Pricing Longevity Swaps via Option Decomposition

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Introduction & Motivation

- A longevity swap involves counterparties swapping fixed for floating payments linked to the survivorship rate in a reference population
- They can be regarded as a portfolio of S-forwards with different maturity dates
- Assuming that there is a positive longevity risk premium, pricing longevity swaps involves the determination of risk-adjusted survival probabilities
- In this paper we show that the fair value of an Index-based Longevity Swap can be decomposed into a basket of long and short positions in European-style longevity options (caplets and floorlets) of different maturities with underlying asset equal to a realized survivor-index and strike equal to the initial fixed survivor schedule
- An analytical formula for the mark-to-market price of a longevity swap is obtained since the risk-adjusted survival probability can be expressed in a closed-form

Mathematical preliminaires

- We are given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and concentrate on an individual aged x at time 0
- We model his/her random lifetime as an $\mathbb{F}$-stopping time τ_x admitting a random intensity μ_x. Specifically, we consider τ_x as the first jump-time of a nonexplosive $\mathbb{F}$-counting process N recording at each time $t \geq 0$ whether the individual has died $(N_t \neq 0)$ or survived $(N_t = 0)$
- By further assuming that N is a Cox (or doubly stochastic) process driven by a subfiltration $\mathbb{G}$ of $\mathbb{F}$, with $\mathbb{F}$-predictable intensity μ, we can express the $T - t$-year "physical" survival probability of an $x + t$ year old individual as

$$ {}_{T-t}p_{x+t}(t) := \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\int_t^T \mu_{x+s}(s)\, ds \right) \middle| \mathcal{F}_t \right], \qquad (1) $$

Longevity Swap: contract design

Consider a fixed-rate payer index longevity swap by which two counterparties agree at time 0 to exchange at predetermined future calendar times an amount equal to the difference between the realized survival rate of a given population and a fixed survival rate agreed at contract inception (fixed leg), multiplied by a notional amount N

At predetermined future calendar dates $t_1 < t_2 < ... < t_n$, the owner of a fixed-rate payer longevity swap agrees to:

- Pay ${}_{t_k}p_x\,(0) \times N$, for $k = 1, 2, ..., n$, where $p_k^{BE} := {}_{t_k}p_x\,(0) \in \,]0, 1[$ is the time-0 risk-adjusted best estimate of future survival probabilities of an individual aged x at time 0 t—fixed leg;
- Receive ${}_{t_k}p_x^{obs}\,(t_k) \times N$, for $k = 1, 2, ..., n$, where

$$ {}_{t_k}p_x^{obs}\,(t_k) := \exp\left(-\int_0^{t_k} \mu_{x+s}\,(s)\,ds\right) \tag{2}$$

  is the time-t_k observable (or realized) survival rate from a "reference population" of individuals aged x at time 0—floating leg

Longevity Swap: option decomposition

The time-t_k payoff of the longevity swap on the kth reset date corresponds to the (terminal) payoff of an S-forward contract maturing at time t_k

$$F^S(t_k) := N \times \left[\exp\left(-\int_0^{t_k} \mu_{x+s}(s)\, ds \right) - p_k^{BE} \right], \tag{3}$$

Following the longevity option decomposition appoach pioneered by Bravo & El Mekkaoui (2018), this is equivalent to the terminal payoff of a portfolio combining a long position in a longevity caplet and a short position in a longevity floorlet with underlying the realized survival rate ${}_{t_k}p_x^{obs}(t_k)$, strike p_k^{BE}, maturity at time t_k, and notional amount N,

$$\begin{aligned} F^S(t_k) &= N \times \left[\exp\left(-\int_0^{t_k} \mu_{x+s}(s)\, ds \right) - p_k^{BE} \right]^+ - \\ & \quad N \times \left[p_k^{BE} - \exp\left(-\int_0^{t_k} \mu_{x+s}(s)\, ds \right) \right]^+, \end{aligned} \tag{4}$$

