Multi-population mortality modelling with Lévy processes

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Introduction

- When hedge is made using a published mortality index, the risk of an imperfect hedging arises due to different realized longevity experience between different populations.
- To measure and manage this risk, multi-population mortality models are applied as give joint characterization of the mortality experience of different populations.
- We proposed a multi-population model with deterministic component modelled via Poisson Generalized Linear Models and correlated stochastic perturbation driven by mixed Lévy processes.
- Ahmadi & Gaillardetz (2015) use a Poisson Generalized Liner Model coupled of Gamma and Variance-Gamma processes to model perturbations and we generalize their approach to multi-population setting.

- In our study, the underlying process is assume to be composed by an idiosyncratic factor and a common factor.
- Each population shares the same common factor but has different idiosyncratic factor.
- We analyse the effect of common factor through different age by locating a loading coefficient on common factor.
- The idiosyncratic factor is assumed as Gamma or Variance Gamma process and common factor as Wiener process.
- The correlation coefficients between populations in different calender year at different ages can be obtained in close-form.

Stochastic Mortality Model

- x is age, t is calendar year.
- $d(x,t)$ is death number, $r(x,t)$ is exposure risk and $\mu(x,t)$ is force of mortality.
- $d(x,t)$ follows Poisson distribution.
- $\mathbb{E}[d(x,t)] = \mu(x,t) \cdot r(x,t)$.
- $\mathbb{E}[d(x,t)|Y_{t'}] =$
  $r(x,t)\exp\left[\beta_0 + \sum_{j=1}^{s}\beta_j L_j(x') + \sum_{i=1}^{r}\alpha_i t'^i + \sum_{i=1}^{k}\sum_{j=1}^{l}\gamma_{ij}L_j(x')t'^i + \theta Y_{t'}\right]$
  where x, t are bounded, L_j is Legendre polynomial. (Ahmadi & Gaillardetz, 2015).
- Transformations $x' = \frac{2x-\max(x)-\min(x)}{\max(x)-\min(x)}$, $t' = \frac{2t-\max(t)-\min(t)}{\max(t)-\min(t)} \in [-1,1]$.

- $Y_{t'}$ is a mean-reverting process driven by a Lévy process $Z_{t'}$

$$dY_{t'} = -aY_{t'}dt' + \omega dZ_{t'},\ t' > -1,$$

with the starting value $Y_{-1} = 0$.

- The unconditional expectation of number of deaths $d(x,t)$

$$\mathbb{E}[d(x,t)] = \mathbb{E}[\mathbb{E}[d(x,t)|Y_{t'}]] = r(x,t)e^{LM} \cdot \mathbb{E}[\exp(\theta Y_{t'})],$$

where LM represents the linear model

$$\beta_0 + \sum_{j=1}^{s} \beta_j L_j(x') + \sum_{i=1}^{r} \alpha_i t'^i + \sum_{i=1}^{k}\sum_{j=1}^{l} \gamma_{ij} L_j(x') t'^i.$$

- $\mathbb{E}[\exp(\theta Y_{t'})] = \exp(\int_{-1}^{t'} k(e^{-a(t'-u)})du)$ by assuming $\theta = \frac{1}{\omega}$. (Eberlein & Raible, 1999)
- $k(\cdot)$ is cumulant transform of variable Z_1 defined as

$$k(u) = \log \mathbb{E}[(\exp(uZ_1))].$$

Multi-population mortality model driven by mixed Lévy processes

- $U_t^{(i)}$ as idiosyncratic factor for i^{th} population, V_t as common factor.
- $Z_t^{(i)} = U_t^{(i)} + c_i V_t$, c_i is loading coefficient.
- Luciano & Semeraro (2010) proposed multi-dimensional Lévy processes

$$\mathbf{Z}_t = \{U_t^{(1)} + c_1 V_t, ..., U_t^{(n)} + c_n V_t\}^T,$$

- Choose $U_t^{(1)}, ..., U_t^{(n)}$ to be mutually independent and also independent with the common factors V_t.
- V_t as Wiener process, with marginal distribution $N(0, \sigma^2)$.
- $U_t^{(i)}$ as Gamma process $G(\mu^{(i)}, \lambda^{(i)})$ or Variance Gamma process $VG(\eta^{(i)}, \frac{1}{\mu^{(i)}}, \theta^{(i)})$.

