Pricing and Risk Management of Variable Annuity Guaranteed Benefits by Analytical Methods

Longevity 10, September 3, 2014

Runhuan Feng, University of Illinois at Urbana-Champaign

Joint work with Hans W. Volkmer, University of Wisconsin-Milwaukee

Variable annuity guaranteed benefits

Variable annuity guaranteed benefits

- Brief introduction
- Guaranteed minimum maturity benefit – Individual model
- Guaranteed minimum maturity benefit - Average model
- Guaranteed minimum withdrawal benefit

Variable annuity guaranteed benefits

Brief introduction

Guaranteed minimum maturity benefit – Individual model

Guaranteed minimum maturity benefit - Average model

Guaranteed minimum withdrawal benefit

Product design

- Arguably the most complex equity-based guarantee available to individual investors;
- Policyholders make contributions (called **purchase payments**) into subaccounts; typically single purchase payment at the inception;
- The term of the product can be broken down into two parts:
  - Accumulation phase; (resembles mutual funds)
  - Income phrase. (various types of guaranteed benefits)

Product design

- The value of each account varies with the performance of the particular fund in which it invests;

$$F_t = F_0 \frac{S_t}{S_0} e^{-mt}, \qquad 0 \leq t \leq T,$$

  where m is the annualized rate of total fees and charges.
  - F_t – total value of subaccounts at time t;
  - S_t – value of equity-index at time t;
- Without any guarantee, equity participation involves no risk to the variable annuity writer, who merely acts as a steward of the policyholders' funds.

- In order to compete with mutual funds, nearly all major variable annuity writers start to offer various types of investment guarantees, which transfer certain financial risks to the insurers. (Liabilities)
  - Guaranteed Minimum Maturity Benefit (GMMB)
  - Guaranteed Minimum Death Benefit (GMDB)
  - Guaranteed Minimum Withdrawal Benefit (GMWB)
  - Guaranteed Minimum Accumulation Benefit (GMAB)
  - Guaranteed Minimum Income Benefit (GMIB)
- Nick-named as the GMxB series. (x=M,D,W,A,I)

- In order to compete with mutual funds, nearly all major variable annuity writers start to offer various types of investment guarantees, which transfer certain financial risks to the insurers. (Liabilities)
  - Guaranteed Minimum Maturity Benefit (GMMB)
  - Guaranteed Minimum Death Benefit (GMDB)
  - Guaranteed Minimum Withdrawal Benefit (GMWB)
  - Guaranteed Minimum Accumulation Benefit (GMAB)
  - Guaranteed Minimum Income Benefit (GMIB)
- Nick-named as the GMxB series. (x=M,D,W,A,I)

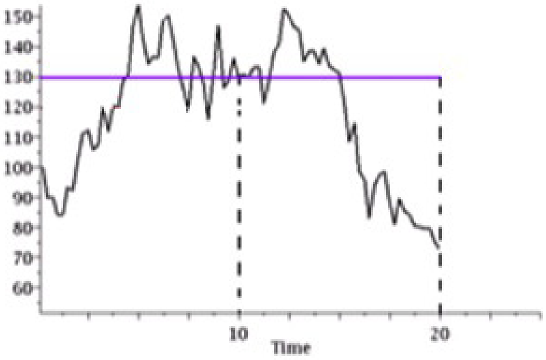
- In order to compete with mutual funds, nearly all major variable annuity writers start to offer various types of investment guarantees, which transfer certain financial risks to the insurers. (Liabilities)
  - Guaranteed Minimum Maturity Benefit (GMMB)
  - Guaranteed Minimum Death Benefit (GMDB)
  - Guaranteed Minimum Withdrawal Benefit (GMWB)
  - Guaranteed Minimum Accumulation Benefit (GMAB)
  - Guaranteed Minimum Income Benefit (GMIB)
- Nick-named as the GMxB series. (x=M,D,W,A,I)

Variable annuity guaranteed benefits

- Brief introduction
- Guaranteed minimum maturity benefit – Individual model
- Guaranteed minimum maturity benefit - Average model
- Guaranteed minimum withdrawal benefit

Guaranteed Minimum Maturity Benefit(GMMB)

- Policyholder receives the greater of a minimum guarantee and account value. The GMMB writer is liable for the difference between the guarantee and account value, should the former exceeds the latter.



