Changes of Long-term Relation in Multi-population Mortality Modeling

Rui Zhou   Guangyu Xing

Warren Centre for Actuarial Studies and Research
University of Manitoba

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Background

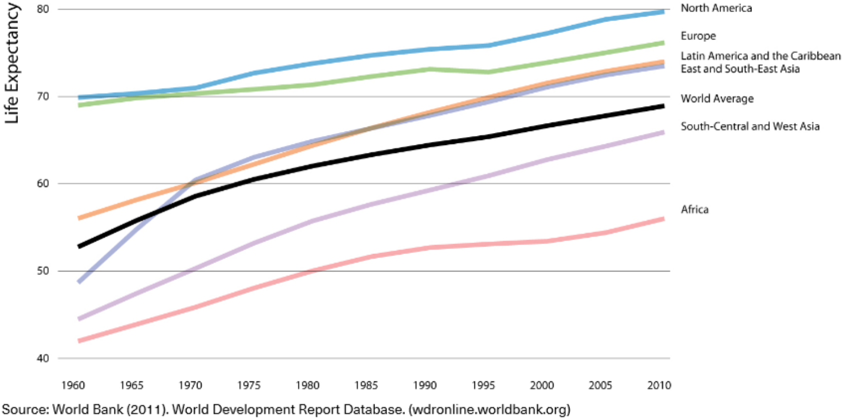


Figure 1: Global Life Expectancy

Background

Longevity-linked securities

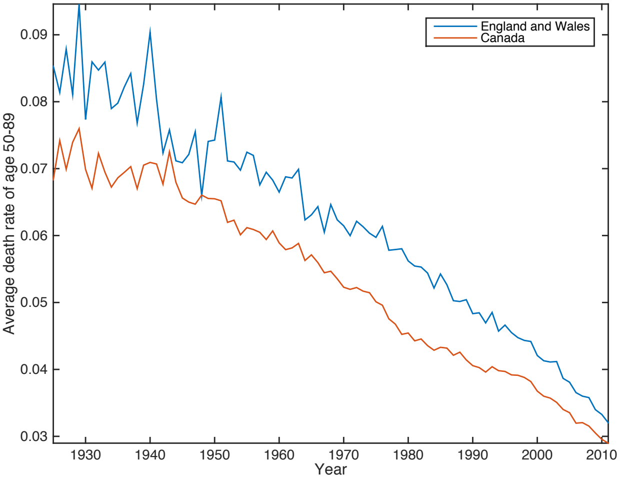
- q-forward, s-forward
- Longevity bond
- Longevity swap

Why multi-population mortality models?

- The longevity/mortality index underlying the security may involve multiple populations.
- The population underlying pension plan or annuity portfolio is different from that associated with the security.

The long-term equilibrium

- Yang et al. (2013) found a long-term relation between mortality rates of multiple populations.



Objectives

- Test and model changes in the long-term relation
- Examine the impact of incorporating these changes on longevity bond pricing

Data

Populations: England and Wales (EW), Canada (CAN)
Age: 50 to 89
Period: 1925 to 2011

Lee-Carter Model (Lee and Carter, 1992)

$$\ln(m_{x,t}) = a_x + b_x k_t + e_{x,t}$$

- $m_{x,t}$: central death rate at age x in year t
- a_x: age effect
- b_x: age sensitivity to period effect
- k_t: period effect
- $e_{x,t}$: error term

Lee-Carter Model

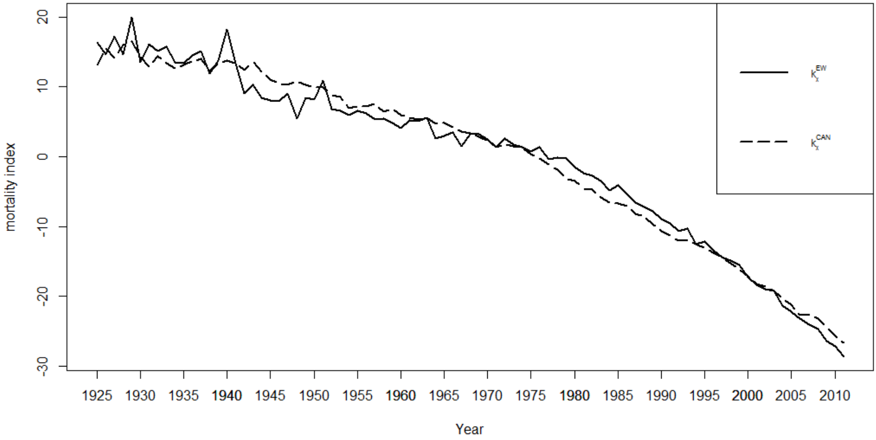


Figure 2: Period effect estimates of EW and CAN

Vector Error Correction Model (VECM, Engle and Granger, 1987)

