Optimal Longevity Hedging Strategy for Insurance Companies Considering Basis Risk

Draft Submission to Longevity 10 Conference

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Abstract

This paper proposes the hedging strategy to deal with longevity risk for life insurer considering basis risk. Different to the existing literature, we build up a hedging strategy for longevity risk considering not only the internal natural hedging but also external hedging by using longevity-linked securities. Extending from the immunization theory, we find the optimal hedging strategy based on the insurer's profit function. We take into account basis risk in the hedging strategy and employ Yang and Wang (2013)'s multi-population mortality model to capture the mortality dynamics for annuity and life insurance business. Instead of using population mortality, we adopt a unique data set of annuity and life insurance policies that enable us to calibrate the multi-population mortality dynamics for different lines of insurance policies and calculate their liabilities in the profit function. In addition, we derive the optimal hedging strategy analytically by applying Taylor expansion on the profit function. As a result, the basis risk is examined empirically and numerical analysis on the optimal hedging strategy.

Keywords: Natural Hedging; Longevity Risk; Basis Risk; Longevity-linked securities

1. Introduction

Recent increase in longevity has increased pressures on defined benefit (DB) pension plan providers and annuity provider. Longevity risk has become non-negligible and its influence is increasing gradually and globally. As a result, hedging longevity risks has taken on an increasingly important role for life insurance companies.

The study of hedging strategies for managing longevity risk has been explored in recent years. In general, the hedging strategy can be categorized as an internal or external method. Natural hedging is regarded as the internal hedging strategy that the insurer can hedge longevity risks with their own business products between life insurance and annuity because these two types of products are sensitive in opposing ways to the changes in mortality rates. If the future mortality of a cohort improves relative to current expectations, life insurers gain a profit because they can pay the death benefit later than initially expected, whereas annuity insurers suffer losses because they must pay annuity benefits for longer than they initially expected. Cox and Lin (2007) find the empirical evidence that annuity writing insurers who have more balanced business in life and annuity risks tend to charge lower premiums than otherwise similar insurers and indicates that insurers who have a natural hedge have a competitive advantage. Wang et al.(2010) investigate the natural hedging strategy to deal with longevity risks for life insurance companies and propose an immunization model to investigate the natural hedging strategy by calculating the optimal life insurance–annuity product mix ratio to hedge against longevity risks. Wang et al. (2013) propose a natural hedging model that can account for both the variance and mispricing effects of longevity risk at the same time. Therefore, life insurance can serve as a dynamic hedge vehicle against unexpected mortality risk.

Alternatively, the life insurer and pension provider can seek to hedge longevity risk externally using capital market solutions. Blake and Burrows (2001) first proposed that issuing survivor bonds could help a pension fund insure against the longevity risk. To utilize the capital market for transferring longevity risk, more recent studies focus on the issue of securitization of longevity risk and a variety of survivor securities and survivor derivatives have been developed in both academic and practice ((e.g., Lin and Cox 2005; Cox et al. 2006; Dowd et al. 2006; Blake et al. 2006; Denuit et al. 2007; Biffis and Blake, 2009; Blake et al. 2010; Dawson et al., 2010). For example, the EIB/BNP longevity bond aimed to transfer longevity risk, though it was never ultimately issued. The world’s first capital market derivative transaction, a q-forward contract between JPMorgan and the U.K. company Lucida, took place in January 2008; the first capital market longevity swap, executed in July 2008, enabled Canada Life to hedge its U.K.-based annuity policies. In December 2010, Swiss Re launched a series of eight-year longevity-based insurance-linked security notes, which it called Kortis notes. Blake et al.(2012) note that the emergence of a traded market in longevity-linked capital market instruments would act a catalyst to help facilitate the development of annuity markets.

