MORTALITY IMPROVEMENT RATE MODELLING AND PROJECTING

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AGENDA

- Motivation
- History
- Model Construction
- Case Study: Modelling Mortality Rates
  Modelling Mortality Improvement Rates
- Conclusions

MODELLING & PROJECTING MORTALITY IMPROVEMENT RATES: HISTORY

MR- mortality rates

Lee & Carter (1992) Brouhns, Denuit, Vermunt (2002) Renshaw & Haberman (2006)	Cairns, Blake, Dowd et al. (2008) Plat (2009) Haberman & Renshaw (2011)

MR- mortality improvement rates

Willets (2004) Richards, Kirkby, Currie (2005) Renshaw & Haberman (2006) CMI

DATA

$\left(d_{xt}, e_{xt}, \omega_{xt}\right)$: age x, period t

$y_{xt} = \hat{m}_{x,t} = \dfrac{d_{xt}}{e_{xt}}$ - empirical mortality rate

$z_{xt} = 2\dfrac{\left(1 - \hat{m}_{x,t} / \hat{m}_{x,t-1}\right)}{\left(1 + \hat{m}_{x,t} / \hat{m}_{x,t-1}\right)}$ - mortality improvement rate

improving mortality (over time) $\Rightarrow z_{xt} > 0$

deteriorating mortality (over time) $\Rightarrow z_{xt} < 0$

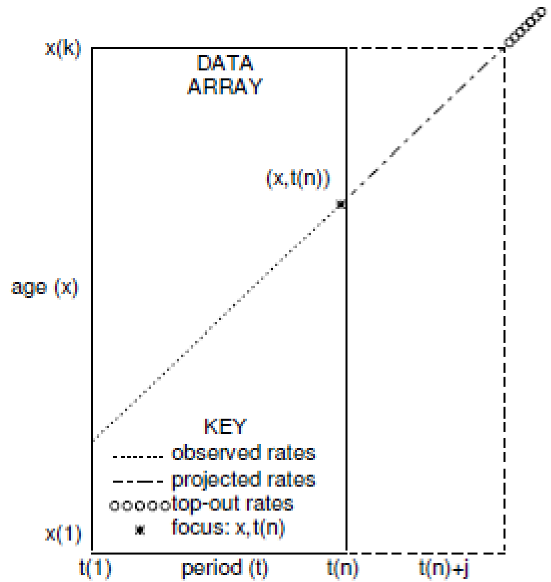
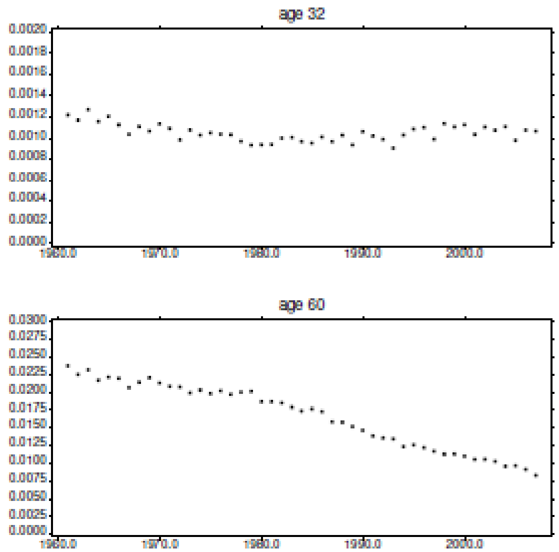


Fig 2. Schematic diagram depicting a rectanglar age-period data array with cohort trajectory, focus at age x, in peripheral year t(n), partitioned in three: comprising observed, projected and top-out mortality rates.



E&W 1961-2007 male mortality.
MR vs period, fixed ages: 32 or 60

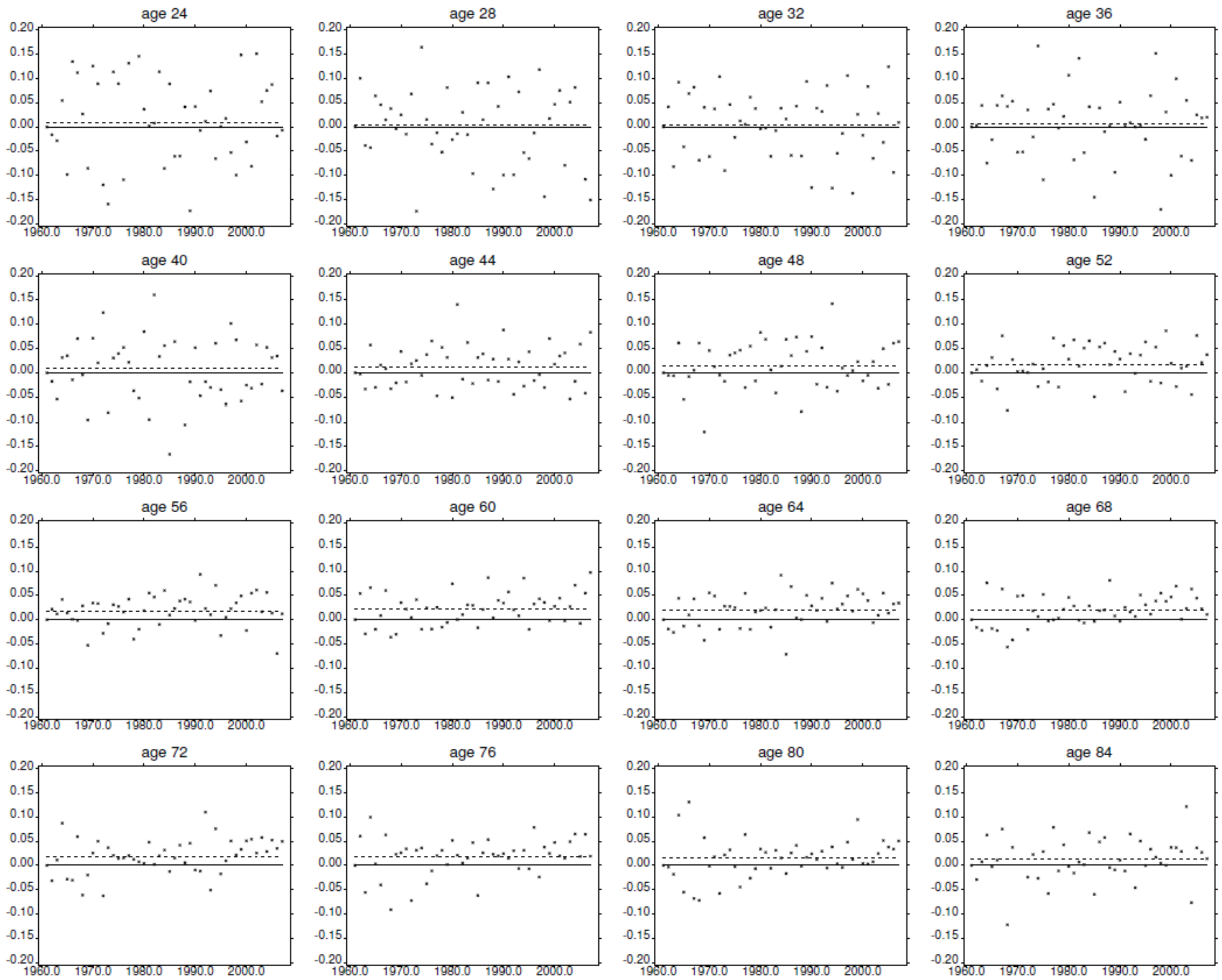


Fig 3a. England & Wales 1961-2007 male mortality experience.
Mortality improvement rates (MIR) and their average over time
(dotted line), plotted against calendar year for various fixed ages.

