities at time t.

The constraint in Eq. (1c) means that short sales are not allowed, while the constraint in Eq. (1d) means that annuities can be bought but not sold. The terminal conditions in Eqs. (1e) and (1f) assert that, at retirement time T, equity, bond and cash holdings are sold off, and all wealth in the retirement plan is annuitized. Eq. (1g) ensures that wealth remains non-negative. The initial condition in Eq. (1h) states that the investor has a known initial wealth at time 0.

2.2. Available assets

The individual can rebalance his portfolio and buy deferred annuities at regular intervals of length Δt years. There are $N \in \mathbb{N}$ such regular intervals in the retirement planning period $[0,T)$, i.e. $T = N\,\Delta t$. Defining $R_{i,t}$ as the accumulated log-return of asset $i \in \{E,C,B\}$ from time $t-\Delta t$ to t, the price $S_{i,t}$ of asset i evolves according to the following:

$$S_{i,t} = S_{i,t-\Delta t}\cdot \exp(R_{i,t}) \qquad \text{for } i\in\{E,C,B\}, \tag{2}$$

where $S_{i,0} = 1$ without loss of generality.

The gross return of the long-term bond fund with a maturity of M years over a holding period of length Δt from time $t-\Delta t$ to t is approximated by

$$R_{i,t} = M\cdot y(\beta_{t-\Delta t},M,\lambda) - (M-\Delta t)\cdot y(\beta_t, M-\Delta t,\lambda), \tag{3}$$

where the asset i is a zero-coupon bond with maturity M at time $t-\Delta t$. The term $y(\beta_t,M,\lambda)$ denotes the M-year spot rate at time t, determined by the Nelson–Siegel term structure model, with parameters β_t and λ to be specified shortly. Accordingly, the dynamics of the bond fund price is obtained by substituting $R_{i,t}$ from Eq. (3) into Eq. (2).

The gross return of the cash fund is defined simply by changing bond maturity M in Eq. (3) to Δt. The cash fund return from time $t-\Delta t$ to t, then, is given by

$$R_{i,t} = \Delta t\cdot y(\beta_{t-\Delta t},\Delta t,\lambda). \tag{4}$$

Of course, this cash fund return at time t does not depend on the current spot rate $y(\beta_t,\Delta t,\lambda)$ at time t, but on the past spot rate $y(\beta_{t-\Delta t},\Delta t,\lambda)$.

For a policyholder aged $\delta+t$ at time t, the fair actuarial price of a deferred annuity contract paying £1 of annual retirement income for lifetime from his retirement at time T is

$$S_{A,t} = \sum_{s=T-t}^{\tau-t} {}_sp_{\delta+t}\cdot \exp\left(-s\cdot y(\beta_t,s,\lambda)\right). \tag{5}$$

We assume static pricing mortality rates here, and we also ignore loading factors (expenses).

3. Financial markets

In order to incorporate interest rate uncertainty into the deferred annuity price, a stochastic term structure model is required. The Nelson–Siegel model is chosen as the term structure model along with a vector autoregressive (VAR) model for stochastic equity and bond returns. Ferstl and Weissensteiner (2011) combine the Nelson–Siegel formulation proposed by Boender et al. (2008) with the VAR model. This allows our model to incorporate asset return predictabilities and to use a seamless yield curve for pricing not only the cash and bond funds, but also annuities.

3.1. Term structure of interest rates

The entire yield curve is determined by a fitted Nelson–Siegel model with three time-varying parameters: $\beta_{1,t}$ (level), $\beta_{2,t}$ (slope), and $\beta_{3,t}$ (curvature)[2]. This parsimonious model is known to avoid over-fitting and to return better out-of-sample predictions than affine term structure models (see Diebold and Li, 2006). The Nelson–Siegel model for the s-year spot rate at time t is as follows:

$$y(\beta_t, s, \lambda) \;=\; \beta_{1,t} + (\beta_{2,t} + \beta_{3,t})\left(\frac{1-e^{-\lambda s}}{\lambda s}\right) - \beta_{3,t}e^{-\lambda s}, \tag{6}$$

where the scaling parameter λ is a constant. Here, $\beta_t = [\beta_{1,t},\ \beta_{2,t},\ \beta_{3,t}]'$.

3.2. Time-varying investment opportunities

To incorporate predictabilities of asset returns and the three parameters in the Nelson–Siegel model, we use a VAR(1) model (for details, see Barberis, 2000; Campbell et al., 2003). In particular, a combined approach of the interest rate model and equity returns, as in Ferstl and Weissensteiner (2011), Pedersen et al. (2013) and Konicz et al. (2016), is applied. Our VAR model is given by

$$z_t = \Phi_0 + \Phi_1 z_{t-1} + \upsilon_t, \tag{7}$$

where $z_t = [r_t, \beta_{1,t}, \beta_{2,t}, \beta_{3,t}]'$. Here, r_t is monthly log-returns on the equity fund. The accumulated return $R_{E,t}$ on the equity fund, defined near Eq. (2) as the log-return from time $t - \Delta t$ to t, is simply a sum of monthly log-returns. In Eq. (7), Φ_0 is a row vector of intercepts, Φ_1 is a 4×4 matrix of the slope coefficients of the VAR model, and υ_t is a row vector of *iid* innovations $\sim N(0, \Sigma_z)$, where $\Sigma_z = \mathbb{E}[\upsilon\upsilon']$.

If all eigenvalues of Φ_1 have moduli less than one, the stochastic process in Eq. (7) is stable with the unconditional expected mean μ_{zz} and covariance Γ_{zz} of z_t in the steady state:

$$\mu_{zz} \;=\; (I - \Phi_1)^{-1}\Phi_0 \tag{8}$$

$$vec(\Gamma_{zz}) \;=\; (I - \Phi_1 \otimes \Phi_1)^{-1} vec(\Sigma_z), \tag{9}$$

where I is an identity matrix and the operator $\otimes$ is the Kronecker product and vec is a vectorization function, which transforms a $K \times K$ matrix into a $K^2 \times 1$ vector.

Using historical yield curves from the Bank of England from January 1993 to December 2013 with 0.5 to 25-year spot rates, and monthly FTSE 100 data over the same period, and by minimizing the sum of squared errors between the fitted and historical yields, we estimate $\lambda = 0.3820$ in Eq. (6) in the Nelson–Siegel model. Our estimates for Φ_0 and Φ_1, in Eq. (7), along with t-statistics, are collected in Table 1. The level of R^2 for the equity return component is low, so it is difficult to confirm that return predictability in the UK equity market exists. Table 2 exhibits the correlations and standard deviations (multiplied by 100) of the residuals. Table 3 presents the unconditional expected mean μ_{zz} of z_t in the steady state.

4. Formulation for multi-stage stochastic programming

4.1. Scenario generation

Stochastic programming is a mathematical framework for optimization problems with uncertain variables, commonly used in operations research. The variables can be economic, financial, and demographic. Multi-stage stochastic programming (MSP) discretizes both the state space and time, and the multiple discrete-time points are known as stages. An MSP model is constructed in a nodal form by using state variables generated in a scenario tree. The scenario tree starts at the initial stage from a unique root node which branches out to a number of children nodes at the second time stage. Each of these children nodes themselves branch out to further nodes at the third time stage, etc. The nodes at the terminal stage are known as leaf nodes. A scenario is the path followed from the root node through descendant nodes to a leaf node. The tree is non-recombining, in general.

Some helpful notation pertaining to the scenario tree is set out below. The root node of the scenario tree is denoted by n_0. Let $\mathcal{N}$ be the set of all nodes in the tree, and $\mathcal{N}_t$ be the set of nodes at time t. For our retirement planning problem, time 0 is the first stage and retirement time T is the terminal stage. Thus, $\mathcal{N}_0 = \{n_0\}$ contains the root node only, $\mathcal{N}_T$ is the set of leaf nodes, and $\mathcal{N} = \bigcup_{t\in[0,T]} \mathcal{N}_t$. The unconditional probability that a node n occurs is $\mathbf{pr}_n$ and, clearly, $\sum_{n\in\mathcal{N}_t} \mathbf{pr}_n = 1$. A node $n \neq n_0$ will branch off a parent node, denoted by n^-, which may itself have its own parent node n^{--}, etc. A node $n \notin \mathcal{N}_T$ will engender a set of children nodes, denoted by $\{n^+\}$, which may themselves have their own children nodes $\{n^{++}\}$, etc.

