Price Bounds of Mortality-Linked Security in Incomplete Insurance Market

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Introduction

- Longevity and mortality catastrophe have increased insurers' risk.
- Mortality securitization or mortality-linked securities (**MLS**) are regarded as a prescription to mitigate these risks.
- Two seminal innovations: the Swiss Re mortality bond issued in 2003 and the EIB/BNP longevity bond in 2004
- The MLS pricing issue also draws much attentions.

Purposes of this article

- Seek reasonable price bounds for MLS in incomplete market and take the Swiss Re mortality bond as an example.
- The gain-loss ratio is imposed as a subjective constraint on the potential MLS return to preclude too good investment opportunities.
- The price bounds will be useful for setting bid-asked spreads and making trading strategies for insurers.

Valuation approaches on MLS

- The risk-adjusted or no-arbitrage pricing method
  - Cairns et al. (2006a), Chen and Cox (2009), Dahl and Moller (2006), Hainaut and Devolder (2008) and Wills and Sherris (2010).
- The Wang transform method
  - Lin and Cox (2008), Denuit et al. (2007) and Cox et al. (2006)
- Instantaneous Sharpe ratio
  - Milevsky et al. (2005), Bayraktar and Young (2007), Young (2008) and Bayraktar et al. (2009)
- Utility-based valuation
  - Hainaut and Devolder (2008), Cox et al. (2010), Dahl and Moller (2006)

MLS Valuation Limitations 1

- The no-arbitrage pricing method
  - Use less model specifications on the economy, such as individual preference and wealth.
  - Require underlying transaction price.
- The utility-based method
  - Use restrictive assumptions such as specific utility function and aggregated wealth distribution
  - Suffering larger model risks and specification errors

Our Solutions 1

- Propose a framework that can accommodate model risk of the utility-based method when the transaction prices are unavailable, and that can keep the no-arbitrage pricing rule hold.

- No-arbitrage pricing still can be applied when we cave the MLS transaction prices.

MLS Valuation Limitations 2

- MLS market is an incomplete market
  - The underlying of MLS are usually not traded in financial markets
  - Insurers only have a limited hedging ability to replicate (can’t perfectly hedge) the MLS payoffs.
  - The no-arbitrage pricing method can only obtain a price range or a **price bound** instead of a single value.

Our Solution 2

- We explore rational price bounds of MLS under the insurers with a limited hedging ability.
- Try to narrow the no-arbitrage price bound by the gain-loss ratio method proposed by Bawa and Lindenberger (1977) and Bernardo and Ledoit (2000).

Intuitions of the gain-loss ratio 1

- The excess payoff

$$y = x - e^{rT} P^* \qquad (1)$$

  – where x is the random payoff of MLS

  – P^* is the benchmark price given by the benchmark model.

- Define

$$\text{Gain-loss ratio} = E^Q[y^+] / E^Q[y^-] \qquad (2)$$

  – where Q is risk-adjusted probability according to the benchmark model

  – $y^+ = \max(y, 0)$ the positive part of the excess payoff

  – $y^- = \max(-y, 0)$ the negative part of the excess payoff

Intuitions of the gain-loss ratio 2

- The benchmark price, $\boldsymbol{P}^*$, is correct or fair, if and only if $E^Q[y]=0$, or equivalently, the gain-loss ratio

$$E^Q[y^+]/E^Q[y^-]=1.$$

- To recognize the mispricing from the benchmark model (model risk or specification errors) and still keep the price in a reasonable range, we allow the gain-loss ratio to deviate from one but not exceed the **ceiling value** L:

$$E^Q[y^+]/E^Q[y^-] \leq L$$

Theoretical parts

- Bernardo and Ledoit (2000) show a duality between the admissible pricing kernels and the gain-loss ratio as

$$\underset{\left\{m_j>0,\ E[m_j x_j^{bi}]=P^{bi}\right\}}{\text{Min}} \frac{\sup\limits_{j=1,\ldots,J} (m_j / m_j^*)}{\inf\limits_{j=1,\ldots,J} (m_j / m_j^*)} = \underset{\substack{y\in\Psi \\ y\neq 0}}{\text{Max}} \frac{E^Q[y^+]}{E^Q[y^-]} \qquad (3)$$

where

m_j is individual' s pricing kernels over state j

m_j^* is benchmark pricing kernels over state j

- Therefore, imposing a restriction on the gain-loss ratio is equal to imposing a restriction on the pricing kernel.