where $a^+ := \max(a, 0)$

Longevity Swap: option decomposition

The time-t_k terminal payoff of each S-forward comprising the longevity swap can be written as

$$F^S(t_k) = N \times {}_k p_{x_0}^{BE} \left[\left(I_x(0,T) - 1 \right)^+ - \left(1 - I_x(0,T) \right)^+ \right] \tag{5}$$

where

$$I_x(0,T) := \frac{\exp\left(-\int_0^T \mu_{x+s}(s)\, ds \right)}{{}_T p_x(0)} \tag{6}$$

is a longevity index that takes values in $\mathbb{R}_+$, because ${}_T p_x^{obs}(T)$, ${}_T p_x(0) \in]0,1[$

The first (second) term on the RHS of equation (5) can be understood as the terminal payoff of a longevity caplet (floorlet) on the longevity index $I_x(0,T)$, with strike 1, maturity at time t_k, and notional amount N

Longevity Swap: Valuation

Denoting by $\{r_t : t \geq 0\}$ the risk-free instantaneous interest rate process, and by $\mathbb{Q}$ the equivalent martingale measure associated to the numeraire "money-market account", the time-0 value of the longevity swap is

$$\begin{aligned} V_{t_0} &= N \times \mathbb{E}_{\mathbb{Q}} \left\{ \sum_{k=1}^{n} \exp\left(-\int_0^{t_k} r_s ds \right) \left[{}_{t_k}p_x^{obs}(t_k) - p_k^{BE} \right] \middle| \mathcal{G}_0 \right\} \\ &= N \times \mathbb{E}_{\mathbb{Q}} \left[\exp\left(-\int_0^{t_k} \left(r_s + \mu_{x+s}(s) \right) ds \right) \middle| \mathcal{H}_0 \right] - \qquad (7) \\ & \quad N \times \sum_{k=1}^{n} P(0, t_k) \times p_k^{BE}, \qquad (8) \end{aligned}$$

where $P(0, t_k)$ is the time-0 price of a default-free (and unit face value) zero-coupon bond with maturity at time t_k

Valuation of longevity caplets

Consider the terminal payoff of a "longevity caplet" on the realized survival rate

$$ {}_T p_x^{obs}(T) := \exp\left(-\int_0^T \mu_{x+s}(s)\, ds\right), \tag{9} $$

with strike $K \in \mathbb{R}_+$, maturity at time T, and notional amount $N = 1$

$$ c_T\left({}_T p_x^{obs}(T), K, T\right) = \left[\exp\left(-\int_0^T \mu_{x+s}(s)\, ds\right) - K\right]^+ . \tag{10} $$

The strike can be stated as some percentage—usually 100%—of the time-0 best estimate for the probability that an individual, aged x at time 0, is still alive at time T, i.e.

$$ K = \kappa \times {}_T p_x(0), \tag{11} $$

for $\kappa \in \mathbb{R}_+$.

Valuation of longevity caplets

The goal is to find the time-0 price for this contract, i.e.

$$
\begin{aligned}
& c_0\left({}_T p_x^{obs}\left(T\right), K, T\right) \\
= \; & \mathbb{E}_{\mathbb{Q}}\left\{\exp\left(-\int_0^T r_s ds\right) \times \left[\exp\left(-\int_0^T \mu_{x+s}\left(s\right) ds\right) - \kappa \times {}_T p_x\left(0\right)\right]\right.
\end{aligned}
$$

Changing the numeraire from the "money-market account" to the zero-coupon risk-free bond with maturity at time T—and time-0 price $P\left(0, T\right)$—that is associated to the equivalent forward measure $\mathbb{Q}_T$, then equation (12) can be restated as

$$
\begin{aligned}
& c_0\left({}_T p_x^{obs}\left(T\right), K, T\right) \\
= \; & P\left(0, T\right) \mathbb{E}_{\mathbb{Q}_T}\left\{\left[\exp\left(-\int_0^T \mu_{x+s}\left(s\right) ds\right) - \kappa \times {}_T p_x\left(0\right)\right]^+ \middle| \mathcal{H}_0\right\}.
\end{aligned}
\tag{13}
$$