There are some discussions regarding external and internal hedging strategies for the life insurer. Cox and Lin (2007) suggest that natural hedging is good but may be too expensive to be effective in the context of internal life insurance and annuity products. They show that insurers that exploit natural hedging by using a mortality swap can charge a lower risk premium than others. In addition, the restriction of using the natural hedging strategy for the insurers is that they must adjust the sales volume of life insurance and annuity products to remain an optimal liability proportion which is sometime not feasible in practice. To overcome such restriction, we propose a longevity hedging framework that considers not only the natural hedging but also the external hedging by using longevity-linked securities. Assume a life insurer with both annuity and life insurance business. Greater longevity risk implies that the insurer can earn profits from selling life insurance policies but suffer losses for selling annuity insurers. The annuity providers usually has more annuity policies than life insurance policies in their liability, i.e. the longevity risk cannot be fully natural hedged and the insurer can consider the external hedging by using the longevity-linked securities to deal with the remaining longevity risk. To deal with the proposed hedging strategy that combines both natural hedging and external hedging, we define the profit function according the entire business of life insurance and annuity business and the optimal hedging strategy is obtained based on the insurer's profit function.

Longevity risk exposure may be different in different lines of business and population groups. The great concern in hedging longevity risk is longevity basis risk. Coughlan et al. (2011) point out the importance of basis risk in longevity hedging because the mortality experience may differ from that of life insurance and annuity portfolio, the hedge will be imperfect and leaving a residual amount of risk, known as basis risk. The use of population-based mortality indices for managing the longevity risk inherent in specific blocks of annuitant liabilities may result in basis risk. However, the existing literature on longevity hedging mainly demonstrate with population mortality experience or treating life insurance and annuity business on the same mortality basis. For example, Wang et al.(2010) employ the same mortality rate measure (population mortality rates) for both life insurance and annuity products in finding the optimal product mix. It may happen the mismatch in mortality rates between life insurance and annuity products. Therefore, to hedge longevity risk, it is important to consider basis risk. To fill the gap, we consider basis risk on finding the optimal hedging strategy. Particularly, we deal with basis risk in the following three aspects. First, instead of population mortality data, we employ a unique experience mortality data[1] from life insurance industry that includes the mortality experience for both life insurance and endowment insurance for men and women separately. With this unique mortality experience, it enables us to capture the longevity risk for different product lines to avoid mismatch in mortality rates. Second, we measure the basis risk for the longevity exposed business based on Yang and Wang (2013)'s multi-population mortality framework. This model is calibrated to capture the basis risk between different lines of insurance policies. The empirical analysis of basis risk for life insurance and annuity policy are studied. Third, we take into the multi-population mortality model in the longevity hedging framework and calculate the profit function for different lines of life insurance business. The optimal hedging strategy is obtained according to the profit function. Therefore, we can benefit from the unique experience mortality data to measure longevity risk and model mortality dynamics in the presence of basis risk.

In addition, we consider not only the natural hedging but also the external hedging in the proposed longevity hedging framework. The life insurer can utilize the longevity-linked securities to hedge longevity risk. Thus, the profit function is calculated based on both the cash flow of the insurer's liability and the cash flow of the longevity-linked securities. The stochastic mortality dynamic is considered to capture the longevity risk and the valuation framework for pricing various longevity-linked securities is also derived. Extending from the immunization theory, the insurer attempts to stabilize its profit in response to the change of mortality improvement. We apply Taylor expansion on the profit function to measure the impact

[1] These incidence data include more than 50,000,000 policies, collected from all Taiwanese life insurance companies.

of mortality improvement and derive the closed-form solution for the optimal hedging strategy under the first order approximation. The contributions of this research are fourfold. First, this paper builds a longevity hedging framework for the insurer that utilizes both natural and external hedging strategy. Second, this paper considers the basis risk in finding the optimal hedging strategy. We employ Yang and Wang (2013)'s multi-population mortality framework and obtain the optimal hedging strategy analytically according the insurer's profit function. Third, we deal with basis risk using a real mortality data from life insurance industry. We can model the mortality dynamics for different lines of business. Fourth, the optimal hedging strategy is examined numerically.