$$\begin{pmatrix} \Delta K_t^{EW} \\ \Delta K_t^{CAN} \end{pmatrix} = c + \alpha\left(K_{t-1}^{EW} - \beta K_{t-1}^{CAN}\right) + \sum_{i=1}^{p-1} \Gamma_i \begin{pmatrix} \Delta K_{t-i}^{EW} \\ \Delta K_{t-i}^{CAN} \end{pmatrix} + \begin{pmatrix} \varepsilon_t^{EW} \\ \varepsilon_t^{CAN} \end{pmatrix}$$

- c : 2×1 constant vector.
- α: adjustment coefficient, 2×1 vector
- β: cointegrating value, constant
- Γ_i: 2×2 matrix
- $\begin{pmatrix} \varepsilon_t^{EW} \\ \varepsilon_t^{CAN} \end{pmatrix}$: follows bivariate normal distribution

Diagnostic checking

	Test statistic	p-value
Mardia's Test (skewness)	0.6813	0.071
Mardia's Test (kurtosis)	10.8636	0.0018
Henze-Zirkler Test	1.5316	0.0012

Table 1: Normality test of residuals from VECM

Structural Change test (Seo, 1998)

Test	Method	Statistic	Critical Value	
			Sig=0.01	Sig=0.05
β	$\text{Ave} - \text{LM}_n^{\beta}$	0.7578726	3.37	2.25
	$\text{Exp} - \text{LM}_n^{\beta}$	0.4561919	3.22	2.02
	$\text{Sup} - \text{LM}_n^{\beta}$	2.902988	9.04	10.52
α	$\text{Ave} - \text{LM}_n^{\alpha}$	**14.26191**	6.07	4.29
	$\text{Exp} - \text{LM}_n^{\alpha}$	**9.842654**	4.72	3.27
	$\text{Sup} - \text{LM}_n^{\alpha}$	**22.90245**	16.44	12.93

Table 2: LM Tests for Structural Change

Threshold VECM

One threshold case:

$$\begin{pmatrix} \Delta k_t^{EW} \\ \Delta k_t^{CAN} \end{pmatrix} = \begin{cases} c_1 + \alpha_1 \left(k_{t-1}^{EW} - \beta k_{t-1}^{CAN} \right) + \sum_{i=1}^{p-1} \Gamma_{1,i} \begin{pmatrix} \Delta k_{t-i}^{EW} \\ \Delta k_{t-i}^{CAN} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1,t}^{EW} \\ \varepsilon_{1,t}^{CAN} \end{pmatrix}, & z_{t-1} > \gamma \\ c_2 + \alpha_2 \left(k_{t-1}^{EW} - \beta k_{t-1}^{CAN} \right) + \sum_{i=1}^{p-1} \Gamma_{2,i} \begin{pmatrix} \Delta k_{t-i}^{EW} \\ k_{t-i}^{CAN} \end{pmatrix} + \begin{pmatrix} \varepsilon_{2,t}^{EW} \\ \varepsilon_{2,t}^{CAN} \end{pmatrix}, & z_{t-1} \leq \gamma \end{cases}$$

- $z_t = k_t^{EW} - \beta k_t^{CAN}$ is the threshold variable
- γ is the threshold value

