# Unravelling the contribution of financial and longevity risks to changes over time in life annuities

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# life annuities

# interest or mortality?

$$\bar{a}_x(t) = \int_0^\infty {}_sp_x(t)v(s,t)ds$$

$$\underbrace{e^{-\int_0^s \mu(x+y,t)dy}}_{\text{mortality}}$$

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$$\boxed{\dot{\bar{a}}_x(t) = \frac{\partial \bar{a}_x(t)}{\partial t}}$$

# Duration

# Duration: sensitivity of $\bar{a}_x(t)$ to constant changes in $\delta$

$$D_x(t) = -\frac{\frac{\partial \bar{a}_x(t)}{\partial \delta}}{\bar{a}_x(t)}$$

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How annuities respond to **changes in interest rates**? (Milevsky, 2013; Charupat, Kamstra, Milevsky, 2015)

# What about changes in mortality?

# Entropy

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They showed that

$$H_x(t) = \frac{\int_0^\infty \mu(x+s,t)_s p_x(t) v(s,t) \bar{a}_x(t) ds}{\bar{a}_x(t)}.$$

# CHANGES OVER TIME IN $\mu(x, t)$ AND $\delta(s, t)$

**Rate of mortality improvement**

$$\rho(x, t) = -\frac{\frac{\partial \mu(x,t)}{\partial t}}{\mu(x,t)}.$$

**Change in the term-structure of interest rates**

$$\varphi(s, t) = -\frac{\frac{\delta(s,t)}{\partial t}}{\delta(s,t)}.$$

# Changes over time in $\bar{a}_x(t)$

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**Relative derivative of $\bar{a}_x(t)$:**

$$\acute{\bar{a}}_x(t) = \frac{\dot{\bar{a}}_x(t)}{\bar{a}_x(t)}$$

# Decomposing $\acute{\bar{a}}_x(t)$

# DECOMPOSITION OF CHANGES OVER TIME IN $\bar{a}_x(t)$

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**Changes over time** in $\bar{a}_x(t)$ are driven by $\bar{\varphi}(t)$ and $\bar{\rho}(t)$, which are **modulated** by $D_x(t)$ and $H_x(t)$.

# Age and term attribution

# AGE AND TERM DECOMPOSITION

$$\bar{\acute{a}}_x(t) = \underbrace{\sum_{j=1}^{m} \bar{\varphi}(t; t_{j-1}, t_j) D_x(t; t_{j-1}, t_j)}_{\text{financial component}} + \underbrace{\sum_{i=1}^{n} \bar{\rho}_x(t; x_{i-1}, x_i) H_x(t; x_{i-1}, x_i)}_{\text{longevity component}},$$

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where

- $\bar{\rho}_x(t; x_{i-1}, x_i)$: weighted average improvement in the age group $[x_{i-1}, x_i)$,
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Analogous to term-attribution (Daul, Sharp and Sørensen, 2012) and key-durations (Ho, 1992) in fixed income.

# Cause of death decomposition

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where,

- $H_x^i(t)$:the cause-specific entropy,
- $\bar{\rho}_x^i(t)$: average rate of mortality improvement of cause $i$.

# Assuming a single interest rate $\delta(t)$

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It suffices to use the **modified duration** ($D_x(t)$) and **entropy** ($H_x(t)$) together with $\dot{\delta}(t)$ and $\bar{\rho}(t)$ to determine the contribution of **financial and longevity risks** to changes over time in **life annuities**.

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**No assumptions about the functional form of $\delta$ and $\mu$ (entirely data-driven).**

| Decomposition | Financial component | Longevity component |
|---|---|---|
| General | $\bar{\varphi}(t)D_x(t)$ | $\bar{\rho}(t)H_x(t)$ |
| Age-Term | $\sum_{j=1}^{m}\bar{\varphi}(t;t_{j-1},t_j)D_x(t;t_{j-1},t_j)$ | $\sum_{i=1}^{n}\bar{\rho}_x^i(t)H_x^i(t)$ |
| Cause of death | $\bar{\varphi}(t)D_x(t)$ | $\sum_{i=1}^{n}\bar{\rho}_x(t;x_{i-1},x_i)H_x^p(t;x_{i-1},x_i)$ |
| Single $\delta$ | $\dot{\delta}(t)D_x(t)$ | $\bar{\rho}(t)H_x(t)$ |