The remainder of this paper is organized as follows. In Section 2, we present a multi-population mortality framework to model the mortality dynamics for different product lines. To avoid basis risk, this multi-population mortality framework is calibrated to the real mortality experience from insurance industry. The mortality data is introduced and the corresponding parameters in the mortality model are obtained. Section 3, we build the proposed hedging framework for the insurance company. The profit function and objective function are introduced. The optimal hedging strategy is derived analytically. In Section 4, the optimal hedging is analyzed numerically. After we provide a numerical analysis of the optimal hedging strategy, we draw some key conclusions and implications.

2. Modeling Mortality dynamics for the Life Insurer Considering Basis Risk

The multi-population mortality dynamic

The insurer has different mortality pattern for life insurance and annuity business.

We consider the basis risk to model longevity risk for different lines of business. Thus, we need a mortality model to project the future mortality rates for different groups of population simultaneously. We employ Yang ad Wang (2013)'s multi-population model which is built on the Lee-Carter framework (Lee and Carter, 1992). We analyze changes in mortality as a function of both age *x* and time *t* on a filtered probability space $\left(\Omega, \Im, P, (\Im_s)_{s=0}^{t+T}\right)$, where P is the physical probability measure and $\Im_t$ is the information available at time *t*. If we consider *N* product lines for life insurance, , then $m_{x,t}^j$, $j=1,...,N$, is the mortality force at age *x* for in the j^{th} insurance product during calendar year *t*. That is,

$$\ln m_{x,t}^j = a_x^j + b_x^j k_t^j + e_{x,t}^j, \; j=1,...,N \,. \tag{1}$$

where the parameters b_x^j and k_t^j are subject to $\sum_x b_x^j = 1$ and $\sum_t k_t^j = 0$, to ensure the model identification. The Lee-Carter model can capture the age–period effect for the *j*^th^ population with a_x^j coefficients. Furthermore, the actual force of mortality for the j^{th} population changes according to an overall mortality index k_t^j, which is modulated by an age response b_x^j. The error term $e_{x,t}^j$ reflects a particular, age-specific historical influence that is not captured by the model. To better present the multi-population mortality forecast clearly, we express the future mortality rates for a person of age *x* at time *t* in matrix form:

$$\begin{bmatrix} \ln m_{x,t}^1 \\ \ln m_{x,t}^2 \\ \vdots \\ \ln m_{x,t}^N \end{bmatrix} = \begin{bmatrix} a_x^1 \\ a_x^2 \\ \vdots \\ a_x^N \end{bmatrix} + \begin{bmatrix} b_x^1 & 0 & \cdots & 0 \\ 0 & b_x^2 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & b_x^N \end{bmatrix} \begin{bmatrix} k_t^1 \\ k_t^2 \\ \vdots \\ k_t^N \end{bmatrix} + \begin{bmatrix} e_{x,t}^1 \\ e_{x,t}^2 \\ \vdots \\ e_{x,t}^N \end{bmatrix}. \tag{2}$$

Lee and Carter (1992) assume that the parameters of a_x and b_x remain constant over time and forecast future mortality by projecting the mortality time index using a standard ARIMA time-series model for a single population. However, to deal with longevity exposure for a pool of policies across different product lines, we need to forecast future mortality for different group of populations. To do so, we adopt a multivariate time-series approach to analyze the mortality time index. That is, we use a co-integration analysis to investigate whether a common stochastic trend appears in the future mortality time trends K_t for populations across countries. We model K_t with a VECM. If each series in K_t is an $I(p)$ process, that is, a nonstationary process with p unit roots, then K_t is co-integrated. According to the Granger representation theorem (Engle and Granger 1987), the VECM of order *p* (or VECM(*p*)) for K_t follows the form:

$$\Delta K_t = \omega_t + \Pi K_{t-1} + \sum_{d=1}^{p-1} \Gamma_d \Delta K_{t-d} + \varepsilon_t \ , \tag{3}$$

where Δ is the first-order difference operator; ω_t, a *N*-by-1 vector, contains deterministic terms (e.g., constant, trend, seasonal dummies); Π is a *N*-by-*N* long-run impact matrix; Γ_d for $d = 1,..., p-1$, is a *N*-by-*N* short-run impact matrix; and $\varepsilon_t = \left[\varepsilon_t^1,...,\varepsilon_t^N\right]'$ is a shock vector, where the $\varepsilon_t, \varepsilon_{t+1}, \varepsilon_{t+2}...$ are mutually independent, each following a *N*-variate normal distribution with 0 mean and a *N*-by-*N* covariance matrix of Σ.