In the operations research literature, scenario trees are generated using three main methods: scenario reduction, state aggregation, and moment matching (see Geyer et al., 2010). We choose the moment matching method (Høyland and Wallace, 2001; Klaassen, 2002) for generating scenario trees of accumulated equity returns and the three Nelson–Siegel term structure parameters. More precisely, we use the sequential approach of Høyland and Wallace (2001) with the moment matching method to generate scenario trees.

A large multi-period scenario tree consists of many small single-period sub-trees. The first-period sub-tree has a number of outcomes corresponding to each child node in the set $\{n_0^+\}$. The outcomes for the first-period sub-tree are obtained by matching the first four moments of the distributions of state variables. For the second-period sub-trees, the conditional outcomes are obtained by matching the first four moments of the conditional distribution properties on outcomes of the first-period sub-tree. This procedure is executed sequentially for the third, fourth etc. sub-tree until the final-period sub-tree. By doing so, we ensure that all conditional distribution properties are fully matched through the multi-period scenario tree.

The scenario tree that we construct in our multi-stage stochastic programming problem has six stages. The time interval between the stages is Δt, so the stages occur at time 0, Δt, $2\Delta t$, ..., $T = 5\Delta t$. At each node n, we store the state variables $[R_n, r_n, \beta_{1,n}, \beta_{2,n}, \beta_{3,n}]$ employing the same notation as before except that we index by node n rather than by time. Thus, if node n occurs at time t, R_n denotes equity log-return over a Δt-long time interval ending at time t (Eq. (2)); r_n denotes equity log-return over a month ending at time t (Eq. (7)); and $\beta_{1,n}, \beta_{2,n}, \beta_{3,n}$ denote the Nelson–Siegel term structure parameters at time t (Eq. (6)). At the root node n_0, the initial state values are set to equal the unconditional expected means in Table 3. In the scenario tree, every non-terminal node branches off to six children nodes. Six outcomes are the minimum to match the first four moments of the five state variables.

The scenarios that are generated are arbitrage-free. This is achieved using the following procedure:

Step 1. Given a node $n \in \mathcal{N} \setminus \mathcal{N}_T$, generate scenarios for its children nodes $\{n^+\}$ by matching the four moments of conditional distributions of five state variables $[R_{n^+}, r_{n^+}, \beta_{1,n^+}, \beta_{2,n^+}, \beta_{3,n^+}]$ given the current state values of $[R_n, r_n, \beta_{1,n}, \beta_{2,n}, \beta_{3,n}]$.

[2] In the Nelson–Siegel model, the long interest rate is given by $\lim_{s\to\infty} y(\beta_t, s, \lambda) = \beta_{1,t}$ and the short interest rate is $\lim_{s\to 0} y(\beta_t, s, \lambda) = \beta_{1,t} + \beta_{2,t}$. See e.g. Boender et al. (2008).

Table 1
Estimated parameters and t-statistics for the VAR(1) model.

	Φ_0	Φ_1				R^2
		r_{t-1}	$\beta_{1,t-1}$	$\beta_{2,t-1}$	$\beta_{3,t-1}$	
r_t	−0.0093	0.0136	0.2446	0.0037	−0.0980	0.0125
t-value	(−0.9961)	(0.2158)	(1.3086)	(0.0266)	(−0.9722)	
$\beta_{1,t}$	0.0070	0.0033	0.8620	−0.0325	0.0229	0.9700
t-value	(4.6116)	(0.3216)	(28.5657)	(−1.467)	(1.4069)	
$\beta_{2,t}$	−0.0044	0.0128	0.0777	1.0008	0.0072	0.9771
t-value	(−4.1225)	(1.7827)	(3.6479)	(63.9633)	(0.6246)	
$\beta_{3,t}$	−0.0024	0.0084	0.0514	0.0206	0.9560	0.9336
t-value	(−1.3018)	(0.6857)	(1.4191)	(0.7742)	(48.9678)	

Monthly data of FTSE 100 and the Bank of England's fitted yield curves are used from January 1993 to December 2013; t-statistics in parentheses.

Table 2
Cross correlations and standard deviations of residuals of the VAR(1) model.

	r	$\beta 1$	$\beta 2$	$\beta 3$
r	[a]4.0371	−0.0354	0.1487	−0.0180
$\beta 1$	−0.0354	[a]0.6518	−0.7944	−0.2002
$\beta 2$	0.1487	−0.7944	[a]0.4599	0.0577
$\beta 3$	−0.0180	−0.2002	0.0577	[a]0.7821

[a]Standard deviations along the leading diagonal are multiplied by 100.

Table 3
Unconditional expected mean μ_{zz} of the VAR(1) model.

	r	$\beta 1$	$\beta 2$	$\beta 3$
μ_{zz}	0.0040	0.0559	−0.0204	0.0028

Table 4
Percentiles of the spot rate for different maturities at the final stage.

	5y	10y	15y	20y	25y	30y
5th perc.	0.0059	0.0160	0.0202	0.0223	0.0235	0.0241
50th perc.	0.0497	0.0530	0.0544	0.0550	0.0555	0.0558
95th perc.	0.1041	0.0993	0.0966	0.0953	0.0943	0.0938

Step 2. Check if the generated scenarios preclude arbitrage opportunities (see Klaassen, 2002). Go back to Step 1 if any arbitrage opportunity is found.
Step 3. Check if each of the generated scenarios has conditional moments over the next stage that are within no-arbitrage bounds (see Geyer et al., 2014)
Step 4. Repeat Steps 1 to 3, if Step 3 meets the always-arbitrage bound.

Step 2 can be subsumed within Step 1. The four moments of the conditional distributions in Step 1 are obtained as in Appendix A. The above procedure is applied from the first to the penultimate stage in a sequential way. Validating arbitrage opportunities among the three financial assets (equity, cash and bond funds) in Steps 2 and 3 is dealt with by using the two methods of Klaassen (2002) for two arbitrage types *ex-post* and the method of Geyer et al. (2014) for no-arbitrage bounds *ex-ante*.

Since there are six children nodes for every non-terminal node and there are six stages (five periods), there are $6^5 = 7{,}776$ scenarios and $\sum_{j=0}^{5} 6^j = 9{,}331$ nodes. To improve the stability of our results, we aggregate two independently-generated scenario trees, with an identical root node, into one large scenario tree (see Høyland and Wallace, 2001). So, the total number of scenarios is 15,552 and the total number of nodes is 18,661. Generating each scenario tree takes just under one hour with Matlab by using parallel package *parfor* on a laptop with Intel CPU i7-7700 3.60 Ghz and 32 Gbyte memory.

In order to verify how well the combined scenario tree describes the fitted VAR model of Section 3.2, we compare unconditional cumulative distributions of the state variables generated on the scenario tree with the actual cumulative distributions from the VAR model: see Fig. B.5 in Appendix B). This shows that, although the scenario tree discretizes the state space of variables, it replicates the distributions very closely, particularly in the later stages.

Table 4 shows the percentiles of the spot rates for different maturities at time T from the scenario tree. This is consistent with the empirical evidence that short-term interest rates are more volatile than long-term interest rates. Fig. 1(a) displays the initial term structure starting from the steady state with the unconditional expected values in Table 3. Fig. 1(b) shows twelve different realizations of the term structure at the second stage. The thickness of the lines is linked to the probability that the term structure occurs.

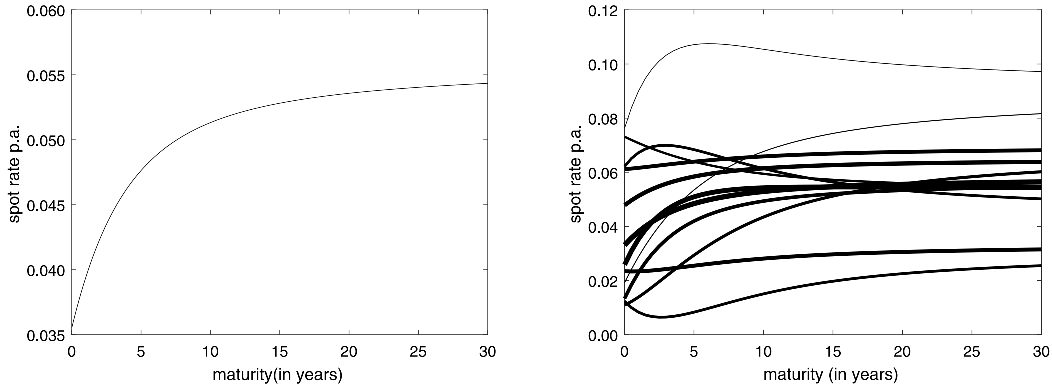
From the generated outcomes on each node, the asset prices given in Eq. (2) to Eq. (5) can be rewritten in a nodal form. Recall that any node n in the scenario tree (except for the root node n_0) branches off a parent node n^- at the previous time stage. The asset price in Eq. (2), for example, is transformed into the nodal form simply by replacing t with n and $t - \Delta t$ with n^- as follows:

$$S_{i,n} = S_{i,n^-} \cdot \exp(R_{i,n}) \quad \text{for } n \in \mathcal{N} \setminus \{n_0\} \text{ and } i \in \{E, C, B\},$$

with $S_{i,n_0} = 1$. Other pricing formulas are transformed in a similar way.