Loadings on Common Factor

- Loading factor $c(x)$ is assumed as

$$c(x) = \frac{e^{\delta_0+\delta_1(x'+1)}}{1+e^{\delta_0+\delta_1(x'+1)}}.$$

- $0 < c(x) < 1$.

Data Description

- The mortality dataset being considered in this article includes UK males and females, both calender years 1984-2013 and both ages 50-100, obtained from Human Mortality Data (www.mortality.org).
- We make the plots for crude mortality rates $q(x,t)$ of both male and female, for the ages 55, 65, 75, 85 and 95.

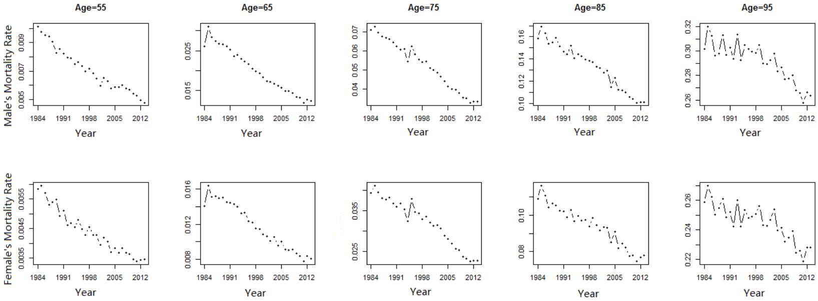


Figure. Crude mortality rate for UK male and female during 1984-2013

Model Estimation

- To manipulate model selection and parameters estimation, we look for the optimal values by minimizing residual deviance.
- The multi-population residual deviance is derived as

$$2\sum_x\sum_t\left(d^{(i)}(x,t)\log\left(\frac{d^{(i)}(x,t)}{\hat{d}^{(i)}(x,t)}-[d^{(i)}(x,t)-\hat{d}^{(i)}(x,t)]\right)\right).$$

- Then by this criterion, we can find the estimates of the parameters by three steps.

Step 1

- Determine the number of terms of α, β, γ without considering the Lévy perturbations and remove statistically insignificant coefficients.
- To avoid redundancy of parameters, we search the each of optimum values for (r, s, k, l) only from 0 to 8 and obtain

$$r_m = 4, s_m = 2, k_m = 4, l_m = 2 \text{ for male population,}$$
$$r_f = 4, s_f = 1, k_f = 3, l_f = 3 \text{ for female population.}$$

- Remove insignificant coefficients UK's male and female populations.

Step 2

- Remove the corresponding coefficients to make the plot of fitted lines of force of mortality satisfies the necessary requirements: 1. When age is fixed, the projection of mortality rate should decrease as time increases;
  2. When time is fixed, the projection of mortality rate should increase as age increases. (Sithole et al.(2000))

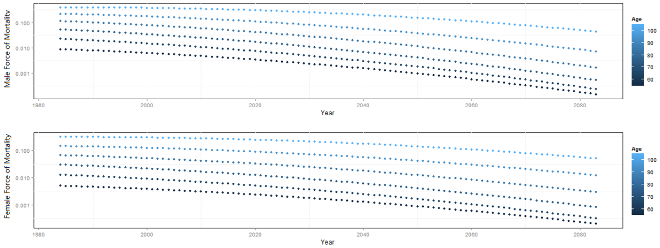


Figure. Plot of fitted force of mortality for UK male and female from 1984 to 2013 and their projections till 2083

Step 3

- Estimate all the parameters from the model by adding the Lévy perturbation.
- We obtain

$$\mu^{(i)}(x,t) = \exp(LM^{(i)}) \cdot \mathbb{E}[\exp(\theta^{(i)} Y_{t'})]$$

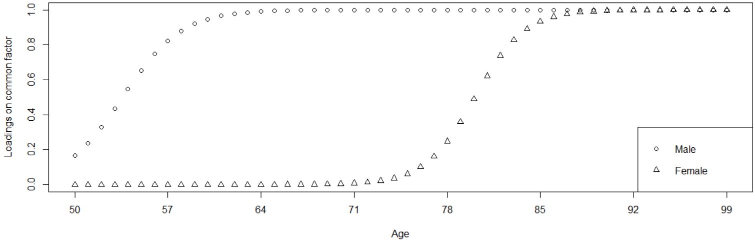
$\mathbb{E}[\exp(\theta^{(i)} Y_{t'})]$ can be calculated based on Gamma Process and Variance Gamma Process, respectively.