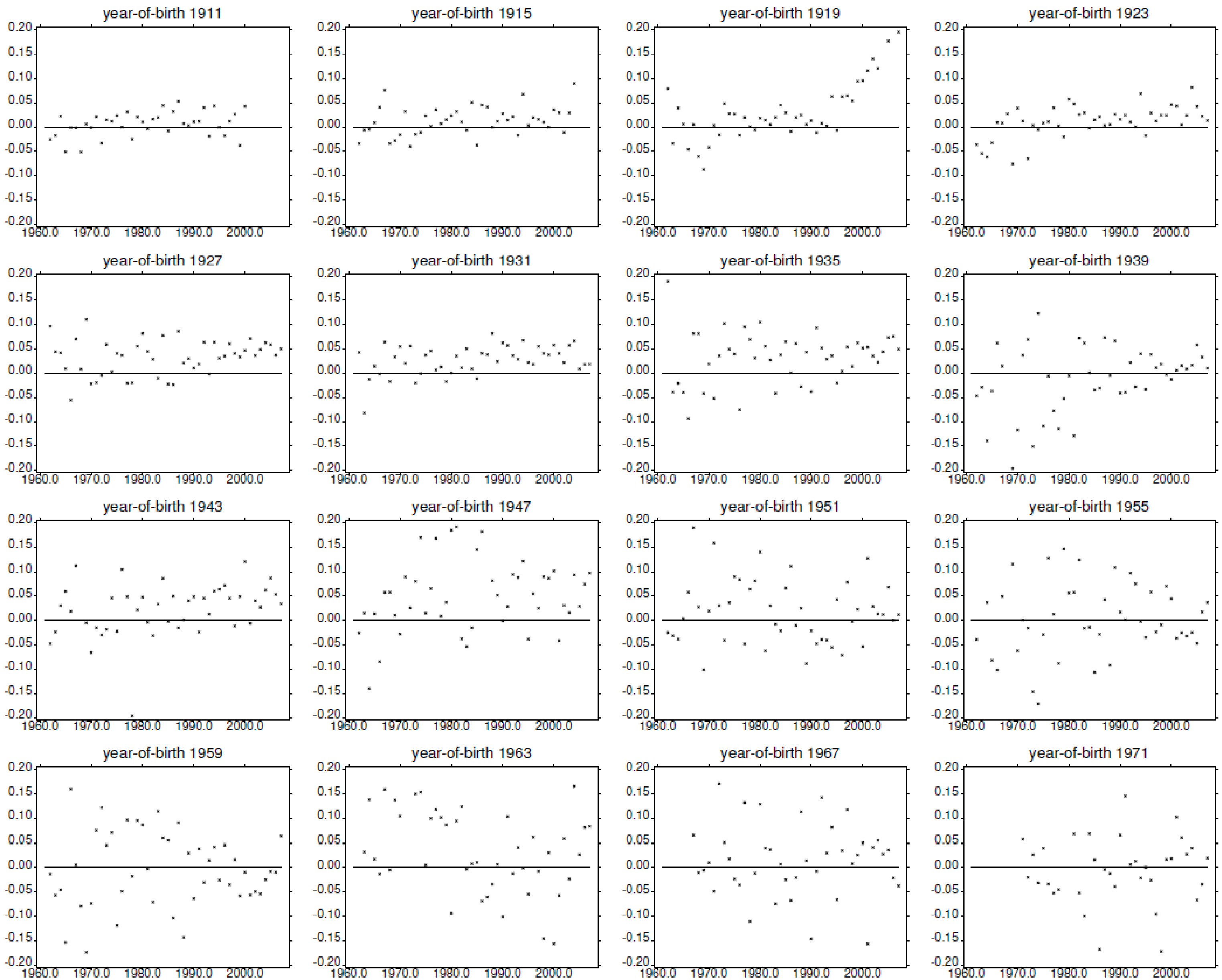
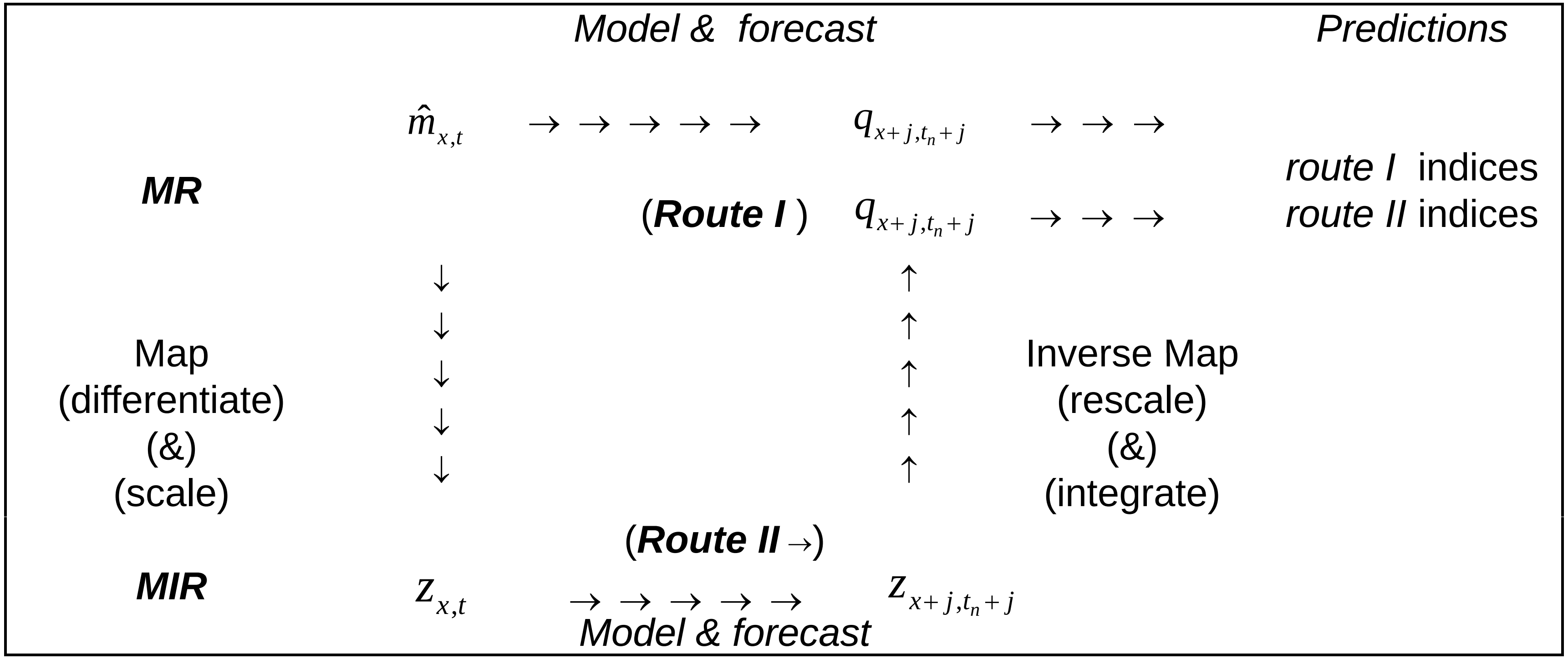


Fig 3b. England & Wales 1961-2007 male mortality experience.
Mortaility improvement rates (MIR) plotted against year of observation
for various fixed years-of-birth (cohorts).