4.2. Optimization problem

The objective function and constraints set out in Eq. (1) for the general problem can now be formulated on the scenario tree as a multi-stage stochastic programming problem. The notation transfers in a straightforward way, except that we index by node rather than time. For example, $X_{i,n}$ refers to the number of units



(a) Steady state at initial stage (b) After 5 years

Fig. 1. Left panel (a) shows the initial term structure starting from the steady state. Right panel (b) shows twelve different realizations of the term structure after 5 years (the thicker the line, the higher the probability of occurrence).

of asset $i \in \{E, C, B, A\}$ held at node n in the scenario tree. We also distinguish between buy and sell decisions, so that $X^{buy}_{i,n}$ is the number of units of asset i to buy at node n and $X^{sell}_{i,n}$ is the number of units of asset i to sell at node n. Recalling that deferred annuities cannot be sold, the decision variable for the individual at node n is therefore $X_n = \left[X^{buy}_{E,n}, X^{sell}_{E,n}, X^{buy}_{C,n}, X^{sell}_{C,n}, X^{buy}_{B,n}, X^{sell}_{B,n}, X^{buy}_{A,n}\right]'$. The decision variable must be chosen at every node n in the scenario tree.

The objective function in Eq. (1a) is rewritten in a nodal form as follows:

$$\max_{\{X_n, n\in\mathcal{N}\setminus\mathcal{N}_T\}} \left[\sum_{t\in[T,\tau]} \sum_{n\in\mathcal{N}_T} {}_t p_\delta \; u(t, X_{A,n})\, \mathbf{pr}_n \right.$$
$$\left. + \sum_{t\in[0,T)} \sum_{n\in\mathcal{N}_{t+\Delta t}} {}_t p_\delta \; {}_{\Delta t} q_{\delta+t} \; \kappa^\gamma u(t+\Delta t, W_n)\, \mathbf{pr}_n \right], \tag{10}$$

where it is implicit that summations occur over the time stages in the scenario tree during the planning phase when $t \in [0, T]$.

A cash balance constraint in Eq. (11) controls cash inflows and outflows. Below, φ^s_i and φ^u_i indicate a percentage selling fee and upfront fee respectively for asset $i \in \{E, C, B, A\}$, and w_0 is non-random positive initial wealth.

$$\mathbb{1}_{\{n=n_0\}} w_0 + \mathbb{1}_{\{n\notin\mathcal{N}_T\}} \phi \cdot L_n + \sum_{i\in\{E,C,B\}} X^{sell}_{i,n} S_{i,n} (1-\varphi^s_i)$$
$$= \sum_{i\in\{E,C,B,A\}} X^{buy}_{i,n} S_{i,n} \left(1+\varphi^u_i\right), \quad \forall n. \tag{11}$$

An asset inventory constraint appears in Eq. (12) and tracks the number $X_{i,n}$ of units of asset $i \in \{E, C, B, A\}$ held at node n:

$$X_{i,n} = \mathbb{1}_{\{n\neq n_0\}} X_{i,n-} + X^{buy}_{i,n} - X^{sell}_{i,n}, \quad \text{for } n \in \mathcal{N}. \tag{12}$$

Wealth in the retirement plan, which includes equities, cash and bonds, and excludes purchased deferred annuities, satisfies the following equation:

$$W_n = \sum_{i\in\{E,C,B\}} X_{i,n-}\, S_{i,n}\, (1-\varphi^m_i), \quad \text{for } n \in \mathcal{N}\setminus\{n_0\}, \tag{13}$$

where $0 \leq \varphi^m_i < 1$ is a percentage investment management fee for asset $i \in \{E, C, B\}$.

Other constraints appear below and correspond to Eq. (1c) to Eq. (1f):

$$X_{i,n} \geq 0 \quad \text{for } i \in \{E, C, B, A\} \text{ and } n \in \mathcal{N}, \tag{14a}$$
$$X^{sell}_{A,n} = 0 \quad \text{for } n \in \mathcal{N}, \tag{14b}$$
$$X_{i,n} = X^{buy}_{i,n} = 0 \quad \text{for } n \in \mathcal{N}_T \text{ and } i \in \{E, C, B\}, \tag{14c}$$
$$X^{buy}_{A,n} = W_n / S_{A,n} \quad \text{for } n \in \mathcal{N}_T. \tag{14d}$$

The non-negative wealth condition of Eq. (1g) is not imposed as it is satisfied in Eq. (13) since asset prices are positive and no short-selling is allowed in Eq. (14a). Wealth is initialized at the non-random amount w_0 specified on the l.h.s. of Eq. (11).

On every node in the scenario tree, the cash balance, asset inventory and other constraints are set, following Eq. (11) to Eq. (14). Finally, we use an efficient non-linear solver, *MOSEK*, to find optimal investment and deferred annuity choices by maximizing the objective function in Eq. (10) subject to the constraints in Eq. (11) to Eq. (14).

5. Numerical results

5.1. Numerical example

We investigate a hypothetical case in which a 40-year-old individual ($\delta = 40$) intends to retire at age 65 ($T = 25$). His goal is to maximize and secure his retirement benefits in nominal terms and to set aside a portion of his portfolio as a bequest if he dies before retirement. In his retirement plan, he can invest in an equity fund, a bond fund (maturity $M = 20$ years), a cash fund (maturity $M = 5$ years), and in deferred annuities as described in Section 2. To price the deferred annuity, we use a U.K. mortality table based on 2000–2006 experience.[3]

The individual can rebalance his portfolio and buy deferred annuities every 5 years ($\Delta t = 5$) so there are six stages (five periods) in the scenario tree spanning the 25-year planning horizon. The individual has an initial wealth of w_0 = £80,000 which he credits to his retirement plan. His annual wage is fixed at £40,000 in nominal terms throughout. He contributes £4000 p.a. to his retirement plan ($\phi = 10\%$). Because of the incidence of cash flows in our model, the contribution is in effect £20,000 every five years in advance ($\phi \cdot L_n$ = £20,000 for $n \in \mathcal{N}\setminus\mathcal{N}_T$).

In the base case, the individual is a male with risk aversion coefficient $\gamma = 3$, time preference $\rho = 0$, bequest parameter $\kappa = 0$. For the bond and equity funds, upfront and selling fees are $\varphi^u_i = \varphi^s_i = 0.5\%$ for $i \in \{B, E\}$, following Geyer et al.

[3] Institute and Faculty of Actuaries, S1PML/S1PFL—All pensioners (excluding dependants), male/female lives www.actuaries.org.uk/research-and-resources/documents/s1pml-all-pensioners-excluding-dependants-male-lives(www.actuaries.org.uk).

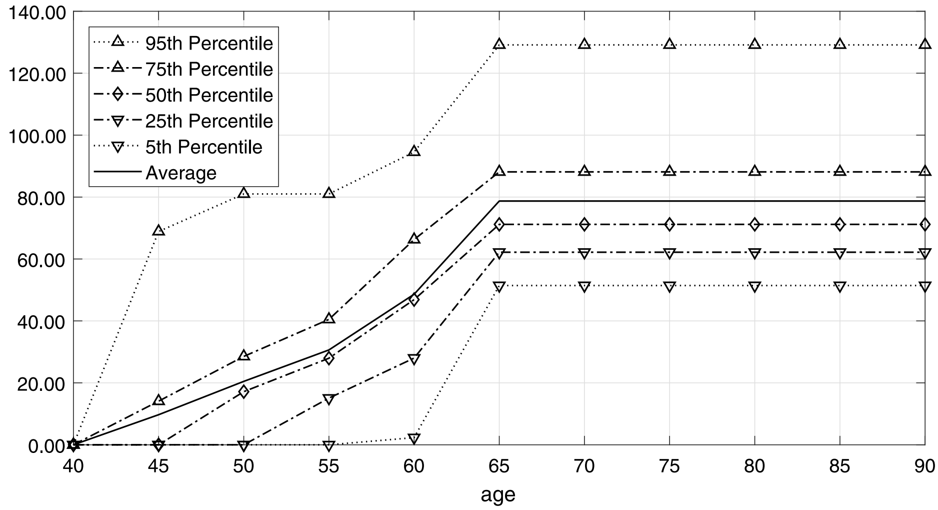


Fig. 2. Percentiles and average of retirement income (×£1000 p.a.) from deferred annuity holdings at various ages. Deferred annuities start paying retirement income from retirement age 65. Parameter values as in base case.