Parameter estimates

$$\begin{pmatrix} \Delta k_t^{EW} \\ \Delta k_t^{CAN} \end{pmatrix} = \begin{pmatrix} -0.9523 \\ -0.7182 \end{pmatrix} + \begin{pmatrix} -0.4883 \\ 0.0296 \end{pmatrix} \left(k_{t-1}^{EW} - 0.9917 k_{t-1}^{CAN} \right)$$
$$+ \begin{pmatrix} -0.4656 & 0.3396 \\ -0.0207 & -0.0486 \end{pmatrix} \begin{pmatrix} \Delta k_{t-1}^{EW} \\ \Delta k_{t-1}^{CAN} \end{pmatrix} + \begin{pmatrix} -0.8236 & -0.3108 \\ -0.1349 & -0.1925 \end{pmatrix} \begin{pmatrix} \Delta k_{t-2}^{EW} \\ \Delta k_{t-2}^{CAN} \end{pmatrix}$$
$$+ \begin{pmatrix} -0.5074 & 0.078 \\ -0.0711 & 0.2052 \end{pmatrix} \begin{pmatrix} \Delta k_{t-3}^{EW} \\ \Delta k_{t-3}^{CAN} \end{pmatrix} + \begin{pmatrix} \varepsilon_t^{EW} \\ \varepsilon_t^{CAN} \end{pmatrix}, \qquad z_{t-1} > -0.1132$$

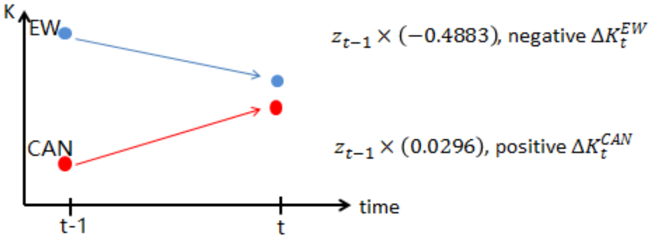
$$\begin{pmatrix} \Delta k_t^{EW} \\ \Delta k_t^{CAN} \end{pmatrix} = \begin{pmatrix} 0.1163 \\ -0.0796 \end{pmatrix} + \begin{pmatrix} -0.1885 \\ 0.0981 \end{pmatrix} \left(k_{t-1}^{EW} - 0.9917 k_{t-1}^{CAN} \right)$$
$$+ \begin{pmatrix} -0.2404 & 0.4099 \\ -0.0194 & 0.1105 \end{pmatrix} \begin{pmatrix} \Delta k_{t-1}^{EW} \\ \Delta k_{t-1}^{CAN} \end{pmatrix} + \begin{pmatrix} 0.2771 & 0.1222 \\ 0.0968 & 0.1481 \end{pmatrix} \begin{pmatrix} \Delta k_{t-2}^{EW} \\ \Delta k_{t-2}^{CAN} \end{pmatrix}$$
$$+ \begin{pmatrix} 0.0323 & 0.7685 \\ 0.1537 & -0.0212 \end{pmatrix} \begin{pmatrix} \Delta k_{t-3}^{EW} \\ \Delta k_{t-3}^{CAN} \end{pmatrix} + \begin{pmatrix} \varepsilon_t^{EW} \\ \varepsilon_t^{CAN} \end{pmatrix}, \qquad z_{t-1} < -0.1132$$

Model interpretation

Upper regime: $z_{t-1} > 0$

$$\begin{pmatrix} -0.4883 \\ 0.0296 \end{pmatrix} \left(K_{t-1}^{EW} - 0.9917 K_{t-1}^{CAN} \right)$$

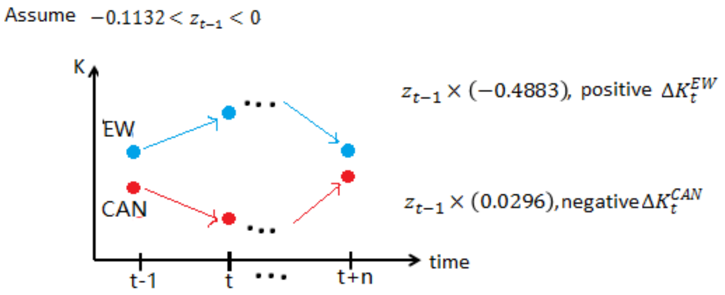
Assume $z_{t-1} > 0$:



Model interpretation

Upper regime: $-0.1132 < z_{t-1} < 0$

$$\begin{pmatrix} -0.4883 \\ 0.0296 \end{pmatrix} \left(K_{t-1}^{EW} - 0.9917 K_{t-1}^{CAN} \right)$$