$m_{x,t}$ - Central mortality rate (MR); $z_{x,t}$ - mortality improvement rate (MIR)

q_{x+j,t_n+j} - predicted probability of death ***MR*** predictions- ***route I***; ***MIR*** predictions- ***route II***

MORTALITY RATES (Route I) APPROACH

TARGET:

m_{xt} - central rate of mortality

PARAMETRIC PREDICTOR STRUCTURES:

$$LC: \eta_{xt} = \alpha_x + \beta_x \kappa_t$$

$$H_1: \eta_{xt} = \alpha_x + \beta_x \kappa_t + \iota_{t-x}$$

$$H_0: \eta_{xt} = \alpha_x + \kappa_t + \iota_{t-x}$$

$$M5: \eta_{xt} = \alpha_x + \kappa_t^{(1)} + \left(x - \overline{x}\right)\kappa_t^{(2)}$$

$$M6: \eta_{xt} = \alpha_x + \kappa_t^{(1)} + \left(x - \overline{x}\right)\kappa_t^{(2)} + \iota_{t-x}$$

$$M7: \eta_{xt} = \alpha_x + \kappa_t^{(1)} + \left(x - \overline{x}\right)\kappa_t^{(2)} + b\left(x\right)\kappa_t^{(3)} + \iota_{t-x}$$

MORTALITY RATES (Route I) APPROACH

$$b(x) = \left\{ (x - \overline{x})^2 - \frac{1}{k} \sum_{i=x_1}^{x_k} (i - \overline{x})^2 \right\}, \ \overline{x} = \frac{1}{k} \sum_{i=x_1}^{x_k} i$$

MODEL FITTING:

Let $D_{xt} \sim P(e_{xt} m_{xt})$ i.i.d., with constant dispersion

$$Y_{xt} = \frac{D_{xt}}{e_{xt}}, \ E(Y_{xt}) = m_{xt}, \ Var(Y_{xt}) = \phi \frac{m_{xt}}{\omega_{xt} e_{xt}}$$

log link $\log m_{xt} = \eta_{xt}$, parametric predictor η_{xt}, weights $\omega_{xt} e_{xt}$,

scale parameter ϕ, variance function $V(u) = u$

To fit: minimise the model deviance.

MORTALITY IMPROVEMENT RATES (Route II) APPROACH

MOTIVATION:

$$\log m_{xt} = \eta_{xt} \Rightarrow \frac{1}{m_{xt}} \frac{\partial m_{xt}}{\partial t} = \frac{\partial \eta_{xt}}{\partial t}$$

Notational convention: re-define symbols appropriately

TARGET:

η_{xt} - mortality improvement rate

MORTALITY IMPROVEMENT RATES (Route II) APPROACH

DUAL PARAMETRIC PREDICTOR STRUCTURES:

$$LC: \eta_{xt} = \beta_x \kappa_t$$

$$H_1: \eta_{xt} = \beta_x \kappa_t + \iota_{t-x}$$

$$H_0: \eta_{xt} = \kappa_t + \iota_{t-x}$$

$$M5: \eta_{xt} = \kappa_t^{(1)} + \left(x - \overline{x}\right)\kappa_t^{(2)}$$

$$M6: \eta_{xt} = \kappa_t^{(1)} + \left(x - \overline{x}\right)\kappa_t^{(2)} + \iota_{t-x}$$

$$M7: \eta_{xt} = \kappa_t^{(1)} + \left(x - \overline{x}\right)\kappa_t^{(2)} + b(x)\kappa_t^{(3)} + \iota_{t-x}$$

LINK FUNCTION: IDENTITY

MORTALITY IMPROVEMENT RATES (Route II) APPROACH

MODEL FITTING (Single stage):

Assume $Z_{xt} \sim N\left(\eta_{xt}, \sigma^2\right)$ i.i.d., constant dispersion

$$E\left(Z_{xt}\right) = \eta_{xt},\ Var\left(Z_{xt}\right) = \sigma^2 \frac{1}{\omega_{xt}}$$

identity link, parametric predictor η_{xt}, weights ω_{xt}

scale parameter σ^2, variance function $V\left(u\right) = 1$.

To fit: minimise the model deviance.

MORTALITY IMPROVEMENT RATES (Route II) APPROACH

MODEL FITTING (Joint Modelling with 2 stages):

Stage 1:

Assume $Z_{xt} \sim N\left(\eta_{xt}, \phi_{xt}\sigma^2\right)$ i.i.d., variable dispersion

$$E\left(Z_{xt}\right) = \eta_{xt},\ Var\left(Z_{xt}\right) = \sigma^2 \frac{\phi_{xt}}{\omega_{xt}}$$

identity link, parametric predictor η_{xt}, weights ω_{xt}/ϕ_{xt}

scale parameter σ^2, variance function $V(u) = 1$

Yields: residuals $r_{xt} = z_{xt} - \hat{\eta}_{xt}$ to form Stage 2 responses.

MORTALITY IMPROVEMENT RATES (Route II) APPROACH

Stage 2:

Model squared residuals R_{xt}^2 as follows

$$E\left(R_{xt}^2\right)=\phi_{xt},\ Var\left(R_{xt}^2\right)=\rho\frac{\phi_{xt}^2}{\omega_{xt}}$$

log link $\log\left(\phi_{xt}\right)=\varsigma_x$, predictor ς_x, weights ω_{xt}

scale parameter ρ, variance function $V\left(u\right)=u^2$

Yields: fitted values $\hat{\phi}_{xt}$ which are then used as Stage 1 weights

To fit: minimise the model deviances.

FITTING THE MODELS

- Use England and Wales male mortality experience for 1961-2007 ages 20-89

- Fit Route I, Route II models (single stage and joint stages)