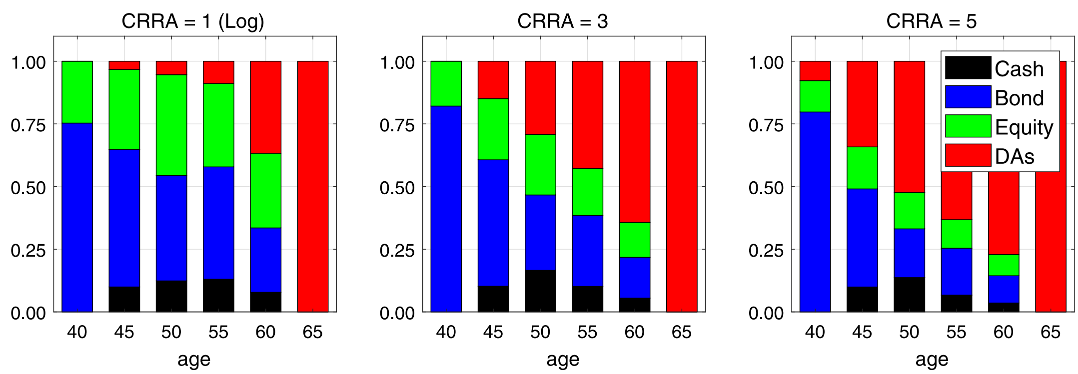


Fig. 3. Optimal investment and deferred annuity allocations of overall wealth on average at various ages over the planning horizon and for different risk aversion (CRRA) coefficients (γ). 'DAs' in the legend indicates Deferred Annuities. Parameter values as in base case except as stated.

(2009) and Konicz et al. (2014). Expense loadings on annuities are $\varphi_A^u = 3.0\%$, comparable with Horneff et al. (2010) and Huang et al. (2017). The cash fund has no fees, $\varphi_C^u = \varphi_C^s = 0$, and management fees for all assets are ignored, $\varphi_i^m = 0$ for $i \in \{B, C, E\}$. Variations on the base case are also considered below, with female mortality and with other parameter values.

5.2. Optimal investment and deferred annuity choices

Our numerical results show that the optimal strategy to secure retirement income involves buying deferred annuities regularly during working lifetime, starting fairly early, and accelerating in the years before retirement. This is illustrated in Fig. 2 which shows the secured retirement income from deferred annuity holdings at various ages, with constant retirement income from annuity payments after retirement for remaining lifetime. The average and various percentiles are shown in Fig. 2.

This result is robust for different investor profiles. Fig. 3 shows average optimal asset allocations including deferred annuities (DA) over 25-year planning horizons for three different constant relative risk aversion (CRRA) coefficients. Deferred annuities are purchased early and continually by investors. Unsurprisingly, the more risk-averse the investor, the greater the holding of deferred annuities, the faster deferred annuity holdings are built up, and the lower equity and bond holdings are at every age, on average. Regardless of the levels of risk aversion, the proportion of deferred annuity holding in overall wealth increases on average, as retirement approaches.

Average holdings of bond and equity decline in general. Bonds clearly play a significant role in the investor's portfolio. This is because they hedge price changes in future deferred annuities, i.e. future retirement income. Similar hedging strategies with long-term bonds are found by Cairns et al. (2006) and Koijen et al. (2011). However, deferred annuity is not available to the investor in their models, so the hedging demand is stronger and bond holding is greater, as retirement draws closer. Our case shows that the hedging demand appears to weaken as annuity holdings increase and retirement draws closer.

The optimal allocation to cash increases initially and then decreases with age, on average. The cash account can be interpreted as providing liquidity to fund future deferred annuity purchases. The decrease near retirement occurs because of the reduced hedging demand.

In Table 5, average and standard deviation values of the optimal asset allocations over 25-year planning horizons are presented for different risk aversion γ, bequest motive κ, time impatience ρ parameter values, and there are no transaction costs and fees. Comparing panels A and B (or D and E), we find that deferred annuity holdings decrease, if bequest has greater importance to the investor. Annuitizing wealth means that less wealth is available to heirs if death occurs before retirement, so this result is sensible. A greater bequest motive increases bond allocation on average, with only a small increase in equity allocation: the investor postpones buying deferred annuities as much as possible, as they reduce inheritance if the individual

Table 5
Optimal investment and deferred annuity allocations (%) on average for different risk aversion, time preference and bequest parameters. Parameter values as in base case except as stated, no transaction costs and fees.

Age	A. $\gamma = 3.0$, $\rho = 0.0$, and $\kappa = 0.0$				B. $\gamma = 3.0$, $\rho = 0.0$, and $\kappa = 2.0$				C. $\gamma = 3.0$, $\rho = 0.04$, and $\kappa = 2.0$			
	Cash	Bond	Equity	DA	Cash	Bond	Equity	DA	Cash	Bond	Equity	DA
40	0.0	81.4	18.6	0.0	0.0	81.1	18.9	0.0	0.0	80.9	19.1	0.0
	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)
45	9.2	52.0	24.1	14.6	9.3	54.6	24.2	11.9	9.3	56.3	24.2	10.2
	(25.8)	(37.9)	(33.7)	(24.7)	(25.8)	(37.0)	(33.7)	(19.4)	(25.8)	(37.0)	(33.6)	(16.7)
50	16.3	32.1	24.8	26.8	16.7	35.4	25.4	22.5	17.0	37.4	25.8	19.8
	(29.8)	(30.2)	(32.1)	(27.1)	(30.0)	(31.2)	(32.3)	(22.0)	(30.1)	(32.2)	(32.6)	(19.6)
55	10.6	32.1	19.2	38.1	11.8	36.0	19.9	32.3	12.3	37.9	20.3	29.5
	(24.1)	(33.2)	(28.8)	(26.8)	(25.0)	(33.6)	(29.4)	(21.6)	(25.6)	(34.2)	(29.8)	(19.9)
60	5.9	18.3	14.9	60.9	8.2	23.6	16.6	51.6	8.8	24.7	17.0	49.5
	(18.4)	(29.9)	(23.0)	(31.9)	(19.7)	(30.0)	(23.7)	(25.4)	(20.4)	(30.6)	(24.1)	(25.0)
65	0.0	0.0	0.0	100.0	0.0	0.0	0.0	100.0	0.0	0.0	0.0	100.0
	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)
Age	D. $\gamma = 5.0$, $\rho = 0.0$, and $\kappa = 0.0$				E. $\gamma = 5.0$, $\rho = 0.0$, and $\kappa = 2.0$				F. $\gamma = 5.0$, $\rho = 0.04$, and $\kappa = 2.0$			
	Cash	Bond	Equity	DA	Cash	Bond	Equity	DA	Cash	Bond	Equity	DA
40	0.0	80.8	13.1	6.1	0.0	83.2	13.4	3.4	0.0	86.6	13.4	0.0
	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)
45	10.5	41.9	16.2	31.3	11.3	49.9	16.3	22.6	12.4	53.6	15.9	18.0
	(23.7)	(34.5)	(25.6)	(29.4)	(24.4)	(33.0)	(25.4)	(20.6)	(25.3)	(33.6)	(24.7)	(19.1)
50	14.1	20.5	14.9	50.4	15.6	27.9	16.1	40.4	16.6	30.4	16.3	36.7
	(26.3)	(17.8)	(20.1)	(25.7)	(27.2)	(20.5)	(20.8)	(18.0)	(28.1)	(22.1)	(20.9)	(16.7)
55	6.9	21.6	11.6	59.9	9.0	28.0	12.9	50.1	9.7	29.7	13.2	47.4
	(16.4)	(25.2)	(18.1)	(22.4)	(18.4)	(26.3)	(19.4)	(16.8)	(19.3)	(27.1)	(20.0)	(16.3)
60	3.8	12.2	9.0	75.0	6.8	18.3	10.8	64.1	7.3	19.2	11.0	62.5
	(12.2)	(20.7)	(13.9)	(21.9)	(14.6)	(22.0)	(15.3)	(16.8)	(15.2)	(22.7)	(15.6)	(17.0)
65	0.0	0.0	0.0	100.0	0.0	0.0	0.0	100.0	0.0	0.0	0.0	100.0
	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)

The expected values are averaged from 15,552 scenarios with standard deviations appearing in parentheses.

dies before retirement, and bonds are held to hedge the eventual purchase of annuities at or close to retirement.