- M&E charge, rider charge are made on a daily basis as a certain percentage of subaccount values. (Incomes)

$$M_t = m_x F_t, \qquad 0 \leq t \leq T,$$

where m_x is the charge allocated to fund GMxB rider.($m_x < m$)

Guaranteed Minimum Maturity Benefit(GMMB)

- Gross liability: (T is the maturity date)

$$e^{-rT}(G - F_T)_+ I(\tau_x > T),$$

where τ_x is the future lifetime of policyholder aged x at issue. (put-option-like payoff)

- Net liability= Gross liability - Margin offset income

$$L_0 = e^{-rT}(G - F_T)_+ I(\tau_x > T) - \int_0^{T \wedge \tau_x} e^{-rs} m_e F_s ds.$$

(exotic option?) Mostly negative and rarely positive.

F investment fund value; G guaranteed benefit; T maturity date; r risk-free interest rate; m_e GMMB fee rate.

Commonly used risk measures

- Quantile risk measure (Value-at-Risk)

$$V_\alpha := \inf\{y : \mathbb{P}[L_0 \leq y] \geq \alpha\}.$$

The minimum capital required to ensure that there is sufficient fund to cover future liability with the probability of at least α.

- Conditional tail expectation (Expected Shortfall)

$$\mathrm{CTE}_\alpha := \mathbb{E}[L_0 | L_0 > V_\alpha].$$

Capital required to cover the liabilities exceeding the quantile measure with the probability of at most $1 - \alpha$.

How do we model the equity-index process?

- The model depends on the assumptions on the dynamics of the equity-index.

$$F_t = F_0 \frac{S_t}{S_0} e^{-mt}, \qquad 0 \leq t < T.$$

- Net liability= Gross liability - Margin offset income

$$L_0 = e^{-rT}(G - F_T)_+ I(\tau_x > T) - \int_0^{T \wedge \tau_x} e^{-rs} m_e F_s ds.$$

- There is standard life table model for the future lifetime τ_x; No consensus on equity-index/asset price process S_t.

F investment fund value; *G* guaranteed benefit; *T* maturity date; *r* risk-free interest rate; *M* margin offset (fees).

- In principle, insurance companies are allowed to use their own stochastic models for equity returns. However, the stochastic models should meet the calibration criteria set by the American Academy of Actuaries.

Tabelle : Calibration Standard for Total Return Wealth Ratios

Percentile	1 Year	5 Years	10 Years	20 Years
2.5%	0.78	0.72	0.79	n/a
5.0%	0.84	0.81	0.94	1.51
10.0%	0.90	0.94	1.16	2.10
90.0%	1.28	2.17	3.63	9.02
95.0%	1.35	2.45	4.36	11.70

Common equity-return models listed in the AAA guildeline

- Independent Lognormal (ILN) (Geometric Brownian motion)
- Monthly Regime-Switching Lognormal Model with 2 Regimes (RSLN2)
- Monthly Regime-Switching Lognormal Model with 3 Regimes (RSLN3-M)
- Daily Regime-Switching Lognormal Model with 3 Regimes (RSLN3-D)
- Stochastic Log Volatility with Varying Drift (SLV)

“Essentially, all models are wrong, but some are useful...” — *George E.P. Box*

- For simplicity, we assume the dynamics of equity prices is driven by geometric Brownian motion.

$$S_t = S_0 \exp(\mu t + \sigma B_t), \qquad t \geq 0.$$

(If the subaccounts are automatically rebalanced (with fixed porportion) and each account is driven by a GBM, then the total fund is also driven by a GBM.)