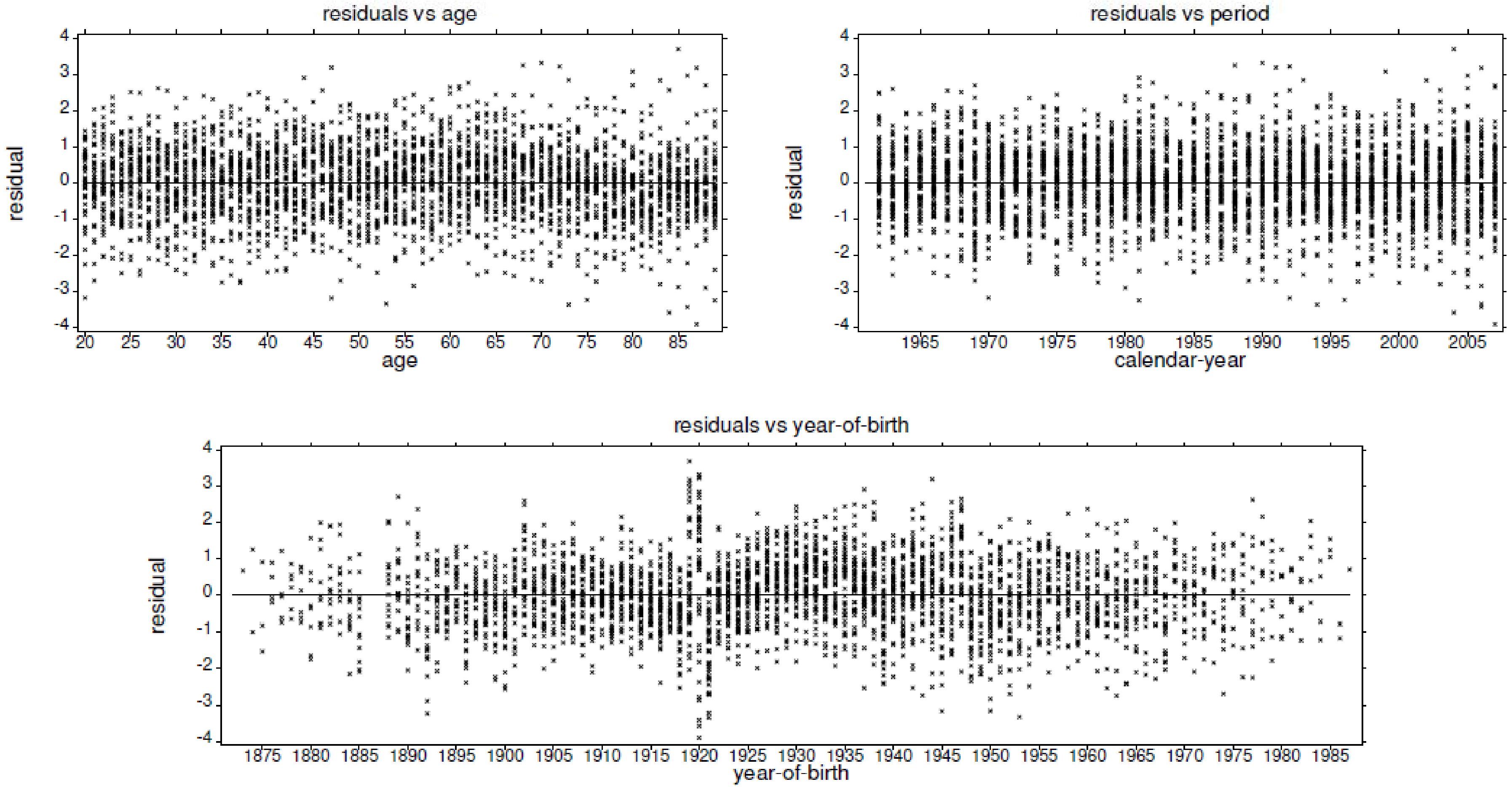


Fig 4b. England & Wales 1961-2007 male mortality experience, ages 20-89.
MIR JLC modelling: standardised residual plots.

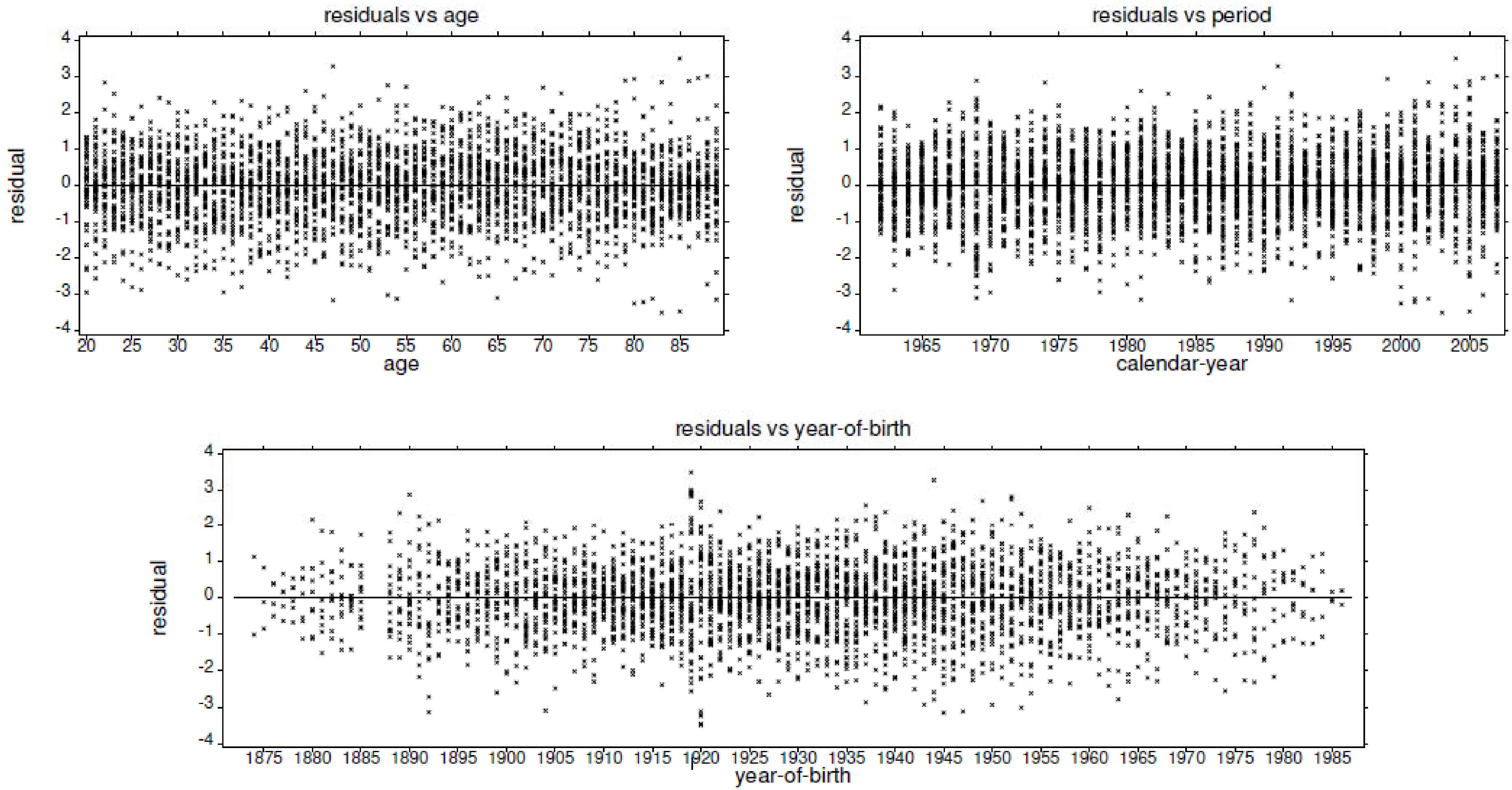


Fig 4d. England & Wales 1961-2007 male mortality experience, ages 20-89.
MIR JH1 modelling: standardised residual plots.