Calibration of mortality model

To calibrate the mortality model for capturing the mortality dynamics of annuity and insurance business, we employ a unique experience mortality data from the life insurance industry in Taiwan. This data set includes the actual mortality experience for both life insurance and endowment insurance and for men and women separately. Thus, we can capture the actual mortality pattern for different lines of business rather than demonstrating with population mortality rates. These data was collected from all Taiwanese life insurance companies. The data covers more than 50,000,000 policies issued by the life insurance companies in Taiwan from the period of 1972 to 2008 (27 years of data)[2]. To capture the mortality pattern for life insurance and annuity policies, we divide all policies into two groups: with or without endowments (single endowment or serial periodic endowments). The insurance company in Taiwan has a special product strategy that selling endowment insurance instead of annuity business. Therefore, we use the policies with endowments as the proxy for annuity policies. Table 1: Summary of policy numbers derived from Taiwan insurance data

Table 1: Policy Summary

Insurance Type	Female	Male
With Endowments	7,175,200	7,509,730
No Endowments	18,254,681	18,072,776

In addition, we demonstrate that the insurer uses q-forward contracts as the external hedging instrument. Assume the underlying mortality index for the q-forward contract is based on the UK male population mortality. To deal with basis risk, we calibrate the multi-population model to each line of business and the underlying mortality index. In other words, we have five populations: Taiwan female annuity(TW-fa), Taiwan male annuity(TW-ma), Taiwan female life(TW-fl), Taiwan male life(TW-ml) and UK male population mortality(UK-male). In order to model the basis risk among different populations, we apply Co-integration analysis to capture the interrelated mortality improvement among populations according to the mortality experience.

[2] The experience data were stored by the Insurance Agency Association of the Republic of China (CIAA) before 2007, and TII has been in charge of the data since 2007. TII now maintains all of the experience data available from 1972 until the present.

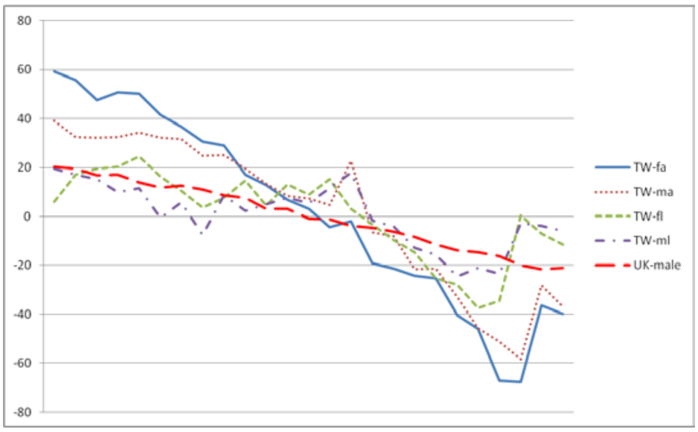

Figure 1: Mortality Time Trend(k_t^j) for Different Population Groups

Figure 1 exhibits the relationships of mortality time trend(k_t) among the 5 different group of populations. We can observe that the mortality improvement for annuity populations seems more faster than for life insurance populations and UK male population. However there is an interesting behavior that the process of all k_t's move in a similar pattern for different groups of populations. Based on the co-integration analysis, it indicates that there is a long-term equilibrium relationships among all k_t's and the k_t's have cointegrated effect. Then we can write down the estimated parameters for the error correction model as follows.

