

OPTION PRICING WITH LEGENDRE POLYNOMIALS

J. Hok

joint work with T.L.R Chan

CASS Business school at London, March 2018

"The opinions expressed in this presentation and on the following slides are solely those of the presenter and not necessary those of CA-CIB. CA-CIB does not guarantee the accuracy or reliability of the information provided herein."

INTRODUCTION AND PLAN

Feynman-Kac formula establishes a link between parabolic partial differential equations (PDEs) and stochastic conditional expectation of payoff function under risk neutral measure:

- PDE representation

$$\frac{\partial V}{\partial t}(t, x) + \mu(t, x) \frac{\partial V}{\partial x}(t, x) + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 V}{\partial x^2}(t, x) - rV(t, x) = 0 \quad (1)$$

for $x \in \mathbb{R}$ and $t \in [0, T]$ subject to the terminal condition $V(T, x) = \psi(x)$.

- Probabilistic representation

$$V(t=0, x) = e^{-rT} \mathbb{E}^{\mathbb{Q}}[V(X_T, T) | X_0 = x] = e^{-rT} \int_{\mathbb{R}} V(y, T) \tilde{f}(y|x) dy \quad (2)$$

with \mathbb{Q} the probability measure s.t X is an Ito process given by

$$dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dW^{\mathbb{Q}}(t) \quad (3)$$

$W^{\mathbb{Q}}(t)$ the Wiener process.

INTRODUCTION AND PLAN

- probability densities $\tilde{f}(y|x)$ are usually unknown.
- Their Fourier transform or characteristic functions are available:

For examples: by Levy-Khinchine theorem (see e.g Cont and Tankov 04) the characteristic functions of Levy processes are known. Pure diffusion context with stochastic volatility (Heston 93) and with stochastic rates (Bakshi and Chen 97).

⇒ the Fourier transform methods for option pricing (see Carr and Madan 99).

Subsequently, some new numerical methods are proposed: QUAD method (Andricopoulos et al 03), CONV method (Lord et al 08), fast Hilbert transform approach (Feng and Linetsky 08).

Highly efficient COS technique, based on Fourier-cosine series expansion of the density function, was proposed by Fang and Oosterlee (08) and has generated other developments by Hurn et al (13) or by Ding et al (11).

Recently, Necula et al (16) have employed the modified Gram-Charlier series expansion, known as the Gauss-Hermite expansion, for the density function and obtained a closed form pricing formula for European option.

Here, we consider an alternative and propose to expand the probability density function $\tilde{f}(y)$ using **Legendre polynomials** when the characteristic function is known:

- Approximating non periodic function on a finite interval: usually we use either Chebyshev polynomials or Legendre polynomials. Legendre polynomial offers tractability allowing to compute analytically many quantities of interests. For example its Fourier transform has analytical formula which is instrumental and used to recover the coefficients in the series expansion of the density function.

INTRODUCTION AND PLAN

- They are widely used for variety of mathematical and numerical solutions. Ex:
 - **Physics:** for the determination of wave functions of electrons in the orbits of an atom (Dicke and Wittke 60, Hollas 92), for the determination of potential functions in the spherically symmetric geometry (Jackson 62).
 - **Numerical analysis,** Legendre polynomials are used to efficiently calculate numerical integrations by Gaussian quadrature method (Mughal et al 06).
- Quantitative finance: Pulch and Emmerich (09) consider the fair price of options as the expected value of a random field. The Polynomial chaos theory using Legendre polynomial yields an efficient approach for calculating the required fair price. They are used to described the terms structure of interest rates in arbitrage free interest rate models (Almeida 04, Ibsen and Almeida 05, Duarte et al 09).

SERIES EXPANSION OF DENSITY FUNCTION WITH LEGENDRE POLYNOMIALS

Objective: Expansion of probability density function using Legendre polynomials when the characteristic function is known and European type option pricing.