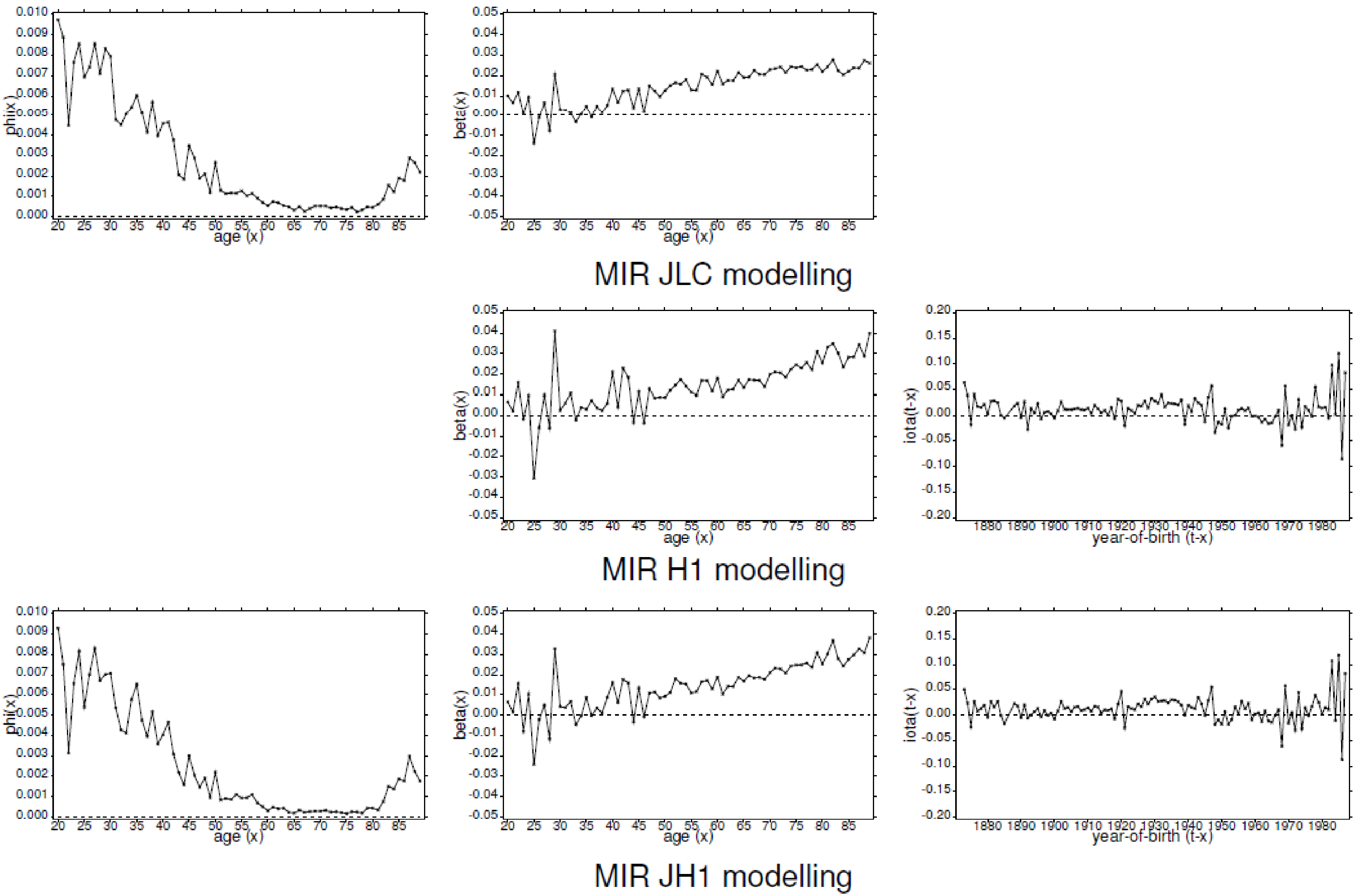


Fig 6. E&W males, ages 20-89. MIR models (rows), fitted parameters
left panels:- phi(x); centre panels:- beta(x); right panels:- iota(t-x).

MODEL DYNAMICS

Recall for dual modelling

$$MR : \kappa_t \mapsto MIR : \frac{\partial \kappa_t}{\partial t}$$

Consequently

$$MR : ARIMA(p,1,q) \mapsto MIR : ARMA(p,q)$$

UNIVARIATE CASE *AR*(1):

$$\kappa_t - \mu = \varphi\left(\kappa_{t-1} - \mu\right) + \varepsilon_t;\ \varepsilon_t \sim N\left(0, \tau^2\right)\ i.i.d.$$

Forecasts

$$\kappa_{t_n+j} = \mu + \varphi^j\left(t_n - \mu\right);\ j = 1,2,3,\ldots$$

MAPPING *MIR* PREDICTIONS TO *MR* PREDICTIONS

Compute *MIR* predictions z_{x,t_n+j} and

convert to *MR* predictions using -

$$m_{x,t_n+j} = m_{x,t_n+j-1}\frac{\left(2-z_{x,t_n+j}\right)}{\left(2+z_{x,t_n+j}\right)};\ j=1,2,3,\ldots$$

Need starter values m_{x,t_n}

Once converted, compute

$$q_{x,t_n+j} \approx 1-\exp\left(-m_{x,t_n+j}\right)$$

TOPPING OUT BY AGE

Extrapolate model projections

$$q_{x+j,t_n+j} : j = 1, 2, \ldots, x_k - x; \;\; (x < x_k)$$

using

$$\log\left(q_{x+j,t_n+j}\right) = a + b\left\{ j - \left(x_k - x - 1\right)\right\}$$
$$+ c\left\{ j - \left(x_k - x - 1\right)\right\}\left\{ j - \left(x_k - x\right)\right\};$$
$$j = x_k - x - 1, x_k - x, x_k - x + 1, \omega - x$$

requiring

$$q_{x_k-1,t_n+x_k-x-1}, \;\; q_{x_k,t_n+x_k-x}, \;\; q_{\omega,t_n+x_k-x}$$

to determine *a*, *b*, *c*.