Analytical methods

- Conditioning on that the policyholder's death occurs after maturity

$$\begin{aligned} L_0 &= e^{-rT}(G - F_T) - \int_0^T e^{-rs} m_e F_s \, \mathrm{d}s \\ &= e^{-rT} G - \left(e^{-rT} F_T + m_e \int_0^T e^{-rs} F_s \, \mathrm{d}s \right) \\ &\sim e^{-rT} G - X_T \qquad \text{(identity in distribution)} \end{aligned}$$

where X is determined by

$$\mathrm{d}X_t = \left[(\mu - \frac{\sigma^2}{2} - m - r) X_t + m_e \right] \mathrm{d}t + \sigma X_t \, \mathrm{d}B_t.$$

- Risk measures

$$
\begin{aligned}
\mathbb{P}(L_0 > V_\alpha) &= \mathbb{P}(L_0 > V_\alpha | \tau_x > T)\mathbb{P}(\tau_x > T) \\
&= \mathbb{P}(e^{-rT}G - X_T > V_\alpha)\mathbb{P}(\tau_x > T) \\
&= \mathbb{P}(X_T < K)\mathbb{P}(\tau_x > T)
\end{aligned}
$$

for some constant K.
Similarly, one can show that

$$
\mathrm{CTE}_\alpha = \frac{1}{\alpha}\mathbb{E}[(e^{-rT}G - X_T)I_{\{X_T<K\}}]\mathbb{P}(\tau_x > T).
$$

- Using spectral expansion, we obtained

$$\mathbb{P}(L_0 > V_\alpha) = \frac{x_0}{2\pi^2} \exp\left(-\frac{1}{4wx_0}\right) w^{\frac{\nu+1}{2}} \exp\left(\frac{1}{4x_0}\right) \int_0^\infty e^{-(\nu^2+p^2)t/2}$$
$$\times W_{-\frac{\nu+1}{2},\frac{ip}{2}}\left(\frac{1}{2wx_0}\right) W_{\frac{1-\nu}{2},\frac{ip}{2}}\left(\frac{1}{2x_0}\right) \left|\Gamma\left(\frac{\nu+ip}{2}\right)\right|^2 \sinh(\pi p) p \, \mathrm{d}p$$

where W is the Whitter function of the second kind and

$$t := \frac{\sigma^2 T}{4}, \nu = \frac{2(\mu - m - r)}{\sigma^2}, x_0 = \frac{\sigma^2}{4m_e}, K = x_0 w, w = \frac{e^{-rT}G - V_\alpha}{F_0}.$$

- We also obtained similar results for $\mathbb{E}(L_0 I_{\{L_0 > V_\alpha\}})$, which determines the conditional risk measure CTE_α.

Comparative study: GMMB (Joint work with Hans W. Volkmer)

- 10-year GMMB with full fund of initial deposit $G/F_0 = 1$;
- Mean and standard deviation of log-returns per annum $\mu = 0.09, \sigma = 0.3$;
- Risk-free discount rate per annum $r = 0.04$;
- M&E charges and rider charges per annum $m = 0.01$;
- GMMB rider charge 35 basis points of account value $m_e = 0.0035$.

Methods	Direct integration	Inverse Laplace	Monte Carlo
$V_{95\%}/F_0$	28.935%	28.935%	29.111%
Initial value	33%	$(28\%, 33\%)$	-
Time (mins)	3.7916	3.54375	396.224
$\mathrm{CTE}_{95\%}/F_0$	40.041%	40.042%	40.029%
Time (mins)	1.9325	0.28775	-

Comparative study: GMMB (Joint work with Hans W. Volkmer)

The same valuation assumptions.

Methods	Integration	Inverse Lap	Spectral	Green
$V_{90\%}$	12.55036%	12.55036%	12.55035%	12.55036%
Initial	10%	$(12\%, 14\%)$	$(12\%, 14\%)$	$(12\%, 14\%)$
Time	3.674(mins)	5.032(mins)	51.579(secs)	0.172(secs)
$\mathrm{CTE}_{90\%}$	30.29643%	30.29648%	30.29643%	30.29648%
Time	1.867(mins)	0.285(mins)	2.953(secs)	0(secs)

Tabelle : A comparison of computational methods for the GMMB rider

Variable annuity guaranteed benefits

- Brief introduction
- Guaranteed minimum maturity benefit – Individual model
- Guaranteed minimum maturity benefit - Average model
- Guaranteed minimum withdrawal benefit