In our two-population study, $i \in \{m, f\}$.

Parameters Estimation

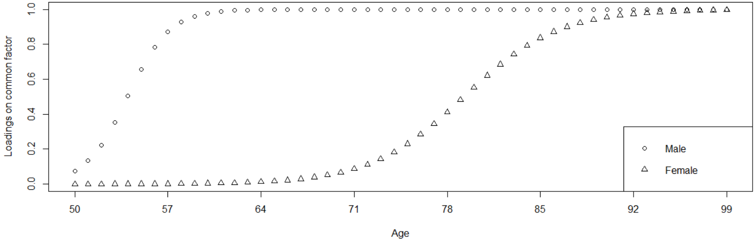
- Mixture of Gamma Process and Wiener Process:
  $\hat{\delta}_0^{(m)} = -1.61, \hat{\delta}_1^{(m)} = 11.02, \hat{\delta}_0^{(f)} = -16.13, \hat{\delta}_1^{(f)} = 13.14, \hat{\sigma} = 2.95$
- Mixture of Variance Gamma Process and Wiener Process:
  $\delta_0^{(m)} = -2.50, \delta_1^{(m)} = 15.43, \delta_0^{(f)} = -8.30, \delta_1^{(f)} = 6.95, \hat{\sigma} = 3.15$

Loading Factors Comparison based on Gamma Process



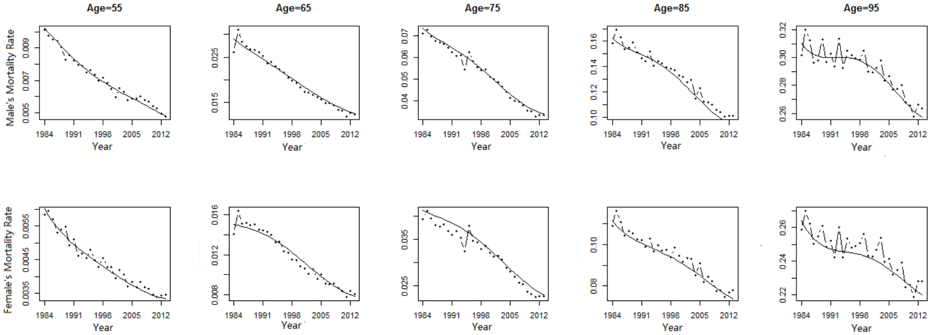
Male: empty dots; Female: overplotted points and lines