$$\begin{pmatrix} \Delta k_{1,t} \\ \Delta k_{2,t} \\ \Delta k_{3,t} \\ \Delta k_{4,t} \\ \Delta k_{5,t} \end{pmatrix} = \begin{pmatrix} -0.9736 \\ 0.6833 \\ -0.7639 \\ 0.0224 \\ -1.9252 \end{pmatrix} + \begin{pmatrix} 7.5470 \\ 9.2306 \\ -0.0714 \\ 2.5930 \\ -0.4821 \end{pmatrix} \left[-0.2570 + \sum_{i=1}^{5} \beta_i k_{i,t-1} \right] + \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{1,t} \\ \varepsilon_{1,t} \\ \varepsilon_{1,t} \\ \varepsilon_{1,t} \end{pmatrix}$$

where

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{pmatrix} = \begin{pmatrix} -0.0250 \\ -0.0845 \\ 0.0744 \\ 0.0053 \\ 0.1729 \end{pmatrix}$$

and

$$\Sigma = \mathrm{Cov}\begin{pmatrix}\varepsilon_{1,t}\\ \varepsilon_{2,t}\\ \varepsilon_{3,t}\\ \varepsilon_{4,t}\\ \varepsilon_{5,t}\end{pmatrix} = \begin{pmatrix} 30.11 & 7.3671 & -8.0381 & 3.3607 & 2.5366 \\ 7.3671 & 27.3574 & -13.4597 & 8.0873 & 1.775 \\ -8.0381 & -13.4597 & 97.9874 & 43.1154 & -0.1343 \\ 3.3607 & 8.0873 & 43.1154 & 67.068 & -0.8885 \\ 2.5366 & 1.775 & -0.1343 & -0.8885 & 1.7289 \end{pmatrix}$$

We can treat the first difference of k_t as a time-varying multivariate normal distribution. That is

$$\begin{pmatrix}\Delta k_{1,t}\\ \Delta k_{2,t}\\ \Delta k_{3,t}\\ \Delta k_{4,t}\\ \Delta k_{5,t}\end{pmatrix} \sim N(\mu(k_{t-1}), \Sigma)$$

where the mean $\mu(k_{t-1})$ is adjusted by the error correction term as follows

$$\mu(k_{t-1}) = \begin{pmatrix}-0.9736\\ 0.6833\\ -0.7639\\ 0.0224\\ -1.9252\end{pmatrix} + \begin{pmatrix}7.5470\\ 9.2306\\ -0.0714\\ 2.5930\\ -0.4821\end{pmatrix}\left[-0.2570 + \sum_{i=1}^{5} \beta_i k_{i,t-1}\right]$$

3. Optimal Hedging Strategy for longevity risk

Profit Function of the Insurer

The purpose of this research is to study the optimal hedging strategy of the insurer to hedge longevity risk using both natural hedging and external hedging instrument. We consider a life insurance company with both life insurance and annuity policies. The insurance company is assumed to have more annuity policies than life insurance policies in their liability, i.e. the longevity risk cannot be fully natural hedged. Thus, the insurer also considers the external hedge instrument and attempts to find the optimal level of longevity securities to effectively reduce longevity risk. In other words, the insurer can use the longevity securities to stabilize the profit function in response to a change mortality improvement. Thus, we extend the immunization theory to find the optimal hedging strategy based on the insurer's profit function. The profit function is calculated according to the cash flow of the insurer's liability and the cash flow of the hedging instrument. Let $U(t)$ denote the insurer's profit at time t. The profit function of the insurance company including all life and annuity policies at time t is defined as follows:

$$U(t) = M \times V^S(t) - \sum_j c_{L,j} V^L(x_j, t) - \sum_j c_{A,j} V^A(x_j, t), \quad (4)$$

where $V^S(t)$ is the value of the external hedge instrument at time t and M is the amount of the hedge instrument, $V^L(x_j, t)$ is the value of jth life insurance policy for the insured aged xj with face amount $c_{L,j}$; $V^A(x_j, t)$ is the value of jth annuity policy for the insured aged xj with face amount $c_{A,j}$. Equation (4) is a general form of the profit function for the life insurance company. If the insurer only provides annuity business or the pension provider, we can exclude the term for life insurance policies in Equation (4).