Valuation of longevity caplets

- Since ${}_Tp_x^{obs}(T)$ only takes values in $]0,1[$, it is not possible to define the Fourier transform of the expectation contained on the right-hand side of equation (13) with respect to the strike

- Following (5), the option payoff can be rewritten as

$$\begin{aligned} & c_0\left({}_Tp_x^{obs}(T), K, T\right) \\ = \quad & P(0,T)\,{}_Tp_x(0)\,\mathbb{E}_{\mathbb{Q}_T}\left\{ \left[I_x(0,T) - \kappa \right]^+ \middle| \mathcal{H}_0 \right\}, \end{aligned} \tag{14}$$

where

  - $I_x(0,T)$ is the longevity index defined in (6), and
  - $P(0,T)\,{}_Tp_x(0)$ is the usual actuarial discount factor

Valuation of longevity caplets

Since both $l_x(0, T)$ and κ take values in $\mathbb{R}_+$, their logs will lie over the entire real line, and equation (14) becomes

$$c_0\left({}_Tp_x^{obs}(T), K, T\right) = P(0, T)\ {}_Tp_x(0)\ V(0, T; \omega), \tag{15}$$

where

$$V(0, T; \varpi) := \mathbb{E}_{\mathbb{Q}_T}\left\{\left[e^{z_x(0,T)} - e^{\varpi}\right]^+ \middle| \mathcal{H}_0\right\}, \tag{16}$$

with

$$z_x(0, T) := \ln l_x(0, T), \tag{17}$$

and

$$\varpi := \ln \kappa. \tag{18}$$

Valuation of longevity caplets

Define the characteristic function of the log longevity index as

$$g\left(t, T; \phi; \mu_{x+t}(t)\right) := \mathbb{E}_{\mathbb{Q}_T}\left[e^{i\phi z_x(t,T)} \middle| \mathcal{H}_t \right], \tag{19}$$

Following Carr and Madan (1999) and Lee (2004), if the characteristic function of the risk-neutral density is known analytically we can derive an analytic expression for the Fourier transform of the option value

Proposition 1: The time-0 fair value of the longevity caplet is given by

$$c_0\left({}_T p_x^{obs}(T), K, T\right) = P(0, T)\ {}_T p_x(0)\ V(0, T; \omega)$$

with

$$\begin{aligned} & V(0, T; \varpi) \\ = \quad & R(\alpha) + \frac{e^{\varpi}}{\pi} \int_{0-i(\alpha+1)}^{\infty-i(\alpha+1)} \mathrm{Re}\left[e^{-iz\varpi} \varsigma(0, T; z + i(\alpha+1); \alpha) \right] dz, \end{aligned} \tag{20}$$

where ϖ is defined by equation (18), $\alpha \in \mathbb{R}$,

Valuation of longevity caplets

$$\varsigma\left(0,T;u;\alpha\right):=\frac{g\left(0,T;u-i\left(1+\alpha\right);\mu_{x}\left(0\right)\right)}{\left(\alpha+iu\right)\left(\alpha+1+iu\right)},\text{ for }u\in\mathbb{R},\text{ and}\qquad(21)$$

and

$$R\left(\alpha\right):=\begin{cases}g\left(0,T;-i;\mu_{x}\left(0\right)\right)-e^{\varpi}\Longleftarrow\alpha<-1\\ g\left(0,T;-i;\mu_{x}\left(0\right)\right)-\frac{1}{2}e^{\varpi}\Longleftarrow\alpha=-1\\ g\left(0,T;-i;\mu_{x}\left(0\right)\right)\Longleftarrow-1<\alpha<0\\ \frac{1}{2}g\left(0,T;-i;\mu_{x}\left(0\right)\right)\Longleftarrow\alpha=0\\ 0\Longleftarrow\alpha>0\end{cases}.\qquad(22)$$

where α denotes the dampening parameter such that the dampened expectation $e^{\alpha\varpi}V\left(0,T;\varpi\right)$ is square integrable with respect to ϖ over the entire real line