Plan:

- 1 Series expansion of density function with Legendre polynomials
- 2 A new computational method for option pricing
- 3 Error analysis
- 4 Numerical experiments
- 5 Conclusions and discussions

SERIES EXPANSION OF DENSITY FUNCTION WITH LEGENDRE POLYNOMIALS

Generalized Fourier Series-Legendre polynomials

$(P_n(t))_{n \geq 0}$ form a complete basis on $[-1, 1]$ and is defined by

$$P_n(t) = \frac{1}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k C_n^k C_{2n-2k}^n t^{n-2k} \quad (4)$$

$\lfloor r \rfloor$ the floor function and $C_n^k = \frac{n!}{k!(n-k)!}$.

Or for an arbitrary interval $[a, b]$, we have

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k C_n^k C_{2n-2k}^n \left[\frac{(2x - (a+b))}{(b-a)} \right]^{n-2k}. \quad (5)$$

via a change of variable $t = \frac{(2x-(a+b))}{(b-a)}$.

Orthogonality relation:

$$\int_a^b P_n \left(\frac{2x - (a+b)}{b-a} \right) P_m \left(\frac{2x - (a+b)}{b-a} \right) dx = \delta_{n=m} \frac{(b-a)}{2m+1} \quad (6)$$

SERIES EXPANSION OF DENSITY FUNCTION WITH LEGENDRE POLYNOMIALS

For suitable $f(x)$ on $[a, b]$, we have the generalized Fourier series representation:

$$f(x) = \sum_{n=0}^{\infty} A_n P_n \left(\frac{2x - (a + b)}{b - a} \right) \quad (7)$$

with

$$A_n = \frac{2n + 1}{b - a} \int_a^b f(x) P_n \left(\frac{2x - (a + b)}{b - a} \right) dx \quad (8)$$

Approximate risk-neutral probability density function using standard Fourier series

For a suitable function $f(x)$ on $[a, b]$, its complex Fourier series representation is given by

$$f(x) = \sum_{k=-\infty}^{\infty} B_k e^{i(\frac{2\pi}{b-a}x)k}, \quad \text{with } B_k = \frac{1}{b - a} \int_a^b f(x) e^{-ik(\frac{2\pi}{b-a}x)} dx. \quad (9)$$

SERIES EXPANSION OF DENSITY FUNCTION WITH LEGENDRE POLYNOMIALS

By replacing (9) in (8) + change order of integration:

$$A_n = \frac{2n+1}{b-a} \sum_{k=-\infty}^{+\infty} B_k \int_a^b P_n \left(\frac{2x - (a+b)}{b-a} \right) e^{i2\pi \left(\frac{xk}{b-a} \right)} dx \quad (10)$$

and

$$\int_a^b P_n \left(\frac{2x - (a+b)}{b-a} \right) e^{i2\pi \left(\frac{xk}{b-a} \right)} dx = \begin{cases} i^n \frac{(b-a)}{2} e^{\frac{i\pi k(a+b)}{b-a}} \sqrt{\frac{2}{k}} J_{n+\frac{1}{2}}(\pi k), & k \neq 0, \\ (b-a)\delta_{n=0}, & k = 0. \end{cases} \quad (11)$$

with $J_\nu(z)$ Bessel function of first kind (see Boisvert et al. 2010).

$$\Rightarrow A_n = \frac{2n+1}{\sqrt{2}} \left[\sum_{k \neq 0} B_k i^n e^{\frac{i\pi k(a+b)}{b-a}} \frac{J_{n+\frac{1}{2}}(\pi k)}{\sqrt{k}} + B_0 \sqrt{2} \delta_{n=0} \right] \quad (12)$$

SERIES EXPANSION OF DENSITY FUNCTION WITH LEGENDRE POLYNOMIALS

Couple of definitions:

- A function $f(x)$ is said to be *piecewise smooth* on the interval $[a, b]$ if either $f(x)$ and its derivative are both continuous on $[a, b]$, or they have only a finite number of *jump discontinuities* on $[a, b]$.
- If x_0 is a point of discontinuity of a function $f(x)$ and if the right-hand and left-hand limits exist, x_0 is said to be a point of *jump discontinuity*.