- In practice, valuation actuaries use spreadsheets to run simulations based on traditional life insurance reserving methods. In each period, they compute

  Change in Surplus = Fee Income - Guaranteed Benifits
  - Expenses + Interest on Surplus

- The spreadsheet calculations are in essence the difference equation from a mathematical point of view. It is easy to show that the net liability used in practice for the GMMB is

$$L_0^* = e^{-rT}(G - F_T)_+ \, {}_Tp_x - \int_0^T {}_sp_x e^{-rs} m_e F_s \, \mathrm{d}s.$$

Connection between individual and average models

- Individual model:

$$L_0 = e^{-rT}(G - F_T)_+ I(\tau_x > T) - \int_0^{T\wedge\tau_x} e^{-rs} m_e F_s ds.$$

- Average model:

$$L_0^* = e^{-rT}(G - F_T)_+ \, {}_Tp_x - \int_0^T {}_sp_x e^{-rs} m_e F_s \, \mathrm{d}s.$$

- Connection: Assuming all policies are **of equal size** and the future lifetimes of policyholders are mutually indepedent, we can show that

$$\frac{1}{N}\sum_{i=1}^{N} L_0^{(i)} \longrightarrow L_0^*, \qquad \text{almost surely.}$$

F investment fund value; G guaranteed benefit; T maturity date; r risk-free interest rate; m_d GMMB fee rate.

Differences between individual and average models

- Individual model: (Two sources of randomness)

$$L_0 = e^{-rT}(G - F_T)_+ I(\tau_x > T) - \int_0^{T\wedge\tau_x} e^{-rs} m_e F_s ds.$$

- Average model: (One source of randomness, the mortality risk is fully diversified.)

$$L_0^* = \mathbb{E}[L_0|\mathcal{F}_T] = e^{-rT}(G - F_T)_+ \, {}_Tp_x - \int_0^T {}_sp_x e^{-rs} m_e F_s \, \mathrm{d}s.$$

- They make no difference for pricing since $\mathbb{E}[L_0] = \mathbb{E}[L_0^*] = \mathbb{E}[\mathbb{E}[L|\mathcal{F}_T]]$, yet they are different in terms of tail probabilities.
- We can compute the tail probability $\mathbb{P}(L_0^* > V)$ using numerical PDE methods but it is much slower than the computation of $\mathbb{P}(L_0 > V)$ through Laplace transform.

Comparison of Individual and Average Models

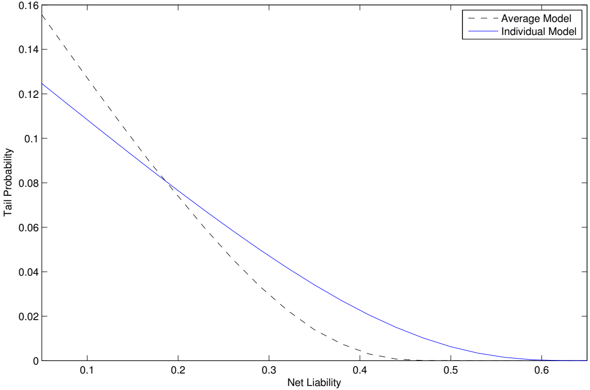


Abbildung : Survival functions of net liabilities

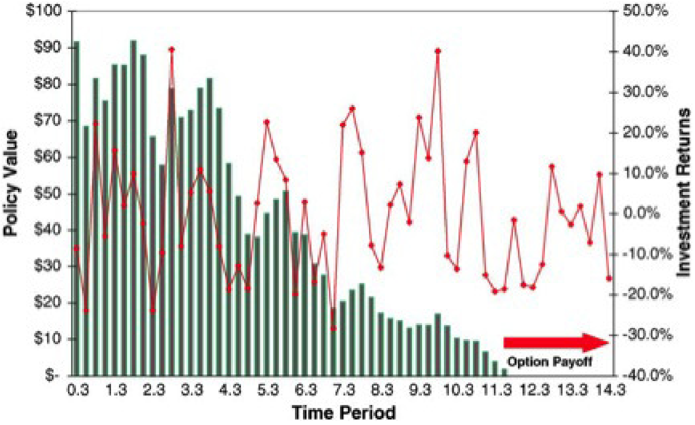
Variable annuity guaranteed benefits

- Brief introduction
- Guaranteed minimum maturity benefit – Individual model
- Guaranteed minimum maturity benefit - Average model
- Guaranteed minimum withdrawal benefit