Valuation of longevity floorlets

The terminal payoff of a "longevity floorlet" on the realized survival rate ${}_T p_x^{obs}(T) := \exp\left(-\int_0^T \mu_{x+s}(s)\,ds\right)$, with strike $K \in \mathbb{R}_+$, maturity at time T, and notional $N = 1$ is

$$p_T\left({}_T p_x^{obs}(T), K, T\right) = \left[K - \exp\left(-\int_0^T \mu_{x+s}(s)\,ds\right)\right]^+, \qquad (23)$$

where the strike is defined as in equation (11), i.e. $K = \kappa \times {}_T p_x(0)$, for $\kappa \in \mathbb{R}_+$

Assuming that $\tau_x > 0$, the time-0 value of this contract is

$$\begin{aligned} & p_0\left({}_T p_x^{obs}(T), K, T\right) \\ = \quad & P(0,T)\,\mathbb{E}_{\mathbb{Q}_T}\left\{\left[\kappa \times {}_T p_x(0) - \exp\left(-\int_0^T \mu_{x+s}(s)\,ds\right)\right]^+ \middle| \mathcal{H}_0\right\} \end{aligned} \qquad (24)$$

Valuation of longevity floorlets

Similarly to equation (14), the option value can be rewritten as

$$p_0\left({}_Tp_x^{obs}(T), K, T\right) = P(0,T)\,{}_Tp_x(0)\,\mathbb{E}_{\mathbb{Q}_T}\left\{\left[\kappa - I_x(0,T)\right]^+ \middle| \mathcal{H}_0\right\}, \tag{25}$$

where the longevity index $I_x(0, T)$ is still defined through equation (6)

Using equations (17) and (18), equation (25) can be restated as

$$p_0\left({}_Tp_x^{obs}(T), K, T\right) = P(0,T)\,{}_Tp_x(0)\,U(0,T;\omega), \tag{26}$$

where

$$U(0,T;\varpi) := \mathbb{E}_{\mathbb{Q}_T}\left\{\left[e^{\varpi} - e^{z_x(0,T)}\right]^+ \middle| \mathcal{H}_0\right\} \tag{27}$$

is given by the following proposition

Valuation of longevity floorlets

Proposition: The time-0 fair value of the longevity floorlet with terminal payoff (23) is given by equation (26) with

$$U(0, T; \varpi) = V(0, T; \varpi) - g(0, T; -i; \mu_x(0)) + e^{\varpi}, \tag{28}$$

where ϖ is defined by equation (18), and $g(\cdot)$ is the characteristic function (19).

Proof.

Using equations (16) and (27), then

$$\begin{aligned} & V(0, T; \varpi) - U(0, T; \varpi) \\ = \ & \mathbb{E}_{\mathbb{Q}_T}\left\{ \left[e^{z_x(0,T)} - e^{\varpi} \right]^+ \middle| \mathcal{H}_0 \right\} - \mathbb{E}_{\mathbb{Q}_T}\left\{ \left[e^{\varpi} - e^{z_x(0,T)} \right]^+ \middle| \mathcal{H}_0 \right\} \\ = \ & \mathbb{E}_{\mathbb{Q}_T}\left[e^{z_x(0,T)} - e^{\varpi} \middle| \mathcal{H}_0 \right] = \mathbb{E}_{\mathbb{Q}_T}\left[e^{z_x(0,T)} \middle| \mathcal{H}_0 \right] - e^{\varpi} \\ = \ & g(0, T; -i; \mu_x(0)) - e^{\varpi}, \end{aligned}$$

Valuation of longevity swaps

Next two propositions provide two alternative valuation approaches for longevity swaps.