Loading Factors Comparison based on Variance Gamma Process

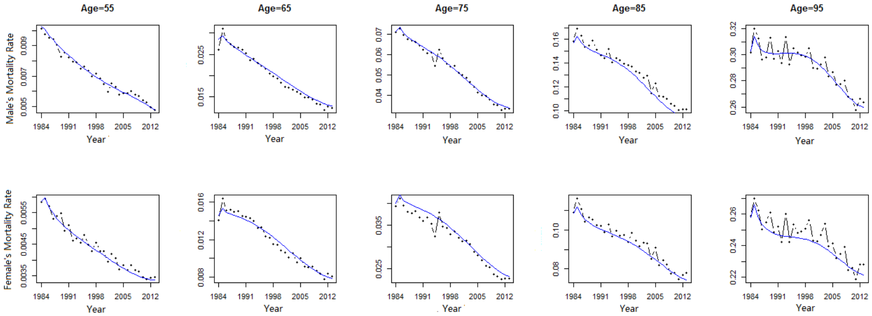


Male: empty dots; Female: overplotted points and lines

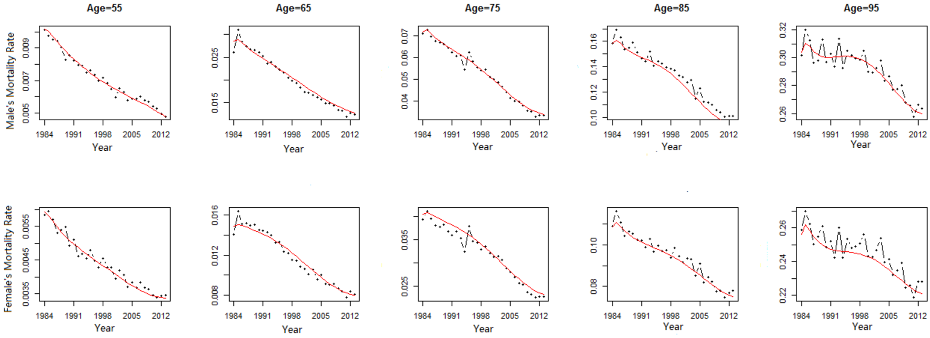
Pure GLM


Age=55
Age=65
Age=75
Age=85
Age=95
Male's Mortality Rate
Female's Mortality Rate
Year
1984
1991
1998
2005
2012

Gamma-Wiener Mixture



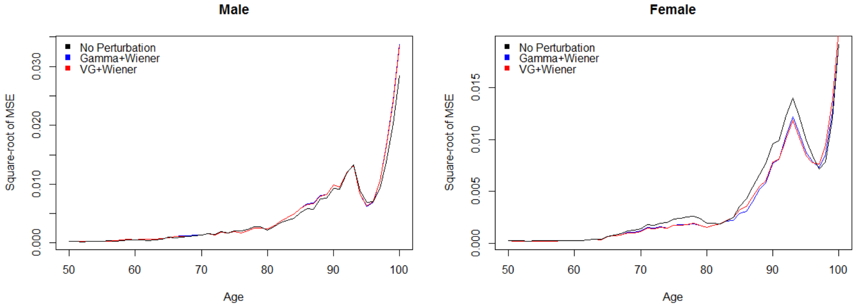
VG-Wiener Mixture



VG-Wiener Mixture

- To compare the model fittings of the three cases mentioned before, we consider the square-root of mean square error which is defined at certain age i by

$$\sqrt{\sum_{t=1984}^{2013} (\text{fitted}_{t,i} - \text{actual}_{t,i})^2/30},$$



Dependence Structure

- We investigate the correlation between expected force of mortality of male and female population and within each population given the underlying processes through different ages at certain years.
- The correlation coefficients can be theoretically expressed in close form from the model settings.

Inter-Population

male \ female	Gamma Process					Variance Gamma Process				
	50	60	70	80	90	50	60	70	80	90
1990										
50	0.9875	0.9648	0.9649	0.9649	0.9441	0.9973	0.9970	0.9970	0.9885	0.9617
60	0.9342	0.9127	0.9130	0.9408	0.9614	0.9451	0.9454	0.9507	0.9828	0.9965
70	0.9274	0.9061	0.9064	0.9361	0.9597	0.9424	0.9428	0.9481	0.9813	0.9961
80	0.9273	0.9060	0.9063	0.9360	0.9597	0.9424	0.9428	0.9481	0.9813	0.9961
90	0.9273	0.9060	0.9063	0.9360	0.9597	0.9424	0.9428	0.9481	0.9813	0.9961
2000										
50	0.9875	0.9648	0.9649	0.9649	0.9441	0.9973	0.9970	0.9970	0.9885	0.9617
60	0.9342	0.9127	0.9130	0.9408	0.9614	0.9451	0.9454	0.9507	0.9828	0.9965
70	0.9274	0.9061	0.9064	0.9361	0.9597	0.9424	0.9428	0.9481	0.9813	0.9961
80	0.9273	0.9060	0.9063	0.9360	0.9597	0.9424	0.9428	0.9481	0.9813	0.9961
90	0.9273	0.9060	0.9063	0.9360	0.9597	0.9424	0.9428	0.9481	0.9813	0.9961
2010										
50	0.9875	0.9648	0.9649	0.9649	0.9441	0.9973	0.9970	0.9970	0.9885	0.9617
60	0.9342	0.9127	0.9130	0.9408	0.9614	0.9451	0.9454	0.9507	0.9828	0.9965
70	0.9274	0.9061	0.9064	0.9361	0.9597	0.9424	0.9428	0.9481	0.9813	0.9961
80	0.9273	0.9060	0.9063	0.9360	0.9597	0.9424	0.9428	0.9481	0.9813	0.9961
90	0.9273	0.9060	0.9063	0.9360	0.9597	0.9424	0.9428	0.9481	0.9813	0.9961