Guaranteed Minimum Withdrawal Benefit(GMWB)

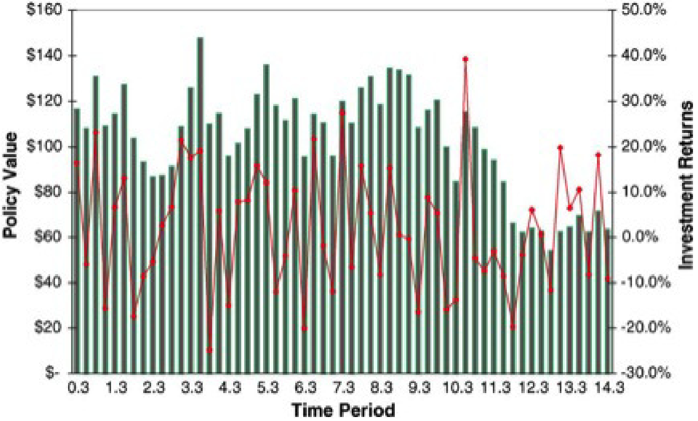
- Contains no life insurance component.
- Provides a minimal payout based on the initial purchase payment.
- For example: A policyholder is guaranteed the ability to withdraw \$7 per annum per \$100 of initial investment until the original \$100 has been fully exhausted. (The benefit expires after $\$100/\$7 \approx 14.28$ years.)

Illustration of a scenario with GMWB payments



Pictures taken from Milevsky and Salisbury (2006)

Illustration of a scenario without GMWB payments



Pictures taken from Milevsky and Salisbury (2006)

Mathematical Formulation

- Assume the dynamics of the equity-index is given by

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}B_t.$$

- The accumulated value of VA subaccounts

$$F_t = F_0 \frac{S_t}{S_0} e^{-mt} - wt, \qquad t \geq 0.$$

- The dynamics of VA investment fund is driven by

$$\mathrm{d}F_t = [(\mu - m)F_t - w] \,\mathrm{d}t + \sigma F_t \,\mathrm{d}W_t, \qquad F_0 = G > 0.$$

Mathematical Formulation (Joint work with Hans W. Volkmer)

- Pricing from an investor's point of view (Milevsky and Salisbury (2006))
  - Present value of guaranteed income (Maturity: $T = G/w$)

$$w \int_0^T e^{-rt}\, \mathrm{d}t = \frac{w}{r}(1 - e^{-rT}).$$

  - Investment income

$$e^{-rT} F_T I(\tau_0 > T), \quad \tau_0 := \inf\{t : F_t < 0\}.$$

  - The fair price m_w is determined by

$$F_0 = \frac{w}{r}(1 - e^{-rT}) + \mathbb{E}^Q[e^{-rT} F_T I(\tau_0 > T)].$$

w withdrawal rate; r risk-free interest rate; T guaranteed period; F fund value; τ_0 first time fund is exhausted.

Mathematical Formulation (Joint work with Hans W. Volkmer)

- Pricing from an insurer's point of view
  - Present value of Income (Assets)

$$m_w \int_0^{\tau_0 \wedge T} e^{-rt} F_t \, \mathrm{d}t.$$

  - Present value of Outgo (Liabilities)

$$w \int_{\tau_0}^{T} e^{-rt} \, \mathrm{d}t \; I(\tau_0 < T) = \frac{w}{r}(e^{-r\tau_0} - e^{-rT}) I(\tau_0 < T).$$

  - The fair price m_w is determined by

$$\mathbb{E}^Q \left[m_w \int_0^{\tau_0 \wedge T} e^{-rt} F_t \, \mathrm{d}t \right] = \frac{w}{r} \mathbb{E}^Q \left[(e^{-r\tau_0} - e^{-rT}) I(\tau_0 < T) \right].$$

w withdrawal rate; r risk-free interest rate; T guaranteed period; F fund value; m_w fees to fund GMWB.