Optimal investment and annuity choices do not appear to be very sensitive to the investor's time preference, at least on average, when we compare panels B and C (or E and F) in Table 5. Time impatience simply enlarges the bequest motive. Nevertheless, in all these cases, deferred annuities are purchased early and continually.

Table 6 presents the optimal investment and deferred annuity allocations with various contribution rates and male/female mortality rates. Comparing panels A and B (or C and D), changing contribution rates, i.e. the proportion of income paid regularly into the retirement plan, does not alter asset allocation significantly, although the number of units of annuities purchased obviously increases with a higher contribution, resulting in higher secured income in retirement. However, a female investor holds on average less annuities, more bonds, and about the same in equities, compared to a male investor who has a higher mortality: compare panels A and C (or B and D) in Table 6.

5.3. Optimal investment and deferred annuity choices: investigating the dynamic solution

The results described in Section 5.2 concern the *average* investment and annuitization choices (with standard deviations also presented), but the optimal strategy is a stochastic and dynamic one. To investigate these results further, we consider how the strategy works at age 45 under different scenarios. In particular, we observe that the dynamic strategy adjusts to hedge price changes in deferred annuities in accordance with realized and expected values of state variables.

In the VAR asset model with Nelson–Siegel term structure model as fitted to UK data in Section 3, short-term interest rates $\beta_{1,t} + \beta_{2,t}$ at time t are strongly negatively correlated with the conditional expectations of equity returns $E_t(R_{t+1})$ as well as with the conditional expectations of long-term interest rates $E_t(\beta_{1,t+1})$. In the following, we suppress dependence on t in the notation as it is implicit.

Table 7 shows how the dynamic strategy works with the predictable and time-varying investment opportunities from ages 40 to 45. We can compare the investment positions at age 40, labelled as case (a) in Table 7, with the positions at age 45, under different scenarios labelled as cases (b)–(f) in the table. At age 40 (the root node of the scenario tree), the optimal allocations are about 0.00%, 81.35%, 18.65%, and 0.00% in cash, bond, equity, and deferred annuity respectively. The prices of the bond and equity fund units are £1000, but the price of the deferred annuity is £2332.90 for £1000 p.a. retirement income.

Case (b) in Table 7 refers to the scenario where the short-term rate $(\beta_1+\beta_2)$ at age 45 is the highest. Since a high short-term rate (7.61%) correlates with a low expected long-term rate (5.02%), deferred annuities are expected to become more expensive in the future, so a large amount of annuities are purchased. Indeed, case (b) coincides with case (e'), i.e. deferred annuities are at their cheapest, hence it is worth buying them to secure income in retirement. The large bond holding at age 40 in case (a) can therefore be sold to buy the annuities. A high short-term rate also correlates with a low expected equity return (5.95%), so equities should be sold as they are relatively unattractive. This is indeed what case (b) in Table 7 shows: almost all the bond and equity funds are sold and the proceeds, together with the contribution to the retirement plan, are used to buy deferred annuities.

On the other hand, case (b') in Table 7 refers to the scenario where the short-term rate at age 45 is the lowest (1.09%). This coincides with a high expectation of return on equities (44.36%), so it is optimal for the investor to buy equities, funded by a sale of bonds as well as by the contribution inflow to the retirement plan. Long-term rates are expected to be high (6.01%), so deferred annuities are expected to be cheap, and less deferred annuities are bought compared to case (b).

Cases (c) and (e) in Table 7 represent identical scenarios where both the long-term bond and the deferred annuity are at their

Table 6
Optimal investment and deferred annuity allocations (%) on average for various contribution rates and male/female mortality. Parameter values as in base case except as stated, $\kappa = 2.0$, no transaction costs and fees.

Age	A. Male, Contribution rate, $\phi = 0.10$					B. Male, Contribution rate, $\phi = 0.20$				
	Cash (%)	Bond (%)	Equity (%)	DA (%)	SRI	Cash (%)	Bond (%)	Equity (%)	DA (%)	SRI
40	0.0	81.1	18.9	0.0	0.0	0.0	80.7	19.3	0.0	0.0
	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)
45	9.3	54.6	24.2	11.9	7.9	8.2	55.0	25.0	11.8	10.4
	(25.8)	(37.0)	(33.7)	(19.4)	(14.7)	(25.7)	(37.7)	(35.5)	(19.2)	(19.6)
50	16.7	35.4	25.4	22.5	15.9	16.3	35.6	26.1	22.0	21.4
	(30.0)	(31.2)	(32.3)	(22.0)	(19.9)	(29.8)	(31.5)	(33.1)	(21.7)	(27.0)
55	11.8	36.0	19.9	32.3	23.5	11.7	36.4	20.1	31.7	32.2
	(25.0)	(33.6)	(29.4)	(21.6)	(20.0)	(25.1)	(33.8)	(29.7)	(21.3)	(27.4)
60	8.2	23.6	16.6	51.6	40.4	8.3	23.8	16.7	51.2	56.7
	(19.7)	(30.0)	(23.7)	(25.4)	(25.3)	(19.9)	(30.2)	(23.8)	(25.6)	(35.4)
65	0.0	0.0	0.0	100.0	83.1	0.0	0.0	0.0	100.0	117.8
	(0.0)	(0.0)	(0.0)	(0.0)	(30.3)	(0.0)	(0.0)	(0.0)	(0.0)	(41.4)
Age	C. Female, Contribution rate, $\phi = 0.10$					D. Female, Contribution rate, $\phi = 0.20$				
	Cash (%)	Bond (%)	Equity (%)	DA (%)	SRI	Cash (%)	Bond (%)	Equity (%)	DA (%)	SRI
40	0.0	81.1	18.9	0.0	0.0	0.0	80.8	19.2	0.0	0.0
	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)
45	9.3	55.2	24.2	11.3	6.7	8.2	55.4	25.0	11.3	8.9
	(25.8)	(37.3)	(33.7)	(19.4)	(13.4)	(25.7)	(38.0)	(35.5)	(19.4)	(18.0)
50	16.8	37.1	25.6	20.5	13.0	16.4	37.1	26.2	20.3	17.7
	(30.1)	(33.3)	(32.5)	(22.5)	(18.6)	(29.8)	(33.5)	(33.2)	(22.3)	(25.4)
55	12.8	43.0	20.2	24.0	15.7	12.7	43.3	20.5	23.5	21.6
	(26.5)	(37.1)	(29.8)	(21.1)	(18.2)	(26.6)	(37.3)	(30.1)	(20.6)	(24.9)
60	10.9	29.1	17.8	42.3	29.2	11.0	29.3	17.9	41.8	41.0
	(24.1)	(35.2)	(25.5)	(29.0)	(24.0)	(24.2)	(35.4)	(25.6)	(29.0)	(33.7)
65	0.0	0.0	0.0	100.0	74.3	0.0	0.0	0.0	100.0	105.3
	(0.0)	(0.0)	(0.0)	(0.0)	(27.7)	(0.0)	(0.0)	(0.0)	(0.0)	(37.8)

The last columns titled SRI are the expected values of the total secured retirement income (×£1000 p.a.) at the given age by purchasing deferred annuities. The expected values are averaged from 15,552 scenarios with standard deviations appearing in parentheses.

Table 7
Optimal investment and deferred annuity choices at ages 40 and 45 under different scenarios. Parameter values as in base case except as stated, $\kappa = 2.0$, no transaction costs and fees.