Let's set

$$f_k^n(x) = B_k P_n \left(\frac{2x - (a + b)}{b - a} \right) e^{i2\pi \left(\frac{xk}{b-a} \right)}, \quad x \in [a, b], \quad k \in \mathbb{Z}, \quad n \in \mathbb{N}. \quad (13)$$

and consider, for a given $n \in \mathbb{N}$, the series of functions

$$\sum_{k=-\infty}^{+\infty} f_k^n(x). \quad (14)$$

SERIES EXPANSION OF DENSITY FUNCTION WITH LEGENDRE POLYNOMIALS

THEOREM

Let's denote by $f(x)$, the restriction of the probability density function $\tilde{f}(x)$ on $[a, b]$ large enough such that $f(a) = f(b)$ and $\varphi(x)$ the characteristic function associated to $\tilde{f}(x)$. Assume that $f(x)$ is a continuous piecewise smooth function and that the series (14) is uniformly convergent on $[a, b]$ for all $n \in \mathbb{N}$. Then we have the following Legendre series representation

$$f(x) = \sum_{n=0}^{\infty} A_n P_n \left(\frac{2x - (a + b)}{b - a} \right) \quad (15)$$

with $A_n = \frac{2n+1}{\sqrt{2}} \left[\sum_{k \neq 0} B_k i^n e^{\frac{i\pi k(a+b)}{b-a}} \frac{J_{n+\frac{1}{2}}(\pi k)}{\sqrt{k}} + B_0 \sqrt{2} \delta_{n=0} \right]$.

SERIES EXPANSION OF DENSITY FUNCTION WITH LEGENDRE POLYNOMIALS

Approximation for A_n

$\tilde{f}(x)$ probability density function, its characteristic function

$$\varphi(u) = \int_{\mathbb{R}} e^{iux} \tilde{f}(x) dx \quad (16)$$

and $\tilde{f}(x) \rightarrow 0$ as $|x| \rightarrow +\infty$

$$\Rightarrow \forall \epsilon > 0, \exists [a, b] \text{ s.t. } \tilde{f}(x) < \epsilon, x \notin [a, b]$$

$f(x)$ is the restriction of $\tilde{f}(x)$ on $[a, b]$:

$$B_k = \frac{1}{b-a} \int_a^b f(x) e^{-ik(\frac{2\pi}{b-a}x)} dx \approx \frac{1}{b-a} \int \tilde{f}(x) e^{-ik(\frac{2\pi}{b-a}x)} dx := \tilde{B}_k \quad (17)$$

By truncating the infinite sum in (12), we get:

$$A_n^M := \frac{2n+1}{\sqrt{2}} \left[\sum_{k=-M, \neq 0}^M \tilde{B}_k i^n e^{\frac{i\pi k(a+b)}{b-a}} \frac{J_{n+\frac{1}{2}}(\pi k)}{\sqrt{k}} + \tilde{B}_0 \sqrt{2} \delta_{n=0} \right] \approx A_n \quad (18)$$

A NEW COMPUTATIONAL METHOD FOR OPTION PRICING

Let's define

$$x := \ln\left(\frac{S_0}{K}\right) \text{ and } y := \ln\left(\frac{S_T}{K}\right), \quad (19)$$

with S_t the underlying price at time t and K the strike price.

The payoff for European options, in log-asset price:

$$V(y, T) = [\alpha.K(e^y - 1)]^+ \text{ with } \alpha = \begin{cases} 1 & \text{for a call,} \\ -1 & \text{for a put,} \end{cases} \quad (20)$$

and

$$V(y, T) = 1_{\alpha y \geq 0} \text{ with } \alpha = \begin{cases} 1 & \text{for a digital call,} \\ -1 & \text{for a digital put,} \end{cases} \quad (21)$$

Here, we focus on the pricing formula for European call option and European digital call option. The valuation for other contracts like asset-or-nothing options, gap options or standard power options can be computed similarly.