Proposition: The time-0 fair value of the longevity swap is with t_k-payoff

$$F^S(t_k) := N \times \left[\exp\left(- \int_0^{t_k} \mu_{x+s}(s)\, ds \right) - {}_{t_k}p_x(0) \right], \tag{29}$$

is

$$V_0 = N \times \sum_{k=1}^{n} P(0, t_k) \times {}_{t_k}p_x(0) \times \left[g(0, t_k; -i; \mu_x(0)) - 1 \right], \tag{30}$$

where $g(\cdot)$ is the characteristic function

$$g(0, t_k; -i; \mu_x(0)) = \frac{\mathbb{E}_{\mathbb{Q}_{t_k}} \left[\exp\left(- \int_0^{t_k} \mu_{x+s}(s)\, ds \right) \middle| \mathcal{H}_0 \right]}{{}_{t_k}p_x(0)}. \tag{31}$$

Valuation of longevity swaps

Alternatively, the fair value of the fixed-rate payer longevity swap can be stated as a portfolio of long positions on longevity caplets and short positions on longevity floorlets

The time-0 fair value of the longevity swap with time-t_k payoff (29) equals

$$V_0 = N \times \sum_{k=1}^{n} \left[c_0 \left({}_{t_k}p_x^{obs}(t_k), {}_{t_k}p_x(0), t_k \right) - p_0 \left({}_{t_k}p_x^{obs}(t_k), {}_{t_k}p_x(0), t_k \right) \right] \tag{32}$$

where $c_0(\cdot)$ and $p_0(\cdot)$ are given by equations (15) and (26), respectively

The implementation of Proposition 1 requires two ingredients: the knowledge of the characteristic function (19) and the identification of the optimal dampening parameter α

Characteristic function

Proposition 2: Assume that the mortality intensity $\mu_{x+t}(t)$ is driven, under the forward probability measure $\mathbb{Q}_T$, by the jump-diffusion process

$$d\mu_{x+t}(t) = m\left(t, \mu_{x+t}(t)\right) dt + n\left(t, \mu_{x+t}(t)\right) dW_t^{\mathbb{Q}_T} + dJ_t^{\mathbb{Q}_T}, \qquad (33)$$

where $m\left(t, \mu_{x+t}(t)\right) \in \mathbb{R}$ and $n\left(t, \mu_{x+t}(t)\right) \in \mathbb{R}$ satisfy the usual *Lipschitz* and growth conditions, $\left\{W_t^{\mathbb{Q}_T} : t \geq 0\right\}$ is a $\mathbb{Q}_T$-measured standard Brownian motion, and

$$J_t^{\mathbb{Q}_T} = \sum_{i=1}^{N_t^{\mathbb{Q}_T}} Y_i^{\mathbb{Q}_T} \qquad (34)$$

is a compound Poisson process such that $\left\{N_t^{\mathbb{Q}_T} : t \geq 0\right\}$ is a $\mathbb{Q}_T$-measured standard Poisson process with intensity $\gamma \in \mathbb{R}$, and the jump sizes $\left\{Y_i^{\mathbb{P}}\right\}_{i=1}^{\infty}$ are i.i.d. random variables with density f_Y and mean $\zeta \in \mathbb{R}$

Characteristic function

Then, the characteristic function (19) solves the PIDE

$$
\begin{aligned}
0 \quad = \quad & -i\phi\mu_{x+t}(t)\, g\left(t, T; \phi; \mu_{x+t}(t)\right) + \frac{\partial g\left(t, T; \phi; \mu_{x+t}(t)\right)}{\partial t} && (35)\\
& + m\left(t, \mu_{x+t}(t)\right) \frac{\partial g\left(t, T; \phi; \mu_{x+t}(t)\right)}{\partial \mu_{x+t}(t)} \\
& + \frac{1}{2} n^2\left(t, \mu_{x+t}(t)\right) \frac{\partial^2 g\left(t, T; \phi; \mu_{x+t}(t)\right)}{\partial \mu_{x+t}(t)^2} && (36)\\
& + \gamma \int_{-\infty}^{\infty} \left[g\left(t, T; \phi; \mu_{x+t}(t) + y\right) - g\left(t, T; \phi; \mu_{x+t}(t)\right) \right] f_Y(y)\, dy,
\end{aligned}
$$

subject to the terminal condition

$$
g\left(T, T; \phi; \mu_{x+T}(T)\right) = \exp\left(-i\phi \ln {}_T p_x(0)\right). \qquad (37)
$$

If the drift and the squared diffusion of the SDE (33) are specified as *affine functions* of $\mu_{x+t}(t)$, the PIDE (35) can be decomposed into a simpler system of ODEs (Bjork, 1998; Duffie, Pan and Singleton, 2000).