Age	Node	Parameters			Net purchases of units[a] (prices in brackets)			
		$\beta_1 + \beta_2$	$E(R)$	$E(\beta_1)$	Cash	Bond	Equity	DA[b]
40	(a) Root node	0.0355	0.2428	0.0559	0.0000	81.3532	18.6468	0.0000
					(1000.00)	(1000.00)	(1000.00)	(2332.90)
45	(b) The highest $\beta_1 + \beta_2$	0.0761	0.0595	0.0502	0.0000	−81.3490	−18.6460	84.0117
					(1268.90)	(631.99)	(570.82)	(976.71)
	(b') The lowest $\beta_1 + \beta_2$	0.0109	0.4436	0.0601	0.0000	−45.9758	35.4381	11.2023
					(1268.90)	(1354.52)	(1456.23)	(2737.74)
	(c) The highest bond price	0.0124	0.3596	0.0593	0.0000	−81.3528	135.1827	0.0000
					(1268.90)	(2171.58)	(1454.81)	(7088.03)
	(c') The lowest bond price	0.0761	0.0595	0.0502	0.0000	−81.3490	−18.6460	84.0117
					(1268.90)	(631.99)	(570.82)	(976.71)
	(d) The highest equity price	0.0191	0.3948	0.0585	0.0000	−63.0181	−18.6463	85.8842
					(1268.90)	(957.31)	(2780.05)	(1538.87)
	(d') The lowest equity price	0.0761	0.0595	0.0502	0.0000	−81.3490	−18.6460	84.0117
					(1268.90)	(631.99)	(570.82)	(976.71)
	(e) The highest DA price	0.0124	0.3596	0.0593	0.0000	−81.3528	135.1827	0.0000
					(1268.90)	(2171.58)	(1454.81)	(7088.03)
	(e') The lowest DA price	0.0761	0.0595	0.0502	0.0000	−81.3490	−18.6460	84.0117
					(1268.90)	(631.99)	(570.82)	(976.71)
	(f) The highest cash hold	0.0621	0.0317	0.0513	119.7519	−81.3518	−18.6467	0.0000
					(1268.90)	(1253.71)	(1606.81)	(3458.22)

[a]Units in funds, or units of deferred annuities (DA). A negative number means that there is a net sale.
[b]DA = deferred annuity. One unit of DA is equivalent to £1000 p.a. of secured retirement income.

most expensive. The short-term rate, however, is very low (1.24%) coinciding with a fairly high expected equity return (35.96%). The optimal strategy is therefore to sell the bond holdings held at age 40 (since bond prices are peaking), to buy no deferred annuity (since it is expensive, about three times higher than at age 40), and instead to buy large amounts of equities.

Finally, case (f) in Table 7 refers to the scenario where it is optimal to hold the most cash. The short-term rate (6.21%) in this case is relatively high and expected return on equities (3.17%) relatively low while the current equity, bond and annuity prices have risen compared to the situation at age 40 in case (a). The optimal strategy therefore calls for selling virtually all equity and bond holdings, in order to enjoy a relatively high risk-free return on cash.

The dynamic optimal investment and annuitization behaviour can therefore be explained by the current level of short-term interest rates, the realized gains of assets, and the change in deferred annuity price. This exploits the predictability in asset

Table 8
Distributional statistics and certainty equivalent values of secured retirement income (×£1000 p.a.) when deferred annuities are available/unavailable and for different values of risk aversion (CRRA) coefficient γ. Parameter values as in base case except as stated, no transaction costs and fees..

γ	Annuitization	Mean	StDev	Mean/StDev	5th Pctl.	95th Pctl.	CE[c]
1 (Log)	Deferred + Immediate[a]	89.1498	42.2555	2.1098	44.3841	164.0076	81.7216
	Immediate only[b]	88.7975	44.1526	2.0112	42.7610	168.5922	80.7718
3	Deferred + Immediate[a]	82.8841	29.7284	2.7880	53.4512	137.5443	72.7528
	Immediate only[b]	83.4228	33.6379	2.4800	49.0865	146.6900	70.5130
5	Deferred + Immediate[a]	77.3079	21.5061	3.5947	56.2560	117.9621	68.5068
	Immediate only[b]	77.4188	25.1553	3.0776	50.6060	125.2448	64.6650

[a]Deferred annuities are available before retirement, and an immediate annuity is available at retirement.
[b]Deferred annuities are not available, but an immediate annuity remains available at retirement.
[c]Certainty equivalent values of secured retirement income from annuities.

returns in our model. This also shows that the dynamic optimal strategy from the multi-stage stochastic programming model is responsive to market changes.

6. Further numerical results

6.1. Availability of deferred annuities

In this section, we consider the situation where deferred annuities are unavailable to the investor, but an immediate annuity remains available at retirement at which point the investor has to annuitize all wealth. This means an additional constraint, $X_{A,t} = 0$ for $t \in [0, T)$, in the general statement of the optimization problem in Section 2.1. Equivalently, in the multi-stage stochastic programming version of Section 4.2, we impose the constraint that $X_{A,n} = 0$ for $n \in \mathcal{N} \setminus \mathcal{N}_T$. To compare the situation with and without deferred annuities, we use the numerical example in Section 5.1 and we compute various statistics for retirement income under all the scenarios in the scenario tree after optimization. This is repeated for three different risk aversion coefficients.

The results are tabulated in Table 8. The individual investor in our model receives a higher average retirement income per unit risk (standard deviation), when deferred annuities are available compared to when they are not, for all three levels of risk aversion: see the fifth column of Table 8. The last column of Table 8 also shows the certainty equivalents of retirement income.[4] In all three cases, the certainty equivalent is higher when deferred annuities are available compared to when they are not.

In fact, our model shows that, for a weakly risk-averse investor ($\gamma = 1$), making deferred annuities available increases the certainty equivalent retirement income by £949.80 per year for the remaining lifetime of the investor. This amount rises to £3841.80 for a strongly risk-averse investor ($\gamma = 5$).[5] The analysis here shows that the availability of deferred annuities raises the mean–variance efficiency of retirement income and increases welfare for the investor.

6.2. Stochastic vs. deterministic investment strategies

It is also helpful to compare our stochastic optimal strategy with traditional deterministic strategies used in practice. For the numerical example in Section 5.1, we evaluate two performance metrics: the expected retirement income per unit risk (which is the mean retirement income from annuities at retirement divided by the standard deviation of retirement income), and the certainty equivalent of retirement income. This is done for twelve different strategies, and the results are displayed in Fig. 4.

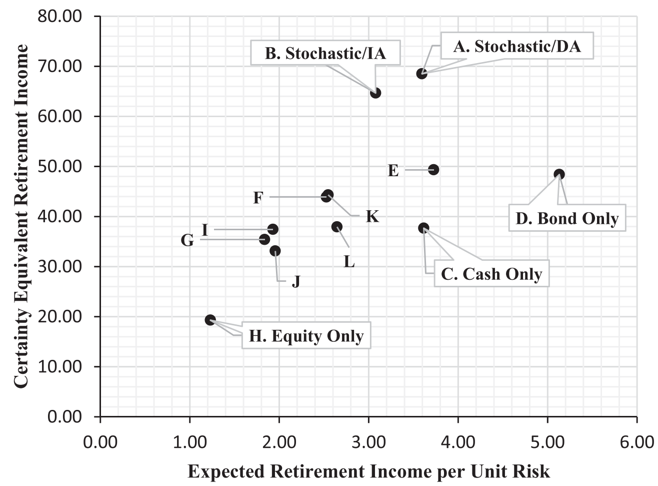


Fig. 4. Performance comparison between 12 different investment strategies labelled **A** to **L**. Strategy **A**, labelled "Stochastic/DA", is our stochastic optimal strategy with deferred annuities (DA). It produces the highest certainty equivalent of retirement income. Strategy **B**, labelled "Stochastic/IA", is the stochastic optimal strategy without deferred annuities but with an immediate annuity (IA) at retirement. The other strategies are described in Section 6.2. Parameter values as in base case except $\gamma = 5.0$, no transaction costs and fees.

The twelve strategies labelled in Fig. 4 are as follows: (**A**) our stochastic optimal strategy with deferred annuities (DA) available including an immediate annuity (IA) at retirement, (**B**) our stochastic optimal strategy without DAs but including an IA at retirement, (**C**) cash only, (**D**) bond only, (**E**) 70/30 bond/equity, (**F**) 50/50 bond/equity, (**G**) 30/70 bond/equity, (**H**) equity only, (**I**) glide path starting from 80/20 equity/bond with equity decreasing and bond increasing by 6% every 5 years, (**J**) glide path starting from 80/20 equity/cash with equity decreasing and cash increasing by 6% every 5 years, (**K**) (100 − age)% in equity and the rest in bond, (**L**) (100 − age)% in equity and the rest in cash. The equity allocations in the glide-path strategies match those suggested by Vanguard (Daga et al., 2016).