A NEW COMPUTATIONAL METHOD FOR OPTION PRICING

European call price:

$$V(x, 0) = e^{-rT} K \mathbb{E}[(e^y - 1)^+] = e^{-rT} K \int_{-\infty}^{+\infty} (e^y - 1)^+ \tilde{f}(y|x) dy \quad (22)$$

$$V(x, 0) \approx V_1(x, 0) = e^{-rT} K \int_a^b (e^y - 1)^+ f(y|x) dy \quad (23)$$

we replace $f(y|x)$ by its Legendre series representation (15) to obtain the following proposition

A NEW COMPUTATIONAL METHOD FOR OPTION PRICING

PROPOSITION

Under the hypotheses of theorem (15), we obtain a valuation approximation given by the following Legendre polynomial pricing formula

$$V_4(x, 0) = e^{-rT} \sum_{n=0}^N A_n^M V_n \quad (24)$$

where A_n^M is given by (18) and V_n by

$$V_n = \begin{cases} K\beta \left[e^{\frac{a+b}{2}} \int_{\alpha}^1 e^{\beta t} P_n(t) dt - \frac{P_{n-1}(\alpha) - P_{n+1}(\alpha)}{2n+1} \right] & \text{for European call} \\ \frac{P_{n-1}(\alpha) - P_{n+1}(\alpha)}{2n+1} & \text{for European digital call} \end{cases} \quad (25)$$

with $\alpha = \frac{a+b}{a-b}$ and $\beta = \frac{(b-a)}{2}$.

A NEW COMPUTATIONAL METHOD FOR OPTION PRICING

Computational procedure

$$U_n = \int_{\alpha}^1 e^{\beta t} p_n(t) dt \quad (26)$$

$$= W_n - \frac{1}{\beta} \int_{\alpha}^1 e^{\beta t} p_n'(t) dt \quad (27)$$

$$W_n = \frac{1}{\beta} (e^{\beta} - e^{\beta \alpha} P_n(\alpha)). \quad (28)$$

Using the Legendre polynomial property

$$(2n + 1)P_n(t) = P'_{n+1}(t) - P'_{n-1}(t), \quad (29)$$

$$p_n'(t) = \begin{cases} \frac{2}{\|p_{n-1}\|^2} p_{n-1}(t) + \sum_{i=0, 2i \leq (n-3)} \frac{2}{\|p_{2i}\|^2} p_{2i}(t) & \text{for odd } n \geq 3, \\ \frac{2}{\|p_{n-1}\|^2} p_{n-1}(t) + \sum_{i=0, 2i+1 \leq (n-3)} \frac{2}{\|p_{2i+1}\|^2} p_{2i+1}(t) & \text{for even } n \geq 2 \end{cases} \quad (30)$$

A NEW COMPUTATIONAL METHOD FOR OPTION PRICING

Using (27) with (30), we obtain the second-order linear difference equation:

PROPOSITION

By posing $Y_n = U_n - W_n$, Y_n satisfies the following second-order linear difference equation

$$Y_{n-1} - \frac{1}{\beta}(2n+1)Y_n - Y_{n+1} = \frac{1}{\beta}(2n+1)W_n. \quad (31)$$

given Y_0 and Y_1 .

The appropriate approach consists to treat the difference equation as a boundary-value problem rather than using initial-value technique: Olver's method (Olver 67) evaluates U_n to machine accuracy in a stable way.