# Changes in life annuities in the UK

## Historical data

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## Yield curve

- UK government bonds, also known as Gilts (Bank of England, 2021),
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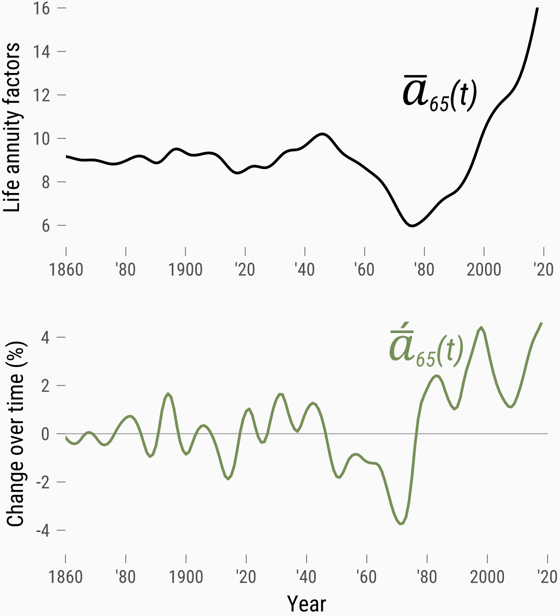
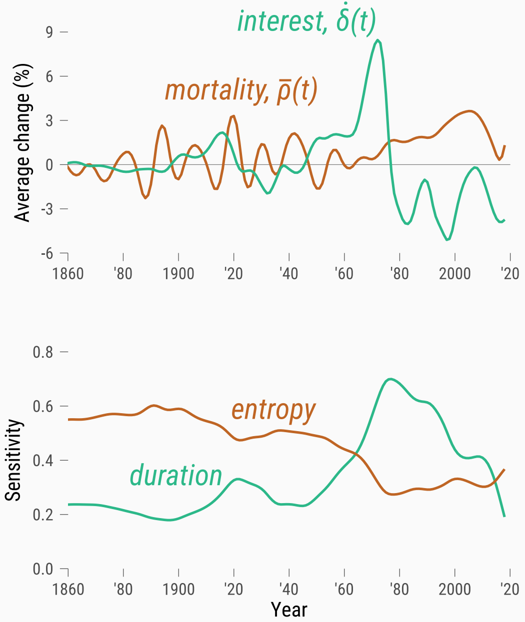
# Life annuity factors

# LIFE ANNUITY FACTORS AT AGE 65. MALES, 1841-2018



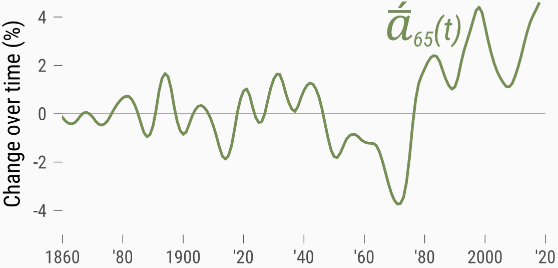
# Changes over time and sensitivities

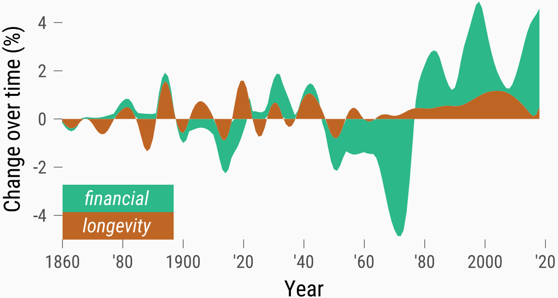
# CHANGES OVER TIME AND SENSITIVITIES. MALES, 1841-2018



# Assuming a single $\delta(t)$

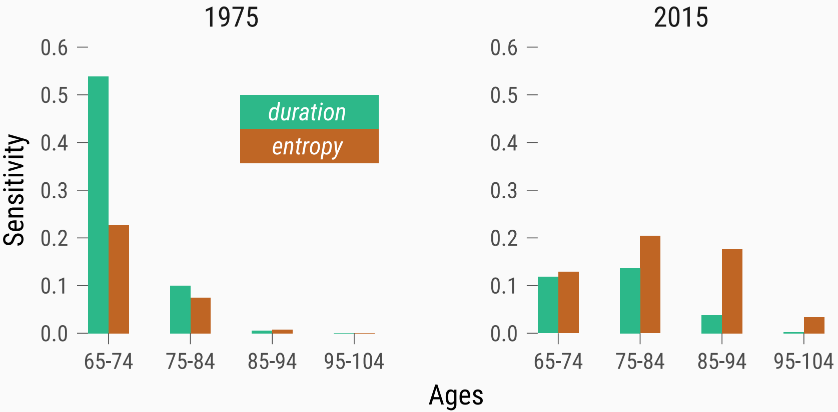
# DECOMPOSITION OF $\bar{\acute{a}}_x(t)$ AT AGE 65. MALES, 1841-2018