The upper bounds

- Define the excess payoff for the upper bound

$$y^u = x^b - x.$$

where $x^b = \sum_i w_i x^{bi}$ is payoff of the basis asset portfolio formed by basis asset b_i with weight w_i.

- The **upper price bound** for the MLS is obtained by solving the minimisation problem:

$$\underset{w_{bi}}{\text{Min}} \quad \sum_i w_{bi} p^{bi} - P^* \geq 0$$

$$s.t \quad \frac{E^Q[(y^u)^+]}{E^Q[(y^u)^-]} \geq L \quad \text{and} \quad x^b = \sum w_{bi} x^{bi} ,$$

where $\sum_i w_{bi} p^{bi}$ is the replication cost and the lowest upper bound.

The lower bound

- Define the excess payoff for the lower bound

$$y^l = x - x^b$$

where $x^b = \sum_i w_i x^{bi}$ is payoff of the basis asset portfolio formed by basis basis asset b_i with weight w_i.

- The **lower price bound** for the MLS is obtained by solving the minimisation problem:

$$\underset{w_{bi}}{\text{Max}} \quad \sum_i w_{bi} p^{bi} - P^* \leq 0$$

$$s.t. \quad \frac{E^Q[(y^l)^+]}{E^Q[(y^l)^-]} \geq L \text{ and } x^b = \sum w_{bi} x^{bi},$$

where $\sum_i w_{bi} p^{bi}$ is the replication cost and highest lower bound.

Example: The Swiss Re Mortality Bonds

- Swiss Re issues the mortality bond, at an amount of $400 million, with the payoff at maturity day:

$$\text{loss}_t = \begin{cases} 1 & \text{if} \quad q_t > K_2 q_0 \\ \left(q_t - K_1 q_0\right)/(K_2 - K_1) q_0 & \text{if} \quad K_1 q_0 \le q_t \le K_2 q_0 \\ 0 & \text{if} \quad q_t < K_1 q_0. \end{cases}$$

where t = 1, 2, 3 for years 2004, 2005 and 2006

- The proportion of principal returned to the bondholders on the maturity date is

$$x = \text{Max}\left(1 - \sum_t \text{loss}_t,\ 0\right).$$

- Following Equation (1), the mortality bond price at time 0 is

$$P = E^Q[x]e^{-rT}.$$

Numerical Study Assumptions

- Selected Mortality Projection Model
  - The Lin and Cox (2008) mortality jump model
  - The Chen and Cox (2009) model
- Basis assets
  - Three basis assets: pure life insurances, pure annuities and risk-free bonds

The Numerical Programming

- Under the three-basis-asset framework, the **upper price bound** for the Swiss Re mortality bond is

$$\underset{w_{b1},\, w_{b2}, w_{b3} \in \mathbb{R}}{\text{Min}} \quad w_{b1} e^{-rT} + w_{b2} P^{b2} + w_{b3} P^{b3} - P^{*} \geq 0$$

$$s.t. \quad \frac{E^{Q}[(x_j^b - x_j)^{+}]}{E^{Q}[(x_j^b - x_j)^{-}]} \geq L \text{ and } x_j^b = w_{b1} + w_{b2}(1 - \sum_t q_{t,j}) + w_{b3} \sum_t q_{t,j}$$

- The replicating cost, $w_1 e^{-rT} + w_2 P^{b2} + w_3 P^{b3}$, is the upper price bound with three basis assets.

The Numerical Programming

- The three-basis-asset framework, the **lower price bound** for the Swiss Re mortality bond is

$$\max_{w_{b1},w_{b2},w_{b3}\in\mathbb{R}} \quad w_{b1}e^{-rT}+w_{b2}P^{b2}+w_{b3}P^{b3}-P^{*}\le 0$$

$$s.t. \quad \frac{E^{Q}[(x_j-x_j^b)^+]}{E^{Q}[(x_j-x_j^b)^-]}\ge L \text{ and } x_j^b=w_{b1}+w_{b2}(1-\sum_t q_{t,j})+w_{b3}\sum_t q_{t,j}.$$

- The replicating cost, $w_1e^{-rT}+w_2P^{b2}+w_3P^{b3}$, is the lower price bound with three basis assets.