Let's write the successive approximations introduced in the derivation of the pricing formula (24):

$$V(x, 0) = \int_{-\infty}^{\infty} V(y, T) \tilde{f}(y|x) dy = V_1(x, 0) + \epsilon_1 \quad (32)$$

$$V_1(x, 0) = \int_a^b V(y, T) \tilde{f}(y|x) dy, \quad \epsilon_1 = \int_{\mathbb{R}-[a,b]} V(y, T) f(y|x) dy \quad (33)$$

$$V_1(x, 0) = \sum_{k=0}^{\infty} A_k V_k = V_2(x, 0) + \epsilon_2 \quad (34)$$

with

$$V_2(x, 0) = \sum_{k=0}^{N-1} A_k V_k, \quad \epsilon_2 = \sum_{k=N}^{+\infty} A_k V_k \quad (35)$$

$$A_k = \frac{2k+1}{b-a} \left[\sum_{m=-\infty}^{+\infty} B_m C_m^k \right], \quad C_m^k = \int_a^b P_k \left(\frac{2y - (a+b)}{b-a} \right) e^{i2\pi \left(\frac{ym}{b-a} \right)} dy$$

$$V_2(x, 0) = V_3(x, 0) + \epsilon_3 \quad (36)$$

$$V_3(x, 0) = \frac{1}{b-a} \sum_{k=0}^{N-1} \sum_{m=-M}^M V_k(2k+1) B_m C_m^k, \quad \epsilon_3 = \sum_{k=0}^{N-1} \frac{V_k(2k+1)}{b-a} \sum_{m \in \mathbb{Z} - [-M, M]} B_m C_m^k$$

$$V_3(x, 0) = V_4(x, 0) + \epsilon_4 \quad (37)$$

$$V_4(x, t) = \frac{1}{b-a} \sum_{k=0}^{N-1} \sum_{m=-M}^M V_k(2k+1) \tilde{B}_m C_m^k, \quad \epsilon_4 = -\frac{1}{b-a} \sum_{k=0}^{N-1} \sum_{m=-M}^M V_k(2k+1) R_m C_m^k$$

$$R_m = \tilde{B}_m - B_m = \frac{1}{b-a} \int_{\mathbb{R} - [a, b]} \tilde{f}(x) e^{-i2\pi \left(\frac{xm}{b-a} \right)} dx \quad (38)$$

ERROR ANALYSIS

To summarize we obtain

$$V(x, 0) = V_4(x, 0) + \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4. \quad (39)$$

The key to bound the errors lies in the decay rate of the generalized Fourier series coefficients. The convergence rate depends on the **smoothness** of the functions on the expansion interval.

Bound for ϵ_1 and ϵ_4 are related to integral truncation

PROPOSITION

$$|\epsilon_1| = \left| \int_{\mathbb{R}-[a,b]} V(y, T) f(y|x) dy \right| \leq \int_{\mathbb{R}-[a,b]} |V(y, T)| \tilde{f}(y|x) dy \quad (40)$$

$$|\epsilon_4| = \left| -\frac{1}{b-a} \sum_{k=0}^{N-1} \sum_{m=-M}^M V_k(2k+1) R_m C_m^k \right| \leq C_{N,M} \cdot \epsilon \quad (41)$$

where $C_{N,M} \equiv \frac{\sum_{k=0}^{N-1} \sum_{m=-M}^M |V_k(2k+1) C_m^k|}{b-a}$ and $\epsilon \equiv \frac{1}{b-a} [\tilde{F}(a) + 1 - \tilde{F}(b)]$ with $\tilde{F}(x)$ the cumulative distribution function of $\tilde{f}(x)$.

PROPOSITION

Let's assume $\int_a^b V^2(y, T) dy < +\infty$ and define

$$g(y) \equiv f\left(\frac{b-a}{2}y + \frac{a+b}{2}\right), \quad \|u\|_T = \int_{-1}^1 \frac{|u'(x)|}{\sqrt{1-x^2}} dx \quad (42)$$

There are two cases:

- 1 If $g, g', \dots, g^{(k)}$ are absolutely continuous on $[-1, 1]$ with $k > 1$ and $\|g^{(k)}\|_T < \infty$, we have

$$|\epsilon_2| \leq \frac{\|g^{(k)}\|_T}{(k-1)(N-\frac{1}{2})(N-\frac{3}{2})\dots(N-\frac{2k-3}{2})} \sqrt{\frac{\pi}{2(N-k)}} \quad (43)$$