Fig. 4 shows that our stochastic optimal strategy **(A)** with deferred annuities produces the highest certainty equivalent of retirement income, which is about 40% higher than the best deterministic strategy **(E)** (70/30 bond/equity). Strategy **(A)** also produces the third highest expected retirement income per unit risk, after strategies **(D)** (bond only) and **(E)** (70/30 bond/equity). Strategy **(H)** (equity only) has the worst performance among the twelve strategies, under both performance metrics.

7. Conclusion

We construct an optimal investment model with deferred annuities for an individual investor who is saving for retirement.

4 The certainty equivalent is calculated by evaluating $u^{-1}\left(\mathbb{E}\left[u\left(T, X_{A,T}\right)\right]\right)$.

5 From the rows for $\gamma = 1$ in Table 8 and the column for certainty equivalent (CE), £949.80 = (81.7216 − 80.7718) × £1000. Likewise, for $\gamma = 5$, £3841.80 = (68.5068 − 64.6650) × £1000.

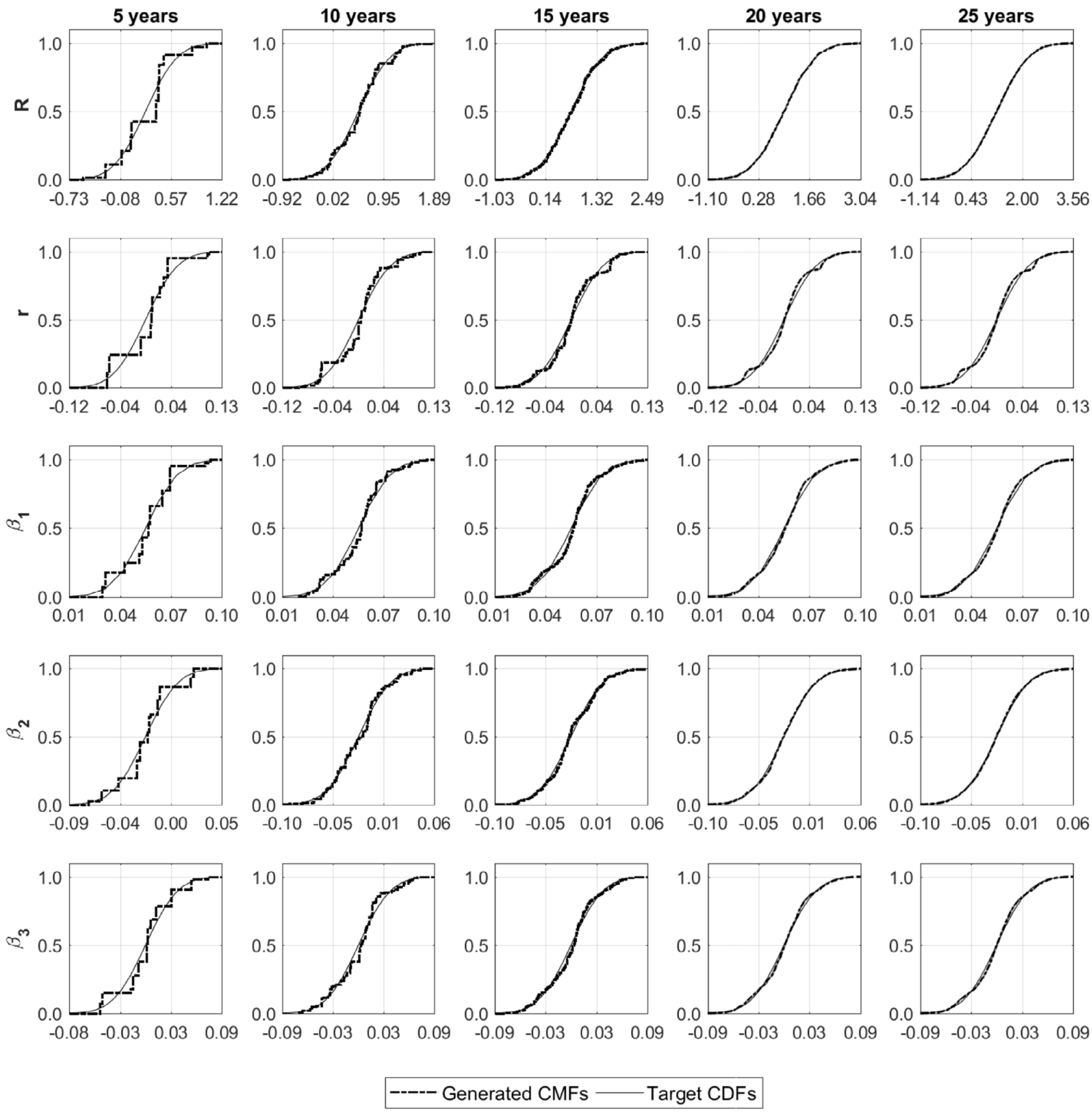


Fig. B.5. Cumulative Density Functions (solid) from VAR model and Cumulative Mass Functions (dot–dash) generated from the scenario tree for state variables R_t, r_t (equity return) and $\beta_{1,t}$, $\beta_{2,t}$, $\beta_{3,t}$ (term structure) at different times during the retirement planning period.

Our results show that buying deferred annuities is an optimal strategy when the objective is to maximize the expected utility of retirement income (as well as bequest), and retirement income is secured by annuitization. Deferred annuity purchase starts fairly early and is made continuously over the working lifetime of the investor, increasing particularly in the years before retirement.

Optimal investment (in equity, cash and bond funds) and optimal deferred annuity choices depend on realized and expected values of state variables, i.e. equity returns and interest rates. We show the links between the time-varying investment opportunities and the optimal investment strategies by investigating representative scenarios. Our results are consistent with previous studies, but also provide support for deferred annuities as a major source of retirement income.

Changes in the optimal portfolio for different preferences (risk aversion, time preference and bequest motive) as well as for different mortality rates, were considered. We also discussed the increased welfare effect of making deferred annuities available to investors in their retirement plan. Our strategy was shown to be superior to typical deterministic strategies, such as glide paths and 100−age strategies, in terms of certainty equivalent of retirement income.

The optimization problem is solved numerically using multi-stage stochastic programming. We use a powerful nonlinear solver to optimize power utility with linear constraints on a scenario tree with a close fit to stock and bond markets data, and we optimize over scenarios that preclude arbitrage by validating scenarios using two different sets of arbitrage-checking methods.

Our model has limitations. We have not taken account of inflation which affects consumer prices, asset prices, and wages. Uncertainties on labour income, labour supply, and political risk are also ignored. Various types of annuity products, such as inflation-protected and other index-linked annuities, are not considered. Other types of insurance products, like life insurance and health insurance covering long-term care and critical illnesses, are also disregarded. They can affect the optimal retirement planning. The results that we have here are influenced by the discretization of state and time in the scenario tree. In practice, individuals will rebalance their portfolio regularly. Practical features such as taxes are also ignored. These limitations will be carefully investigated in future studies.

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Appendix A. Moments of cumulative returns and Nelson–Siegel parameters

Recall from Section 4.1 that $\mathcal{N}$ is the set of all nodes in the scenario tree, and $\mathcal{N}_t$ is the set of nodes at time t. The set of children nodes of node $n \in \mathcal{N} \setminus \mathcal{N}_T$ is denoted by $\{n^+\}$. The time interval between node n and its children nodes $\{n^+\}$ is Δt. At each node $n \in \mathcal{N}_t$, R_n is the equity log-return over a Δt-long time interval ending at time t (Eq. (2)); r_n is the equity log-return over a month ending at time t (Eq. (7)); and $\beta_{1,n}$, $\beta_{2,n}$, $\beta_{3,n}$ are the Nelson–Siegel term structure parameters at time t (Eq. (6)).

We derive the moments of the conditional distributions of the five state variables $(R_{n^+}, r_{n^+}, \beta_{1,n^+}, \beta_{2,n^+}, \beta_{3,n^+})$ on its unique parent node n. Let H be the number of monthly time steps of our vector autoregressive (VAR) model in Eq. (7) which matches the time interval Δt between the parent node and its children nodes in the scenario tree.