Characteristic function

- Under the same assumptions of Proposition 2, and if

$$m\left(t, \mu_{x+t}\left(t\right)\right) = b + a\mu_{x+t}\left(t\right), \tag{38}$$

and

$$n\left(t, \mu_{x+t}\left(t\right)\right) = \sqrt{d + c\mu_{x+t}\left(t\right)}, \tag{39}$$

for $a, b \in \mathbb{R}$ and $a, b \in \mathbb{R}_{+}$, then the characteristic function (19) is

$$g\left(t, T; \phi; \mu_{x+t}\left(t\right)\right) = \exp\left[\alpha\left(t, T; \phi\right) + \beta\left(t, T; \phi\right)\mu_{x+t}\left(t\right)\right], \tag{40}$$

where $\alpha\left(t, T; \phi\right), \beta\left(t, T; \phi\right) \in \mathbb{C}$ solve the complex-valued ODEs:

Characteristic function

$$\frac{\partial \beta(t, T; \phi)}{\partial t} = i\phi - a\beta(t, T; \phi) - \frac{1}{2}c\beta^2(t, T; \phi) \qquad (41)$$

and

$$\begin{aligned}\frac{\partial \alpha(t, T; \phi)}{\partial t} &= -(b + \gamma\zeta)\beta(t, T; \phi) - \frac{1}{2}d\beta^2(t, T; \phi) \qquad (42)\\ &\quad -\gamma \int_{-\infty}^{\infty} \left[e^{\beta(t, T; \phi)y} - 1\right] f_Y(y)\, dy,\end{aligned}$$

subject to the boundary conditions

$$\alpha(T, T; \phi) = -i\phi \ln {}_T p_x(0)$$

and

$$\beta(T, T; \phi) = 0$$

Pricing model: Actuarial market

We assume that, under the real world probability measure $\mathbb{P}$, $\mu_{x+t}(t)$ is driven by a square root process (with no mean reversion) and a "double exponential" compound Poisson process:

$$d\mu_{x+t}(t) = a\mu_{x+t}(t)\,dt + \sigma\sqrt{\mu_{x+t}(t)}\,dW_t^{\mathbb{P}} + d\left(\sum_{i=1}^{N_t^{\mathbb{P}}} Y_i^{\mathbb{P}}\right), \qquad (43)$$

where $\mu_x(0) > 0$, $a > 0$, $\sigma > 0$, $\left\{W_t^{\mathbb{P}} : t \geq 0\right\}$ is a $\mathbb{P}$-measured standard Brownian motion, and $\left\{N_t^{\mathbb{P}} : t \geq 0\right\}$ is a $\mathbb{P}$-measured standard Poisson process with intensity η

The jump sizes $\left\{Y_i^{\mathbb{P}}\right\}_{i=1}^{\infty}$ are i.i.d. random variables with the asymmetric double exponential density (Kou and Wang, 2004):

$$f(y) := \frac{\pi_1}{v_1} e^{-\frac{y}{v_1}} \mathbb{I}_{\{y\geq 0\}} + \frac{\pi_2}{v_2} e^{\frac{y}{v_2}} \mathbb{I}_{\{y<0\}}, \qquad (44)$$

where $\pi_1, \pi_2 \geq 0$, $v_1, v_2 > 0$, and $\pi_1 + \pi_2 = 1$

Pricing model: Actuarial market

To price longevity derivatives, the SDE (43) must be rewritten under the pricing measure $\mathbb{Q}$; For the diffusion component of the longevity risk, we assume

$$dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \lambda_d \frac{\sqrt{\mu_{x+t}(t)}}{\sigma} \tag{45}$$

is a standard Brownian motion increment under the equivalent martingale measure $\mathbb{Q}$, for $\lambda_d \in \mathbb{R}$