Numerical procedures

1. Simulate the mortality projections by 15,000 times from 2004 to 2006. The initial mortality index q_0 starts from 0.007 to 0.014.
2. Specify the benchmark pricing model and determine the benchmark price P^* and basis asset prices P^{b1}, P^{b2}, and P^{b3}.
3. The strike prices are $K_1 = 1.3 \times 0.008453$ and $K_2 = 1.5 \times 0.008453$.

Numerical procedures

3. Substitute P^*, P^{b1}, P^{b2}, and P^{b3} into the optimisation programs.
4. Choose $L=2$ to 5 to solve the optimal weights and obtain the upper and lower bounds for P^*.
5. Repeat the above procedures for the three benchmark pricing models.

Numerical Results 1

- Price Bounds under Lin and Cox Model and Wang Transform Valuation
- Figure 1
- Table 1

Price bounds at q_0=0.008453 under the Lin and Cox (2008) model, L=2 to 5

[Lower bound, Upper bound]

	L=2	L=3	L=4	L=5	Benchmark price
(a) $\lambda = 0$	[0.986, 0.996]	[0.979, 0.998]	[0.972, 0.998]	[0.966, 0.999]	0.988
Range width	0.011	0.019	0.026	0.033	
(b) $\lambda = -0.45$	[0.971, 0.992]	[0.960, 0.995]	[0.946, 0.996]	[0.934, 0.997]	0.983
Range width	0.021	0.035	0.051	0.063	
(c) $\lambda = -1$	[0.943, 0.985]	[0.919, 0.990]	[0.897, 0.992]	[0.877, 0.994]	0.969
Range width	0.041	0.070	0.095	0.117	
(d) $\lambda = -1.3603$	**[0.926, 0.978]**	[0.897, 0.985]	[0.871, 0.989]	[0.848, 0.991]	**0.960**
Range width	**0.053**	**0.088**	**0.118**	**0.143**	

Analysis

- The higher gain-loss ratio, L, the wider the bound.
- Bounds range from the precise price implied by the benchmark pricing model, to the loosest price bounds implied by the pure no-arbitrage method.
- More kinds of basis assets, the tighter the bound.

Numerical Results 2

- Price Bounds under the Chen and Cox Model and no-arbitrage Pricing
- Figure 2
- Table 2

Price bounds at q_0=0.008453 under Chen and Cox model, L=2 to 5

	L=2	L=3	L=4	L=5	Benchmark price
(a) $\lambda_1 = \lambda_2 = \lambda_3 = 0$	[1.000, 1.000]	[1.000, 1.000]	[1.000, 1.000]	[1.000, 1.000]	1.000
Range width	0.000	0.000	0.000	0.000	
(b) $\lambda_1 = \lambda_2 = \lambda_3 = 0.83$	[0.996, 0.999]	[0.994, 0.999]	[0.992, 0.999]	[0.990, 1.000]	0.998
Range width	0.003	0.005	0.007	0.009	
(c) $\lambda_1 = \lambda_2 = \lambda_3 = 1$	[0.983, 0.996]	[0.976, 0.997]	[0.968, 0.998]	[0.962, 0.998]	0.990
Range width	0.012	0.021	0.029	0.036	
(d) $\lambda_1 = \lambda_2 = \lambda_3 = 1.5$	**[0.933, 0.981]**	[0.908, 0.987]	[0.887, 0.990]	[0.868, 0.992]	0.960
Range width	**0.047**	0.078	0.103	0.124	

Numerical Results 3

- The Price Bound with Additional Basis Assets
- Figure 4
- Table 4

Price bounds with an additional basis asset, from L=2 to 5.

	L=2	L=3	L=4	L=5	Benchmark price
(a) The Lin and Cox model with $\lambda = -1.3603$	**[0.947, 0.974]**	[0.932, 0.980]	[0.918, 0.984]	[0.905, 0.987]	0.960
Range width	**0.028**	0.049	0.066	0.082	
(b) The Chen and Cox model with $\lambda_1 = \lambda_2 = \lambda_3 = 1.5$	[0.941, 0.974]	[0.927, 0.980]	[0.914, 0.983]	[0.902, 0.986]	0.960
Range width	0.033	0.053	0.070	0.084	

Conclusions

- We provide a methodology to obtain the rational price bounds which is useful for MLS valuation.
- The price bounds allow model misspecification such as preference, mortality processes and risk adjustment method.
- The price bound is applicable to most existing MLS valuation models; it can be regarded as an add-on analysis tool for MLS pricing
- The price bound can be derived by the utility-based or the no-arbitrage method, and the MLS transaction data are not necessary.