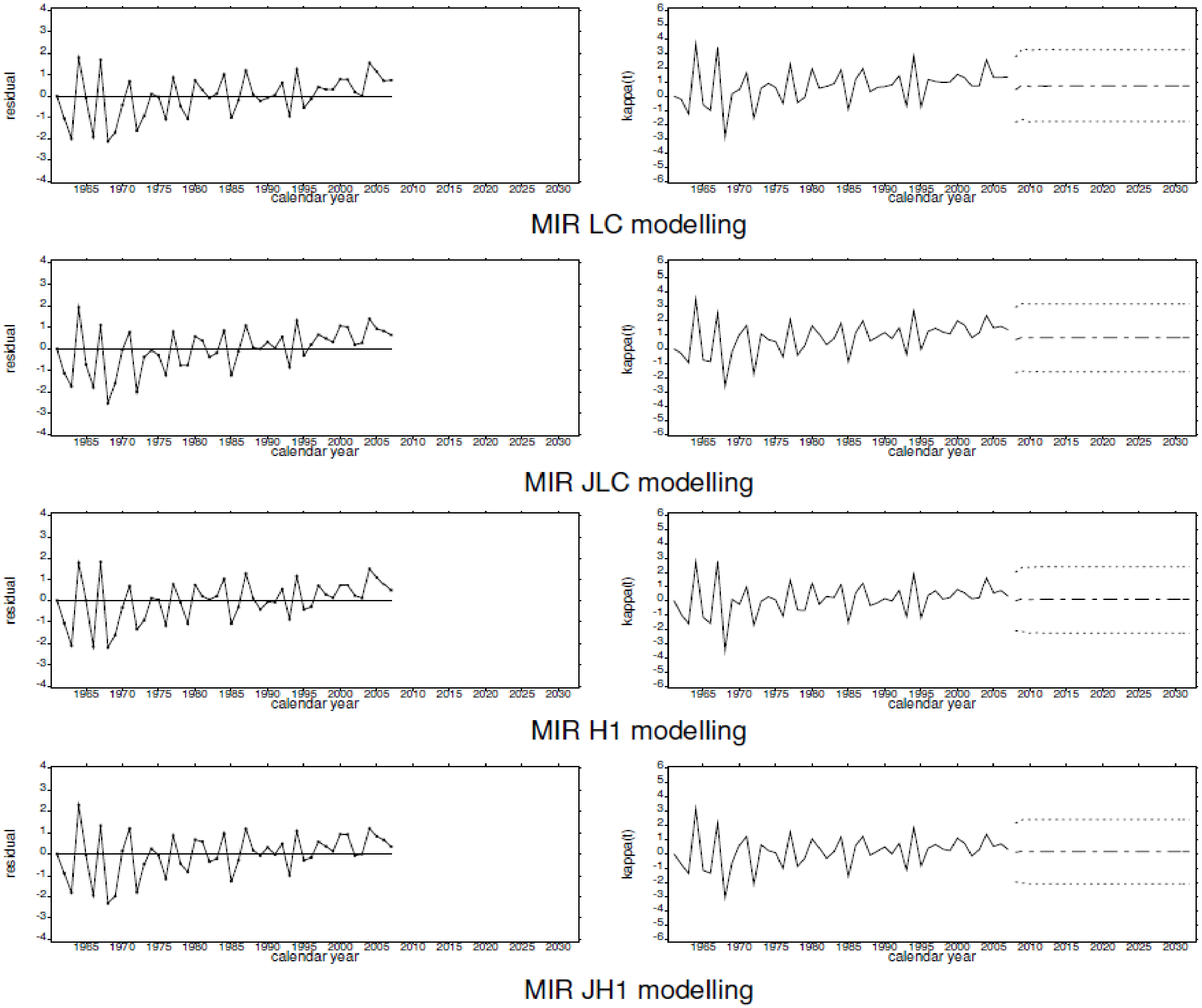


Fig 5. E&W 1961-2007 males, ages 20-89. AR(1) period index processes
with forecasts (RH panels), residuals (LH panels): MIR- models (rows).

INDICES OF INTEREST

Life expectancy predictions-

$$e_x\left(t_n\right)=\frac{\sum\limits_{j\geq 1} l_{x+j}\left(t_n+j\right)\left\{1-\frac{1}{2}q_{x+j,t_n+j}\right\}}{l_x\left(t_n\right)}$$

Fixed rate annuity value predictions-

$$a_x\left(t_n\right)=\frac{\sum\limits_{j\geq 0} l_{x+j}\left(t_n+j\right)v^j}{l_x\left(t_n\right)}$$

ESTIMATING PREDICTION INTERVALS BY SIMULATION: MIR

Generate predictions and prediction intervals.

Period index modelled as an $AR(1)$ process.

(VAR(1) process if multivariate).

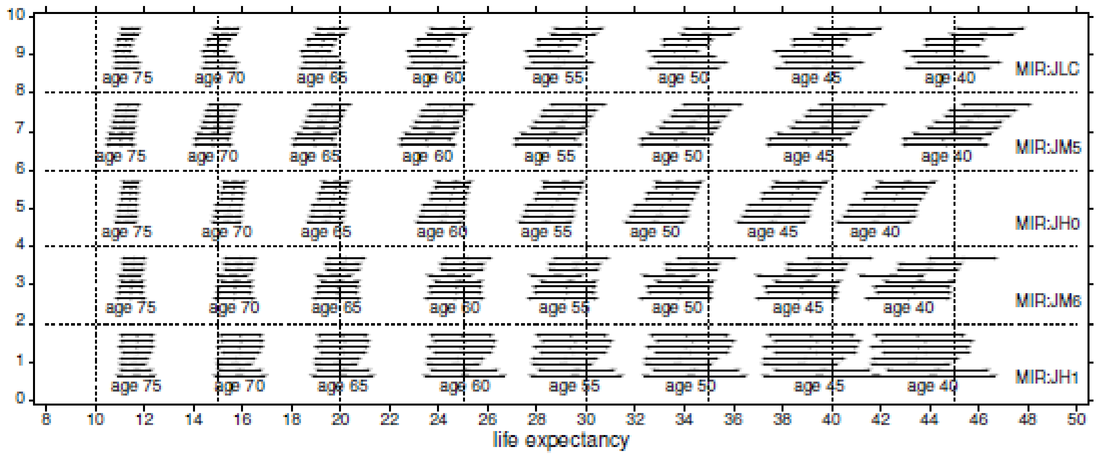
ESTIMATING PREDICTION INTERVALS BY SIMULATION: MIR

ALGORITHM

For simulation $k = 1, 2, \ldots, K$

For $j = 1, 2, \ldots, J$

1. sample $\xi_k^{*(j)}$ from $N(0,1)$
2. compute $\kappa^*_{t_n+j|k} = \kappa_{t_n+j} + \sqrt{mse_{t_n+j}}\,\xi_k^{*(j)}$
3. compute $z^*_{x+j,t_n+j|k}$
4. compute $m^*_{x+j,t_{n+j|k}}$
5. compute $q^*_{x+j,t_{n+j|k}}$
6. apply topping-out
7. compute indices of interest.



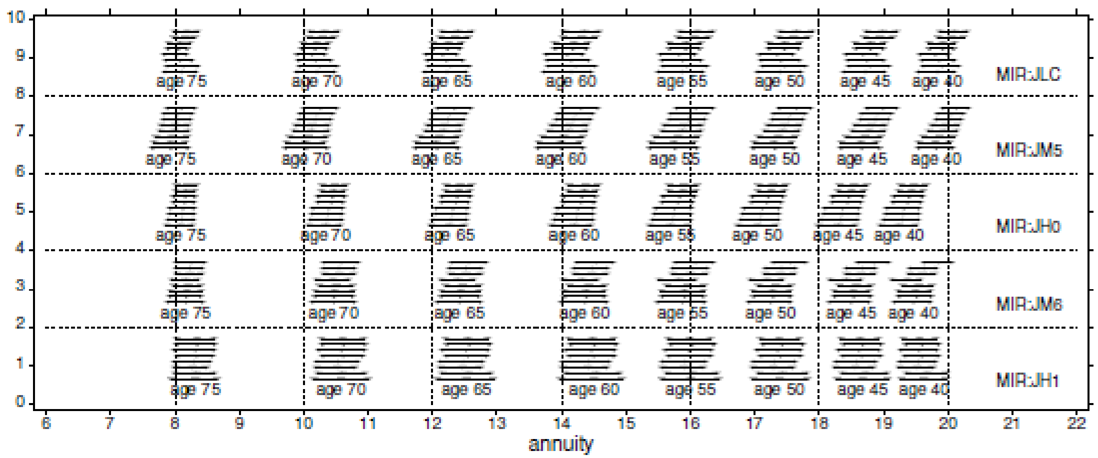


Fig 7a. E&W 1961-2007 males, ages 20-89. Year 2007 life expectancy and 4% annuity simulated 10%,50%,90% quantile predictions, subject to biennial front-end data deletions 1961(02)75 in ascending sequence, ages 40(05)75. MIR Route II approach. Structures- LC, M5, H0, M6, H1 fitted by joint modelling. Topping-out: q(109)=1