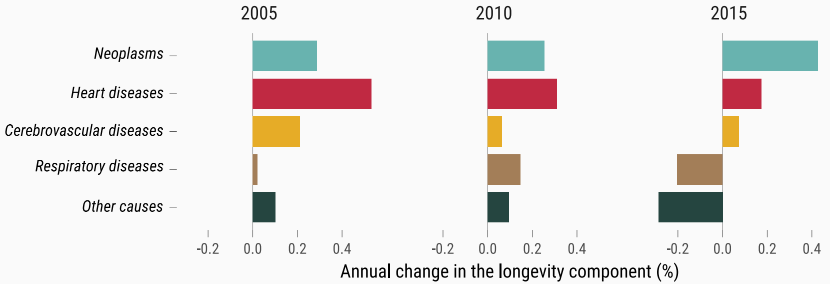
# Age and term attribution

# AGE AND TERM ATTRIBUTION. MALES, 1970-2018



# Causes of death

# CAUSE OF DEATH ATTRIBUTION



# To sum up

## Bringing both perspectives together

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$\rightarrow$ Age-term attribution, causes of death, single $\delta(t)$,

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- **At higher ages** (i.e. age 75 or older ages):
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  - Policies aiming at **increasing retirement ages entail higher longevity risk** (e.g. Denmark, Alvarez et al (2021)).

# Next steps

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# NEXT STEPS

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  - $\bar{\rho}(t)$ and $\bar{\varphi}(t)$ can be modelled as **stochastic processes**,
  - Interest rates: Cox-Ingersol-Ross (CIR), Vasicek,

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# NEXT STEPS

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- Sources of change in the reserve $V(t)$ in a **market-consistent framework** (Møller and Steffensen, 2007).

# Unravelling the contribution of financial and longevity risks to changes over time in life annuities

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**Rate of mortality improvement**

$$\rho(x,t) = -\frac{\frac{\mu(x,t)}{\partial t}}{\mu(x,t)} = -\frac{\dot{\mu}(x,t)}{\mu(x,t)}. \tag{1}$$

**Change in interest rates over time**

$$\varphi(s,t) = -\frac{\frac{\delta(s,t)}{\partial t}}{\delta(s,t)} = -\frac{\dot{\delta}(s,t)}{\delta(s,t)}. \tag{2}$$

**Entropy**

$$H_x^p(t) = \frac{\int_0^\infty \mu(x+s,t)_s|\bar{a}_x(t)ds}{\bar{a}_x(t)} \quad (3)$$

**Duration**

$$D_x^p(t) = \frac{\int_0^\infty \delta(s,t)_s|\bar{a}_x(t)ds}{\bar{a}_x(t)} \quad (4)$$

**Time derivative of $\bar{a}_x(t)$**

$$\dot{\bar{a}}_x(t) = \int_0^\infty \rho(s,t)\mu(s,t)_s|\bar{a}_x(t)ds + \int_0^\infty \varphi(s,t)\delta(s,t)_s|\bar{a}_x(t)ds$$
$$= \int_0^\infty \rho(s,t)_s M_x(t)ds + \int_0^\infty \varphi(s,t)_s W_x(t)ds,$$

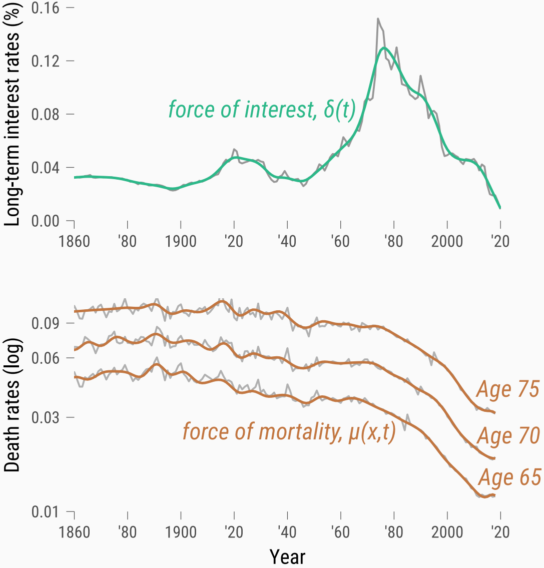
where ${}_sM_x(t) = \mu(s,t)_s|\bar{a}_x(t)$ and ${}_sW_x(t) = \delta(s,t)_s|\bar{a}_x(t)$.

**Relative derivative of $\bar{a}_x(t)$**

$$\acute{\bar{a}}_x(t) = \frac{\dot{\bar{a}}_x(t)}{\bar{a}_x(t)} = \underbrace{\bar{\rho}(t)H_x^p(t)}_{\text{longevity component}} + \underbrace{\bar{\varphi}(t)D_x^p(t)}_{\text{financial component}},$$

where $\bar{\rho}_x(t) = \frac{\int_0^\infty \rho(x+s,t)_s M_x(t)ds}{\int_0^\infty {}_sM_x(t)ds}$ and $\bar{\varphi}(t) = \frac{\int_0^\infty \varphi(s,t)_s W_x(t)ds}{\int_0^\infty {}_sW_x(t)ds}$

# INTEREST AND MORTALITY RATES FOR MALES IN THE UK, 1841-2018



# AGE AND TERM ATTRIBUTION. MALES, 1970-2018