Here we consider q-forward as the hedging instrument against longevity risk. The present value of payoff function of q-forward can be represent as

$$V^S(0) = 1000(1 + r)^{10}(q^f_{UK-male65} - q^{real}_{UK-male65})$$

where $q^f_{UK-male65}$ is the q forward rate determined at the beginning of the contract and the maturity is 10 years. The interest rate is assumed to be 0.02.

Object Function of finding optimal hedging strategy

According to the profit function, the insurer's objective is to find an effective hedging strategy for avoiding unexpected changed on the profit caused by the mortality shifting. Our objective is to determine the optimal value of M, i.e. how many units of longevity-linked securities the insurer should take to reduce the variation for the insurer's profit. To estimate the effect of the mortality change on the profit, we apply Taylor expansion of $U(t)$ with respect to mortality trend (k_t). That is, the change of profit is measured by $\Delta U(t) \approx \nabla U(t)' \Delta k_t$.

We investigate three different hedging strategies based on the mean-variance framework for the insurer's profit process. The objective functions under different hedging strategies are shown below:

(1) Maximizing mean-variance approach:

$$\max_{\mathrm{M}} E[\Delta \mathrm{U}(t)] - \theta Var\big(\Delta U(t)\big)$$

(2) Maximizing mean- value at risk approach:

$$\max_{\mathrm{M}} E[\Delta U(t)] - z_p \sqrt{Var\big(\Delta U(t)\big)}$$

where $\mathrm{z_p}$ is the inverse cumulated distribution function of standard normal distribution.

(3) Maximizing mean- conditional tail expectation approach:

$$\max_{\mathrm{M}} E[\Delta \mathrm{U}(t)] - \frac{\sqrt{Var\big(\Delta U(t)\big)}}{p} \frac{\mathrm{e}^{-\frac{[\mathrm{z_p}]^2}{2}}}{\sqrt{2\pi}}$$

Under the assumption that the change of profit is approximately normal distributed, we can obtain the mean and variance analytically. Therefore, under the first order approximation of $\Delta\mathrm{U}(\mathrm{t})$, we can take the properties of normal distribution to solve the optimal hedging strategy analytically.

Derivation of Optimal Hedging Strategy

We demonstrate how the insurer hedge the longevity risk using q-forward contracts when the insurance company has more annuity product than life contract in their liability. The hedging strategy is for single-period, that is to control the unexpected profit changes form this year to next year under the assumption that mortality dynamics follow Yang and Wang(2013). We define the profit function to analyze and measure the risk of each product, then find the optimal solution to hedge mortality risk for different hedging strategies.

We first estimate the change of $U(t)$ with respect to k_t's using Taylor expansion.

Define

$$\Delta U(t)=\begin{pmatrix}\frac{\partial U(t)}{\partial k_{1,t}}\\ \frac{\partial U(t)}{\partial k_{2,t}}\\ \vdots\\ \frac{\partial U(t)}{\partial k_{5,t}}\end{pmatrix}=\begin{pmatrix}\frac{\partial U(t)}{\partial k_{1,t}}\\ \frac{\partial U(t)}{\partial k_{2,t}}\\ \vdots\\ M\frac{\partial V^S(k_{5,t})}{\partial k_{5,t}}\end{pmatrix}=\begin{pmatrix}D_1\\ MD_2\end{pmatrix}$$

where

$$D_1=\begin{pmatrix}-\frac{\partial U(t)}{\partial k_{1,t}}\\ -\frac{\partial U(t)}{\partial k_{2,t}}\\ \vdots\\ -\frac{\partial U(t)}{\partial k_{4,t}}\end{pmatrix}\quad \text{and}\quad D_2=\frac{\partial V^S(k_{5,t})}{\partial k_{5,t}}$$