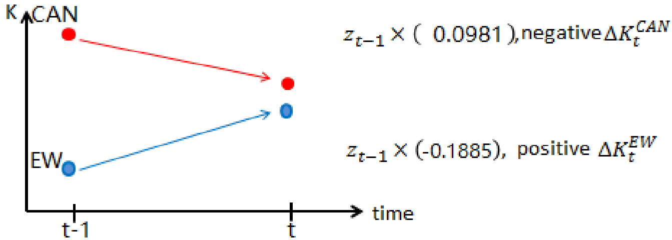


Model interpretation

Down regime: $z_{t-1} < -0.1132$

$$\begin{pmatrix} -0.1885 \\ 0.0981 \end{pmatrix} \left(K^{EW}_{t-1} - 0.9917 K^{CAN}_{t-1} \right)$$

Assume $z_{t-1} < 0$:



Diagnostic Checking

	Test statistic	p-value
Mardia's Test (skew)	3.387762	0.4951482
Mardia's Test (kurtosis)	1.135402	0.2562069
Henze-Zirkler Test	0.5548691	0.4409921

Table 3: Normality test of Residuals

Longevity Bond

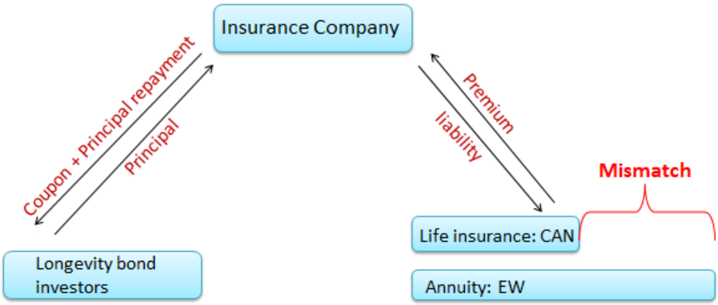


Figure 3: Overview of longevity bond

- The bond is traded at its face value.
- Principal is subject to erosion.

Longevity Bond Structure

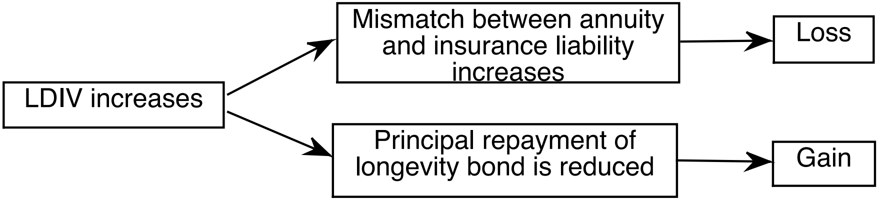
$$\begin{aligned} & \text{Longevity Divergence Index Value (LDIV)} \\ = \ & \text{Mortality improvement of EW annuitants} \\ & -\text{Mortality improvement of CAN insured} \end{aligned}$$

$$\text{Principal reduction factor (PRF)}$$

$$= \begin{cases} 0, & \text{LDIV} < ap \\ \frac{\text{LDIV}-ap}{ap-ep}, & ap < \text{LDIV} < ep \\ 1, & \text{LDIV} > ep \end{cases}$$

where *ap* is attachment point, and *ep* is exhaustion point.

Longevity Bond Structure



Numerical Results

- $ap = 0.065\%$ and $ep = 0.237\%$
- Use multivariate Wang transform to obtain risk-neutral measure
- λ is the market price of risk

		Spread over LIBOR
TVECM	$\lambda = 0$	3.0576%
	$\lambda = -0.1$	3.248%
	$\lambda = -0.3$	3.99949%
	$\lambda = -0.5$	5.371%
VECM	$\lambda = 0$	4.0926%
	$\lambda = -0.1$	4.228%
	$\lambda = -0.3$	5.1314%
	$\lambda = -0.5$	6.638%

Table 4: Spread over 3-month LIBOR using different market prices of risk

Conclusion

- The long-term equilibrium between k_t^{EW} and k_t^{CAN} experienced changes in the past.

- Incorporating potential changes in the long-term equilibrium has a significant impact on the pricing of longevity bond.