The jump component of the longevity risk is accounted for through a new $\mathbb{Q}$-measured standard Poisson process $\left\{N_t^{\mathbb{Q}} : t \geq 0\right\}$ with intensity $\bar{\eta}$, and new i.i.d. jump sizes $\left\{Y_i^{\mathbb{Q}}\right\}_{i=1}^{\infty}$ with a different asymmetric double exponential density

$$f(y) := \frac{\bar{\pi}_1}{\bar{v}_1} e^{-\frac{y}{\bar{v}_1}} \mathbb{I}_{\{y \geq 0\}} + \frac{\bar{\pi}_2}{\bar{v}_2} e^{\frac{y}{\bar{v}_2}} \mathbb{I}_{\{y<0\}}, \tag{46}$$

where $\bar{\pi}_1, \bar{\pi}_2 \geq 0$, $\bar{v}_1, \bar{v}_2 > 0$, and $\bar{\pi}_1 + \bar{\pi}_2 = 1$

Pricing model

Actuarial market

In brief, under the pricing measure $\mathbb{Q}$, the mortality intensity $\mu_{x+t}(t)$ is driven by the following affine-jump-diffusion process (with finite activity):

$$d\mu_{x+t}(t) = (a - \lambda_d)\,\mu_{x+t}(t)\,dt + \sigma\sqrt{\mu_{x+t}(t)}dW_t^{\mathbb{Q}} + d\left(\sum_{i=1}^{N_t^{\mathbb{Q}}} Y_i^{\mathbb{Q}}\right). \tag{47}$$

It can be shown that, for the affine pricing model adopted, it is possible to obtain an explicit solution for the ODEs (41) and (42)

Pricing model

Financial market

A HJM (1992) model guaranteeing an automatic fit of the observed yield curve will be adopted, i.e,

$$\frac{dP(t,T)}{P(t,T)} = r_t dt + \sigma(t,T)' \cdot dZ_t^{\mathbb{Q}}, \tag{48}$$

where $\cdot$ denotes the inner product in $\mathbb{R}^n$, and $\left\{Z_t^{\mathbb{Q}} \in \mathbb{R}^n : t \geq 0\right\}$ is a n-dimensional standard Brownian motion, initialized at zero and generating the augmented, right continuous and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ Following, e.g.,, Schräger (2005), we further "assume independence between financial markets and mortality"

Model Calibration

- Mortality Data:
  - USA Total population 1950-2017, Source: HMD (2019)
  - We follow a cohort approach (generation born in 1950)
- Assumptions for the calibration:
  - Age range $[65, 100]$
  - $x_0 = 65$
  - $\mu_{x_0}(t_0) = -\ln(p_{x_0}(t_0))$

We estimate parameters using a ML approach by minimizing

$$L = \min_{a,\sigma,\eta,\pi_1,\pi_2,v_1,v_2} \left[\sum_{k=1}^{T_x} \left({}_k p_{x_0}^{obs}(t_0) - {}_k\, p_{x_0}^{\text{model}}(t_0) \right)^2 \right] \tag{49}$$