Eqs. (A.1) and (A.2) show the first two conditional moments of the accumulated equity log-return and Nelson–Siegel parameters. Barberis (2000) and Pedersen et al. (2013) apply these equations to scenario generation for their asset–liability management model. Let $\zeta_{n^+} = [R_{n^+}, \beta_{1,n^+}, \beta_{2,n^+}, \beta_{3,n^+}]'$ and $z_n = [r_n, \beta_{1,n}, \beta_{2,n}, \beta_{3,n}]'$, then we have the following equations:

$$\begin{aligned}\mathbb{E}(\zeta_{n^+} \mid z_n) &= \left(\left(\sum_{h=1}^{H-1} (I + J(H-h))\, {\Phi_1}^{h-1} \right) + {\Phi_1}^{H-1} \right) \Phi_0 \\ &\quad + \left({\Phi_1}^{H} + \sum_{h=1}^{H-1} J {\Phi_1}^{h} \right) z_n \qquad \text{(A.1)}\end{aligned}$$

$$\begin{aligned}\mathbb{V}(\zeta_{n^+} \mid z_n) &= \Sigma_z + (J + \Phi_1)\, \Sigma_z\, (J + \Phi_1)' \\ &\quad + \left(J + J\Phi_1 + {\Phi_1}^2\right) \Sigma_z \left(J + J\Phi_1 + {\Phi_1}^2\right)' \\ &\quad + \cdots + \left({\Phi_1}^{H-1} + \sum_{h=1}^{H-1} J {\Phi_1}^{h-1} \right) \\ &\quad \times \Sigma_z \left({\Phi_1}^{H-1} + \sum_{h=1}^{H-1} J {\Phi_1}^{h-1} \right)', \qquad \text{(A.2)}\end{aligned}$$

where $J = diag([1, 0, 0, 0])$. Note that Φ_0, Φ_1 and Σ_z are the coefficients and covariance matrices from the VAR model (Eq. (7)). If $J = 0$ (a 4×4 matrix of zeros), then Eqs. (A.1) and (A.2) simplify to $\mathbb{E}(z_{n^+} \mid z_n)$ and $\mathbb{V}(z_{n^+} \mid z_n)$ respectively. In addition, we can evaluate the covariance $\sigma_{R,r}$ between R_{n^+} and r_{n^+}:

$$\begin{aligned}\sigma_{R,r} &= J^{(1)} \Sigma_z J^{(1)'} + \Phi_1^{(1)} \Sigma_z \left(J^{(1)} + \Phi_1^{(1)} \right)' \\ &\quad + \left(\Phi_1^{(1)} \Phi_1 \right) \Sigma_z \left(J^{(1)} + \Phi_1^{(1)} + \Phi_1^{(1)} \Phi_1 \right)' + \cdots \\ &\quad + \left(\Phi_1^{(1)} {\Phi_1}^{H-2} \right) \Sigma_z \left(J^{(1)} + \sum_{h=2}^{H} \Phi_1^{(1)} {\Phi_1}^{H-h} \right)', \qquad \text{(A.3)}\end{aligned}$$

where $\Phi_1^{(1)}$ is the first row of Φ_1 and $J^{(1)}$ is the first row of J.

Aggregating the moments information from Eq. (A.1) to Eq. (A.3), the conditional expectations and covariances of the five state variables $(R_{n^+}, r_{n^+}, \beta_{1,n^+}, \beta_{2,n^+}, \beta_{3,n^+})$ on its unique parent node n can be evaluated. These are used, as described in Section 4.1, to generate the scenario tree.

Appendix B. Comparing distributions from VAR model and scenario tree

See Fig. B.5.

References

Barberis, N., 2000. Investing for the long run when returns are predictable. J. Finance 55 (1), 225–264.

Boender, G., Dert, C., Heemskerk, F., Hoek, H., 2008. Chapter 18 - A scenario approach of ALM. North-Holland, San Diego, pp. 829–860.

Cairns, A.J.G., Blake, D., Dowd, K., 2006. Stochastic lifestyling: Optimal dynamic asset allocation for defined contribution pension plans. J. Econom. Dynam. Control 30 (5), 843–877.

Campbell, J.Y., Chan, Y.L., Viceira, L.M., 2003. A multivariate model of strategic asset allocation. J. Financ. Econ. 67 (1), 41–80.

Consigli, G., Iaquinta, G., Moriggia, V., di Tria, M., Musitelli, D., 2012. Retirement planning in individual asset-liability management. IMA J. Manage. Math. 23 (4), 365–396.

Daga, A., Schlanger, T., Westaway, P., 2016. Vanguard's approach to target retirement funds in the UK. Technical Report, Vanguard, London, UK, URL: https://www.vanguard.co.uk/documents/adv/literature/trf-research-paper.pdf.

Dempster, M.A.H., Medova, E.A., 2011. Asset liability management for individual households. Br. Actuar. J. 16 (2), 405–439.

Diebold, F.X., Li, C., 2006. Forecasting the term structure of government bond yields. J. Econometrics 130 (2), 337–364.

Duarte, T.B., Valladão, D.M., Veiga, Á., 2017. Asset liability management for open pension schemes using multistage stochastic programming under Solvency-II-based regulatory constraints. Insurance Math. Econom. 77, 177–188.

Ferstl, R., Weissensteiner, A., 2011. Asset-liability management under time-varying investment opportunities. J. Bank. Financ. 35 (1), 182–192.

Geyer, A., Hanke, M., Weissensteiner, A., 2009. Life-cycle asset allocation and consumption using stochastic linear programming. J. Comput. Finance 12 (4), 29–50.

Geyer, A., Hanke, M., Weissensteiner, A., 2010. No-arbitrage conditions, scenario trees, and multi-asset financial optimization. European J. Oper. Res. 206 (3), 609–613.

Geyer, A., Hanke, M., Weissensteiner, A., 2014. No-arbitrage bounds for financial scenarios. European J. Oper. Res. 236 (2), 657–663.

Horneff, W.J., Maurer, R.H., Mitchell, O.S., Dus, I., 2008. Following the rules: Integrating asset allocation and annuitization in retirement portfolios. Insurance Math. Econom. 42 (1), 396–408.

Horneff, W.J., Maurer, R.H., Mitchell, O.S., Stamos, M.Z., 2009. Asset allocation and location over the life cycle with investment-linked survival-contingent payouts. J. Bank. Financ. 33 (9), 1688–1699.

Horneff, W., Maurer, R., Rogalla, R., 2010. Dynamic portfolio choice with deferred annuities. J. Bank. Financ. 34 (11), 2652–2664.

Høyland, K., Wallace, S.W., 2001. Generating scenario trees for multistage decision problems. Manage. Sci. 47 (2), 295–307.

Huang, H., Milevsky, M.A., Young, V.R., 2017. Optimal purchasing of deferred income annuities when payout yields are mean-reverting. Rev. Finance 21 (1), 327–361.

Klaassen, P., 2002. Comment on "generating scenario trees for multistage decision problems". Manage. Sci. 48 (11), 1512–1516.

Koijen, R.S.J., Nijman, T.E., Werker, B.J.M., 2011. Optimal annuity risk management. Rev. Finance 15 (4), 799–833.

Konicz, A.K., Mulvey, J.M., 2013. Applying a stochastic financial planning system to an individual: Immediate or deferred life annuities? J. Retire. 1 (2), 46–60.

Konicz, A.K., Pisinger, D., Rasmussen, K.M., Steffensen, M., 2014. A combined stochastic programming and optimal control approach to personal finance and pensions. OR Spectrum 37 (3), 583–616.

Konicz, A.K., Pisinger, D., Weissensteiner, A., 2016. Optimal retirement planning with a focus on single and joint life annuities. Quant. Finance 16 (2), 275–295.

Maurer, R., Mitchell, O.S., Rogalla, R., Kartashov, V., 2013. Lifecycle portfolio choice with systematic longevity risk and variable investment-linked deferred annuities. J. Risk Insurance 80 (3), 649–676.

Merton, R.C., 2014. The crisis in retirement planning. Harv. Bus. Rev. 92 (7), 42–50.

Milevsky, M.A., Young, V.R., 2007. Annuitization and asset allocation. J. Econom. Dynam. Control 31 (9), 3138–3177.

Pedersen, A.M.B., Weissensteiner, A., Poulsen, R., 2013. Financial planning for young households. Ann. Oper. Res. 205 (1), 55–76.

Scott, J.S., 2008. The longevity annuity: An annuity for everyone? Financ. Anal. J. 64 (1), 40–48.

U.S. Treasury Department, 2014. Treasury issues guidance to encourage annuities in 401(k) plans. p. 1, https://www.treasury.gov/press-center/press-releases/Pages/jl2673.aspx.