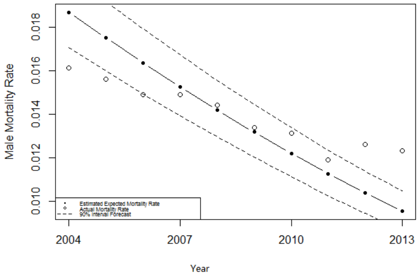
Cross-Population

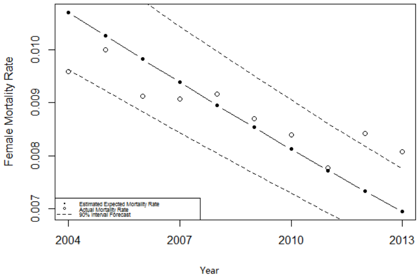
	Gamma					Variance Gamma				
	Male					Male				
	50	60	70	80	90	50	60	70	80	90
50	1	0.9732	0.9693	0.9691	0.9691	1	0.9677	0.9652	0.9652	0.9652
60	0.9732	1	0.9999	0.9998	0.9998	0.9677	1	1.0000	1.0000	1.0000
70	0.9693	0.9999	1	1.0000	1.0000	0.9652	1.0000	1	1.0000	1.0000
80	0.9691	0.9998	1.0000	1	1.0000	0.9652	1.0000	1.0000	1	1.0000
90	0.9691	0.9998	1.0000	1.0000	1	0.9652	1.0000	1.0000	1.0000	1
	Female					Female				
	50	60	70	80	90	50	60	70	80	90
50	1	0.9948	0.9944	0.9888	0.9502	1	1.0000	0.9998	0.9875	0.9576
60	0.9948	1	1.0000	0.9937	0.9634	1.0000	1	0.9999	0.9877	0.9579
70	0.9944	1.0000	1	0.9938	0.9636	0.9998	0.9999	1	0.9903	0.9627
80	0.9888	0.9937	0.9938	1	0.9871	0.9875	0.9877	0.9903	1	0.9907
90	0.9502	0.9634	0.9636	0.9871	1	0.9576	0.9579	0.9627	0.9907	1

Forecasting

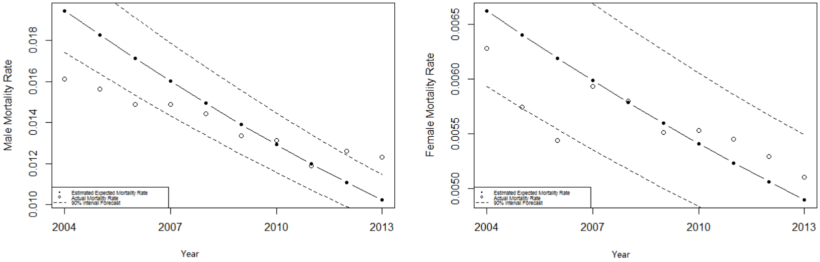
- Training set is selected forty years from 1964 to 2003.
- Test Set is selected from 2004 to 2013 as test data.
- Redo all the procedure, we make plots from 2004 to 2013 including 10 years of model forecasting as well as the corresponding 90 percent confidence intervals.
- The following figures show the forecasting of the age 65.

Gamma Mixture





Variance Gamma Mixture



Conclusions

- The single-population mortality model can be developed into the multi-population mortality model by assuming a mixed perturbation process composed by idiosyncratic and common factors.
- For both male and female, the loading coefficients grow larger when the population's age goes up.
- The values of correlation coefficients are close even if underlying processes are different.
- If we set the loading factor is time dependent, close-form results for correlation coefficients are not available to get but numerical approaches will work.

Thanks for Your Attention.

Reference

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