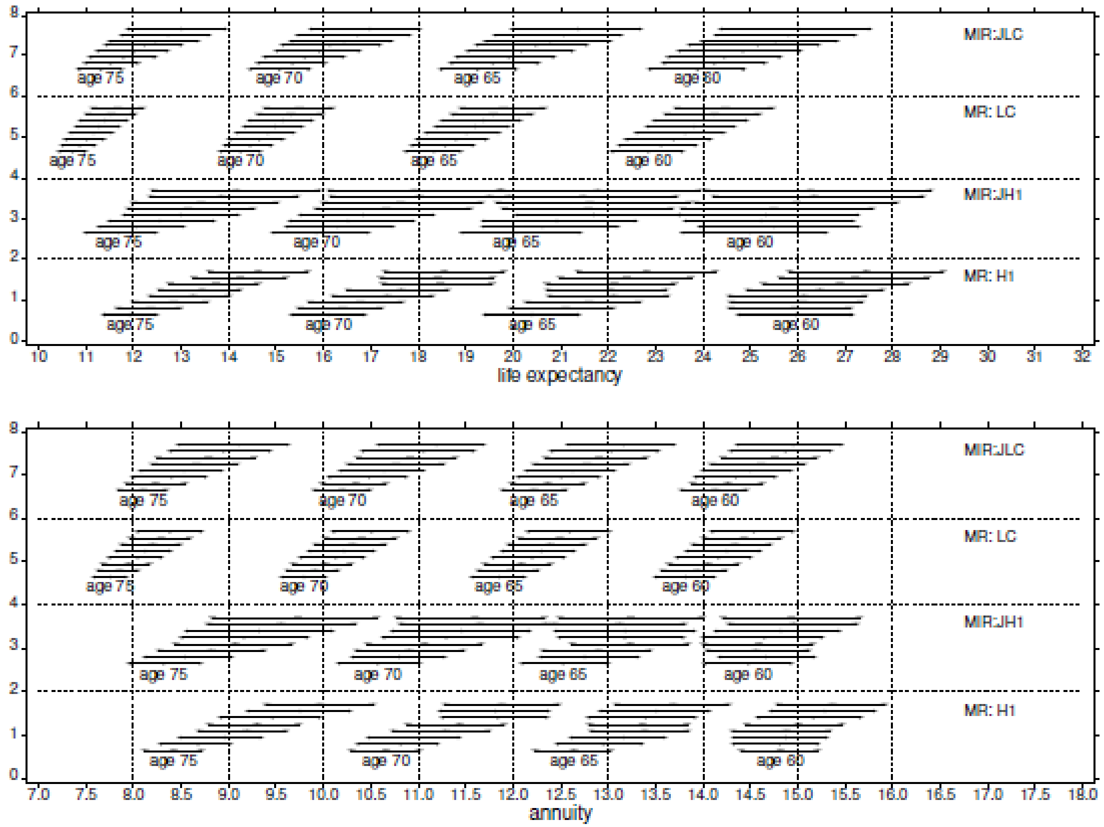


Fig 15. E&W 1961-2007 males, ages 20-89. Evolving biennial 2007(02)21
life expectancy and 4% annuity simulated 10%,50%,90% quantile predictions
in ascending sequence, ages 60(05)75. Adjacent Route II and Route I
approaches comparing MIR:JLC / MR:LC and MIR:JH1 / MR:H1 case structures.
Topping-out: q(109)=1

FORECASTING

- Use England and Wales male experience for 1961-1982, ages 55-89 to fit models and then calculate life expectancies (and annuity values) for age x in 1982 by cohort method where $x = 65, 66, \ldots, 80$

- Compare values for same indices calculated using raw mortality rates for 1983-2007, by depicting (predicted-actual) values against age.

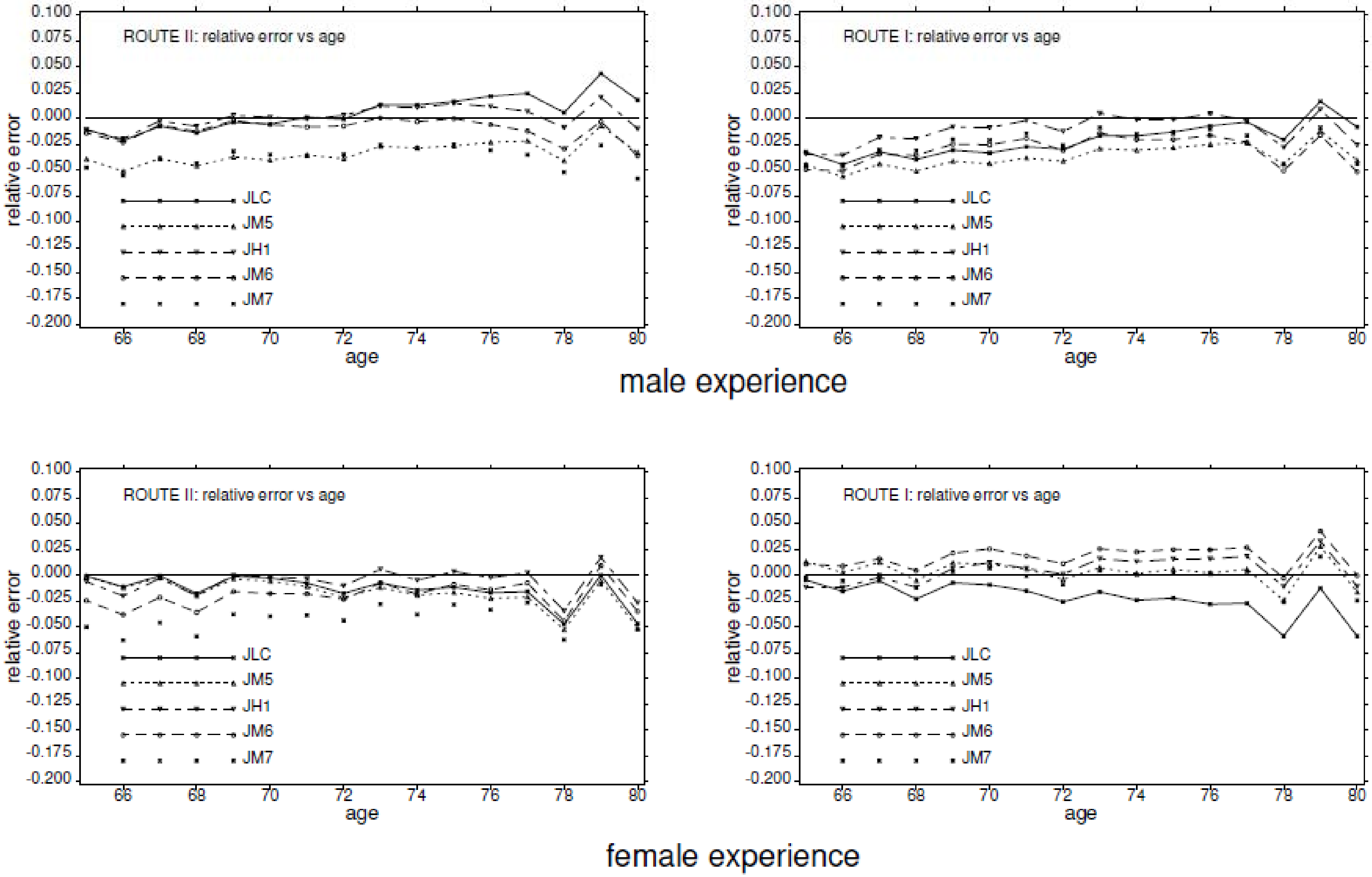


Fig 15a. E&W male and female mortality experiences. Retrospective relative error in 1982 predicted life expectancies.(Route II LH panels; Route I RH panels): ages 65-80.