The Limitations

- We assume a frictionless market, where the model builders can buy and sell an arbitrary amount of the basis assets.

- The price bound still suffers some model risk; if the benchmark model is biased, the ceiling value must be increased to loosen the price bounds to contain such risk.

- We do not consider the counter-party risk and loading fees, which may result in the undervaluation of the MLS.

Thank you for listening!

Mortality projection models

- Many stochastic mortality models have been proposed to capture mortality process pattern.
  - Dowd et al. (2010), Lin and Cox (2008), Chen and Cox (2009)
- Swiss Re mortality catastrophe bond
  - Mortality process with jumps is a significant phenomenon and should be captured in the stochastic mortality model.
  - We take the mortality catastrophe bond as an valuation example

An Example 1

- A two-period economy, three basis assets and four possible states.
- Three basis assets: risk-free bonds, annuities and life insurances; all the prices equal to 1.
- The four possible states have equal (subjective) probabilities: $\pi_j = 1/4$, for $j = 1, \ldots, 4$.
- The payoffs of basis assets at t=1 are:

$$x^{b1} = (1, 1, 1, 1) \text{ for the risk-free bond,}$$
$$x^{b2} = (0, 1, 1, 2) \text{ for the annuity,}$$
$$\text{and } x^{b3} = (3, 2, 0, 0) \text{ for the life insurance.}$$

- A new MLS with a terminal payoff $x = (0, 0, 0, 3)$ is introduced into the market with price P. What is the reasonable bound for P ?

An Example 2

- Utility-based pricing method assigns a pricing kernels in each state

$$m^* = (m_1^*,\ m_1^*, m_3^*, m_4^*)$$

where m^* is the benchmark pricing kernel representing the marginal rate of substitution across different states of the benchmark investor.

- Given m^*, MLS can be priced as

$$E[m^* x] = P^*$$

where $E[\ ]$ is the expectation operator under the subjective probability measure. We call the P^* benchmark price.

- For example, a risk-neutral investor with pricing kernel $m^* = (1, 1, 1, 1)$ will price the MLS by $P^* = 1/4 \times 3 = 3/4$.

An Example 3

- The no-arbitrage pricing rule requires $m_1, m_2, m_3, m_4 > 0$, and pricing kernels correctly price the basis assets:

$$1/4(m_1) + 1/4(m_2) + 1/4(m_3) + 1/4(m_4) = 1 \qquad \text{for the risk-free bond,}$$

$$1/4(m_2) + 1/4(m_3) + 1/4(m_4 \times 2) = 1 \qquad \text{for the annuity}$$

$$1/4(m_1 \times 3) + 1/4(m_2 \times 2) + 0 + 0 = 1 \qquad \text{for the life insurance.}$$

- The admissible pricing kernel sets are

$$m_2 = 2 - 3/2\, m_1,\ \ m_3 = 2 - 1/2\, m_1,\ \ m_4 = m_1 \ \text{ and } 0 < m_1 < 4/3$$

- This set yields the no-arbitrage price bound for the MLS is

$$0 < P < 1.$$

An Example 4

- Narrow the price bound by the gain-loss ratio constraint.
- For example, the risk-neutral investor with regards the excess payoffs of the MLS not to be greater than some value L,

$$\frac{\sup_j (m_1,\ 2-\frac{3}{2}m_1,\ 2-\frac{1}{2}m_1)}{\inf_j (m_1,\ 2-\frac{3}{2}m_1,\ 2-\frac{1}{2}m_1)} \le L \text{ and } m_1 > 0 .$$

- Then, the bounds on m:

$$4/(1+2L) < m_1 < (4L-4)/(3L-1),\ m_2=2-3/2\ m_1,$$

$$m_3=2-1/2\ m_1 \text{ and } m_4 = m_1.$$

- If we set L=3, (the investor regards the gain on the investment not to be better than 3 times of the loss), yield $4/7<m_1<1$, or $1/7<P<1/4$.