- Flat yield curve at 2%

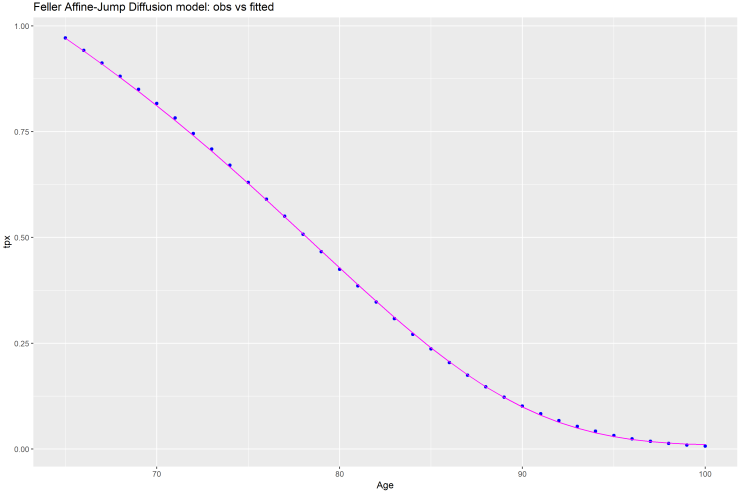
Model Calibration

Parameter estimates

Parameter	Value
a	0.07540775
σ	0.009747797
η	0.09983138
π_1	0.0001000325
π_2	$1-\pi_1$
v_1	0.0010000150
v_2	0.0008243410
$\mu_{65}(0)$	0.02883801
SSE	0.000208508

Model Calibration: Feller AJD

Fitting results



Longevity Swap Pricing Results

Longevity caplets, floorlets & S-Forwards

tenor	caplet			floorlet			S-forward		
yrs	k=0.25	k=0.5	k=0.75	k=0.25	k=0.5	k=0.75	k=0.25	k=0.5	k=0.75
1	5,9	6,1	6,3	5,6	5,4	5,2	0,4	0,7	1,1
2	11,5	12,3	13,0	10,1	9,5	8,8	1,4	2,8	4,2
3	20,9	22,5	24,2	17,8	16,3	14,9	3,1	6,2	9,4
4	32,0	34,9	37,9	26,5	23,9	21,5	5,5	11,0	16,5
5	44,3	48,8	53,5	35,7	31,7	28,1	8,5	17,0	25,4
6	57,4	63,8	70,7	45,2	39,6	34,6	12,1	24,2	36,1
7	71,1	79,8	89,1	54,8	47,4	40,7	16,3	32,4	48,4
8	85,0	96,3	108,3	64,2	54,8	46,4	20,8	41,5	61,9
9	98,9	113,0	128,1	73,1	61,7	51,5	25,8	51,3	76,5
10	112,6	129,6	148,0	81,5	67,9	56,0	31,0	61,7	92,0
11	125,6	145,8	167,6	89,2	73,4	59,7	36,4	72,4	107,9
12	137,9	161,3	186,7	96,0	78,1	62,6	41,8	83,2	124,0
13	149,0	175,7	204,7	101,8	81,8	64,8	47,2	93,8	140,0
14	158,8	188,7	221,4	106,4	84,6	66,1	52,3	104,1	155,3
15	167,0	200,0	236,3	109,8	86,3	66,5	57,1	113,7	169,7

Note: values in basis points

Longevity Swap Pricing Results

Longevity Swaps

Maturity	Longevity Swap		
Yrs	k=0.25	k=0.5	k=0.75
1	0,4	0,7	1,1
2	1,8	3,5	5,2
3	4,9	9,7	14,6
4	10,4	20,8	31,1
5	19,0	37,8	56,5
6	31,1	62,0	92,6
7	47,4	94,4	141,0
8	68,2	135,8	202,9
9	94,0	187,2	279,4
10	125,0	248,8	371,4
11	161,4	321,2	479,4
12	203,3	404,4	603,4
13	250,5	498,2	743,4
14	302,8	602,4	898,7
15	359,9	716,1	1.068,5

Note: values in basis points

Final remarks

- We show that the fair value of an Index-based Longevity Swap can be decomposed into a basket of long and short positions in European-style longevity options (caplets and floorlets) of different maturities with underlying asset equal to a realized survivor-index and strike equal to the initial fixed survivor schedule
- Assuming the mortality intensity $\mu_{x+t}(t)$ is driven by an affine-jump diffusion process, we can derive an analytical solution for the characteristic function of the log longevity index and valuate longevity options using the fast Fourier transform
- An analytical formula for the mark-to-market price of a longevity swap can then be easily derived and priced efficiently
- Thank you!