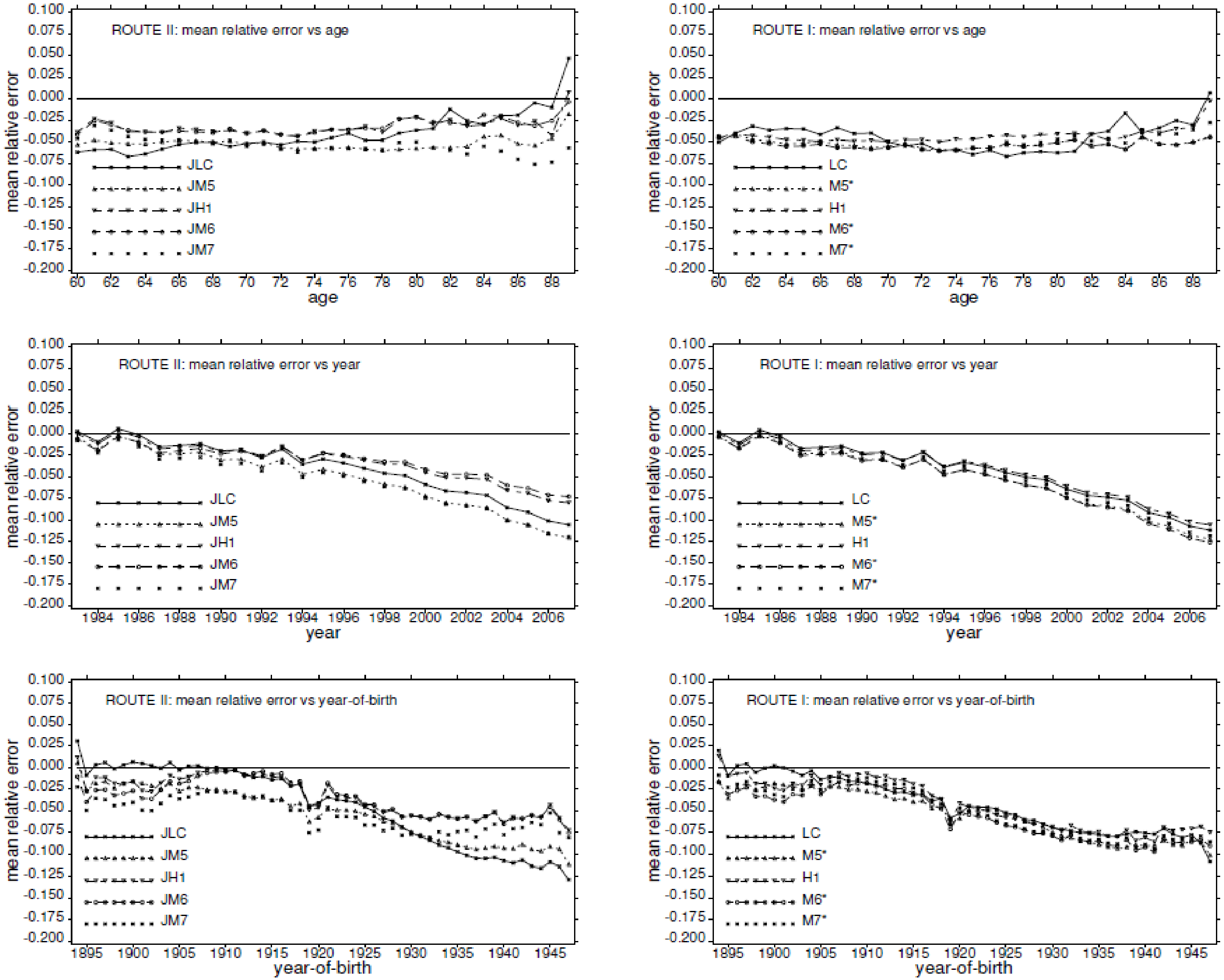


Fig 15b. EW males. Retrospective relative error in 1982 predicted log death rates, averaged over ages, years, cohorts respective. (Route II LH panels; Route I RH panels) based on region bounded by ages 60-89, years 1983-2007.

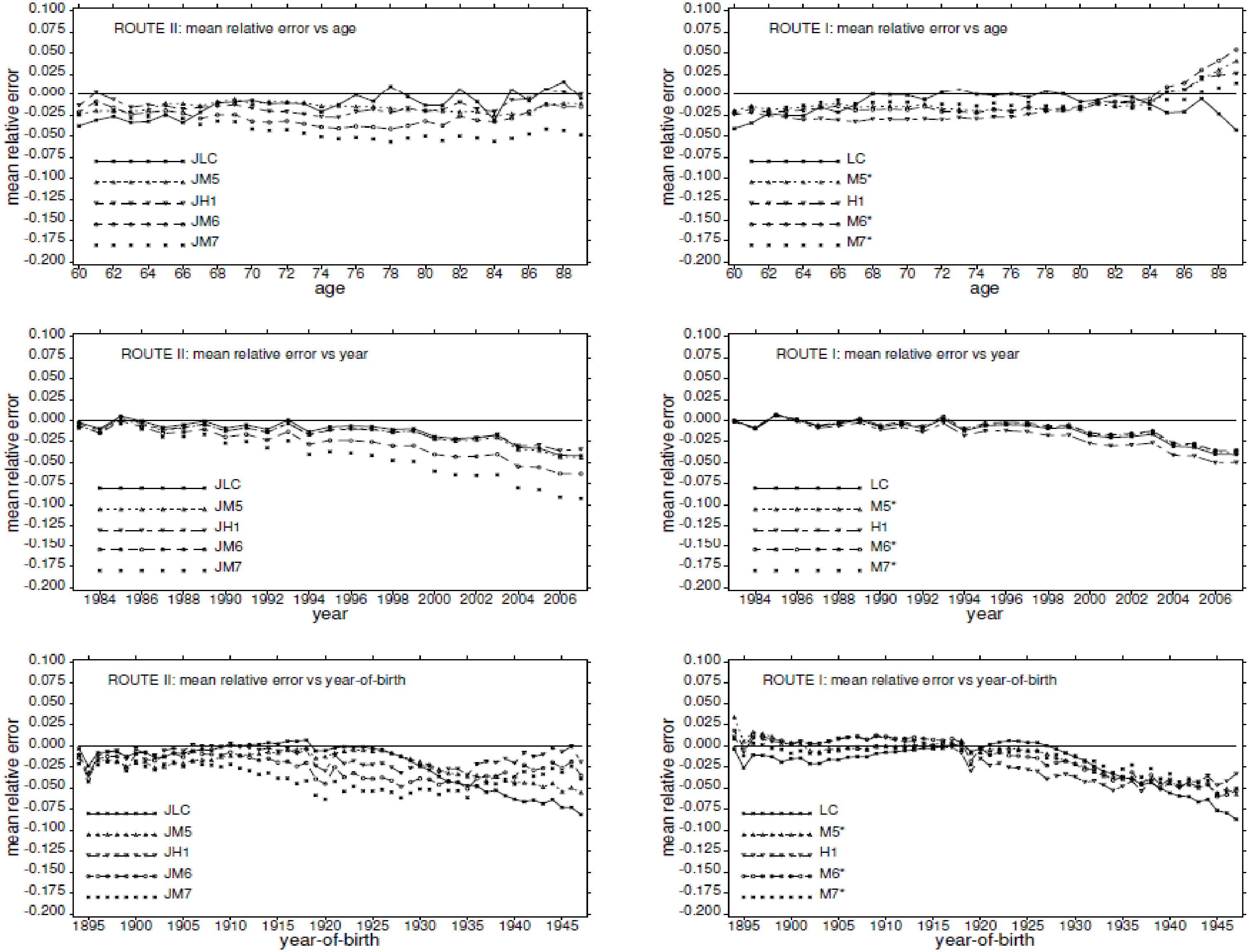


Fig 15c. EW females. Retrospective relative error in 1982 predicted log death rates, averaged over ages, years, cohorts respective.
(Route II LH panels; Route I RH panels) based on region bounded by ages 60-89, years 1983-2007.

CONCLUSIONS

- Differencing the mortality rate series can lead to stationarity.
- Mortality improvement rate models have some advantages
  - age span of data is less restricted (for M5, M6, M7)
  - age index, α_x, plays no role
- Forward predictions show regular and desirable properties.
- Cohort-based version of MIR models is being developed.