We also define Δk_t's

$$\Delta k_t=\begin{pmatrix}\Delta k_{1,t}\\ \Delta k_{2,t}\\ \Delta k_{3,t}\\ \Delta k_{4,t}\\ \Delta k_{5,t}\end{pmatrix}=\begin{pmatrix}\Delta k_t^{(1)}\\ \Delta k_t^{(2)}\end{pmatrix}\sim N\left(\begin{pmatrix}\mu^{(1)}(k_{t-1})\\ \mu^{(2)}(k_{t-1})\end{pmatrix},\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}\right)$$

where

$$\Delta k_t^{(1)}=\begin{pmatrix}\Delta k_{1,t}\\ \Delta k_{2,t}\\ \Delta k_{3,t}\\ \Delta k_{4,t}\end{pmatrix}\quad \text{and}\quad \Delta k_t^{(2)}=\Delta k_{5,t}$$

With these notations, we can derive the distribution of the change of profit. It is approximately normal distributed. That is

$$\Delta U(t)\approx \nabla U(t)'\Delta k_t\sim N(\nabla U(t)'\mu(k_{t-1}),\nabla U(t)'\Sigma\nabla U(t))$$

We can further express the mean and variance of the change of profit as follows:

$$E[\Delta U(t)] = A_1 + MA_2$$

$$Var[\Delta U(t)] = A_3 + MA_4 + M^2 A_5$$

where

$$A_1 = D_1 \mu^{(1)}(k_{t-1})$$

$$A_2 = D_2 \mu^{(2)}(k_{t-1})$$

$$A_3 = D_1' \Sigma_{11} D_1$$

$$A_4 = D_2 \Sigma_{21} D_1 + D_1' \Sigma_{12} D_2$$

$$A_5 = D_2 \Sigma_{22} D_2$$

Proposition 1: Under the maximizing mean-variance approach, the optimal solution is $M^* = \frac{A_2 - \theta A_4}{2\theta A_5}$.

Proposition 2: Under the maximizing mean- value at risk approach, :the optimal solution is $M^* = \frac{-K_1 \pm \sqrt{K_1^2 - 4K_2 K_0}}{2K_2}$, where $K_0 = 4A_2^2 A_3 - z_p^2 A_4^2$, $K_1 = 4A_2^2 A_4 - 4A_4 A_5 z_p^2$, $K_2 = 4A_2^2 A_5 - 4z_p^2 A_5^2$ and $M^* \geq 0$.

Proposition 3: Under the m aximizing mean- conditional tail expectation approach, the optimal solution is $M^* = \frac{-K_1' \pm \sqrt{K_1'^2 - 4K_2' K_0'}}{2K_2'}$, where $K_0' = 4A_2^2 A_3 - K^2 A_4^2$, $K_1' = 4A_2^2 A_4 - 4A_4 A_5 K^2$, $K_2' = 4A_2^2 A_5 - 4K^2 A_5^2$, $K = \frac{e^{-\frac{[z_p]^2}{2}}}{p\sqrt{2\pi}}$ and $M^* \geq 0$,

4. Numerical examples

Assume both the life insurance and annuity contain two policies, one is of male insured aged 55 and the other is of female insured aged 55 respectively. For calculating the value of life insurance contracts, annuities and q-forward, we generate 50,000 sample paths of mortality rate and compute the averaged value of actuarial present value of each simulated mortality.

Table 2 displays the optimal hedging strategies for different objective functions. We can see under mean variance assumption, as θ increases, the optimal hedging units decreases. The optimal hedging strategies of VaR and CTE need fewer units of q-forward to hedge the longevity risk.

Table 2: Optimal Hedging Strategies

	Optimal M
Strategy 1: Delta mv	4.008961($\theta = 0.8$), 3.267354($\theta = 1$), 2.772949($\theta = 1.2$)
Strategy 2: Delta VaR	0.810618
Strategy 3: Delta CTE	1.086048

(To be continued…)

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