- Pricing from an investor's point of view

$$F_0 = \frac{w}{r}(1 - e^{-rT}) + \mathbb{E}^Q[e^{-rT} F_T I(\tau_0 > T)].$$

- Pricing from an insurer's point of view

$$\mathbb{E}^Q \left[m_w \int_0^{\tau_0 \wedge T} e^{-rt} F_t \, \mathrm{d}t \right] = \frac{w}{r} \mathbb{E}^Q \left[(e^{-r\tau_0} - e^{-rT}) I(\tau_0 < T) \right].$$

- Milevsky and Salisbury (2006) used numerical PDE methods to price the GMWB from an investor's perspective. We found closed-form solutions in both cases.
- One can prove using Dynkin's formula that the two methods of pricing are equivalent when $m_w = m$.

Future Work

- Computation of risk measures for the GMWB would be rather difficult. (Importance sampling?)

$$w \int_{\tau_0 \wedge T}^{T} e^{-rt}\,\mathrm{d}t - m_w \int_0^{\tau_0 \wedge T} e^{-rt} F_t\,\mathrm{d}t.$$

- Guaranteed Lifetime Withdrawal Benefit (GLWB): The guarantee lasts until the policyholder's death.
  - Individual model: (T_x is the future lifetime of the policyholder at age x)

$$w \int_{\tau_0 \wedge T_x}^{T_x} e^{-rt}\,\mathrm{d}t - m_w \int_0^{\tau_0 \wedge T_x} e^{-rt} F_t\,\mathrm{d}t.$$

  - Average model:

$$w \int_{\tau_0}^{\infty} e^{-rt}\,{}_tp_x\,\mathrm{d}t - m_w \int_0^{\tau_0} e^{-rt}\,{}_tp_x F_t\,\mathrm{d}t.$$

Future Work

- Computation of risk measures for the GMWB would be rather difficult. (Importance sampling?)

$$w \int_{\tau_0 \wedge T}^{T} e^{-rt} \, \mathrm{d}t - m_w \int_0^{\tau_0 \wedge T} e^{-rt} F_t \, \mathrm{d}t.$$

- Guaranteed Lifetime Withdrawal Benefit (GLWB): The guarantee lasts until the policyholder's death.
  - Individual model: (T_x is the future lifetime of the policyholder at age x)

$$w \int_{\tau_0 \wedge T_x}^{T_x} e^{-rt} \, \mathrm{d}t - m_w \int_0^{\tau_0 \wedge T_x} e^{-rt} F_t \, \mathrm{d}t.$$

  - Average model:

$$w \int_{\tau_0}^{\infty} e^{-rt} \, {}_t p_x \, \mathrm{d}t - m_w \int_0^{\tau_0} e^{-rt} \, {}_t p_x F_t \, \mathrm{d}t.$$

References

- GMMB & GMDB:
  - R. Feng, H.W. Volkmer. Analytical calculation of risk measures for variable annuity guaranteed benefits. *Insurance: Mathematics and Economics* (2012), 51(3), 636-648.
  - R. Feng, H.W. Volkmer. Spectral methods for the calculation of risk measures for variable annuity guaranteed benefits. *ASTIN Bulletin*, (2014), to appear.
  - R. Feng. A comparative study of risk measures for the guaranteed minimum maturity benefit by a PDE method. *North American Actuarial Journal* (2014), to appear.
- GMWB:
  - M.A. Milevsky, T.S. Salisbury. Financial valuation of guaranteed minimum withdrawal benefits. *Insurance: Mathematics and Economics*, 38(1), 21-38.
  - R. Feng, H.W. Volkmer. An identity of hitting times and its application to the pricing of guaranteed minimum withdrawal benefit. Preprint.

Thank you very much for your attention!