- 2 g analytic on $[-1, 1]$. Then we get

$$|\epsilon_2| \leq \frac{(2N\rho + 3\rho - 2N - 1)\ell(E_\rho)M}{\pi\rho^{N+1}(\rho-1)^2(1-\rho^{-2})} \quad (44)$$

where \tilde{g} is the analytic continuation of g on and within E_ρ (Bernstein ellipse with foci ± 1 and major semiaxis and minor semiaxis summing to $\rho > 1$),

$M \equiv \max_{z \in E_\rho} |\tilde{g}(z)|$ and $\ell(E_\rho)$ denotes the length of the circumference of E_ρ .

PROPOSITION

If

①

$$f(b) = f(a), f^{(1)}(b) = f^{(1)}(a), \dots, f^{(l-1)}(b) = f^{(l-1)}(a) \quad (45)$$

② $f^{(l)}(x)$ is integrable

Then

$$\epsilon_3 = \mathcal{O}\left(\frac{C_{N-1}}{M^l}\right) \quad \text{for } |M| \gg 1 \quad (46)$$

with $C_{N-1} \equiv \sum_{k=0}^{N-1} \frac{|V_k|(2k+1)}{b-a}$.

In particular if the function f is differentiable to all orders and (45) is satisfied for any l , then ϵ_3 decreases faster than $\frac{1}{|M|^l}$ for any finite power of l . This is the exponential convergence property.

REMARK

Our error analysis relies on the smoothness of the density function and not on the regularity of the payoff function, which is particularly relevant in quantitative finance.

NUMERICAL EXPERIMENTS

- European call options and European digital call options (continuous and discontinuous payoffs)
- Models:
 - Black Scholes Model;
 - Merton Jump Diffusion Models;
 - Heston Stochastic Volatility Model.
- Short, standard and long maturities (0.1, 1, 3, 10 years) and in/at/out of the money options
- Truncation Range

We use the following formula from (Fang and Oosterlee 08)

$$[a, b] := \left[c_1 - L\sqrt{c_2 + \sqrt{c_4}}, c_1 + L\sqrt{c_2 + \sqrt{c_4}} \right] \quad (47)$$

c_n are the cumulants of $X = \log\left(\frac{S_T}{K}\right)$ (see Cont and Tankov 04).

NUMERICAL EXPERIMENTS: BLACK-SCHOLES MODEL

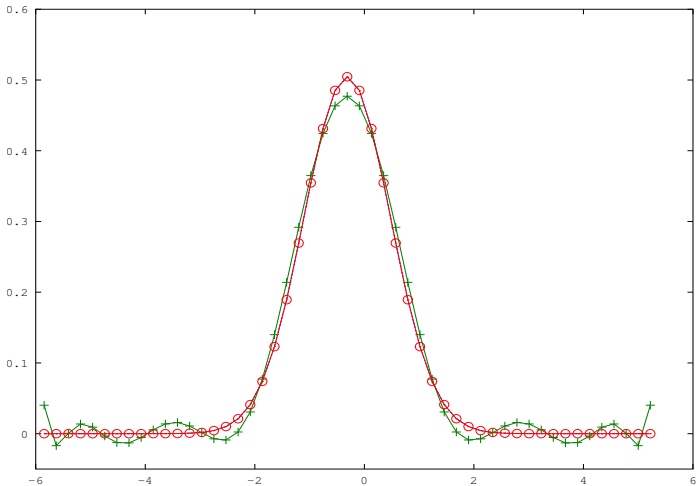


FIGURE: Comparison of the true Gaussian density (solid line) and its approximation based on $N = M = 12$ (solid line with '+' marker) and $N = M = 32$ (solid line with 'o' marker) for maturity $T = 10$.

NUMERICAL EXPERIMENTS: BLACK-SCHOLES MODEL

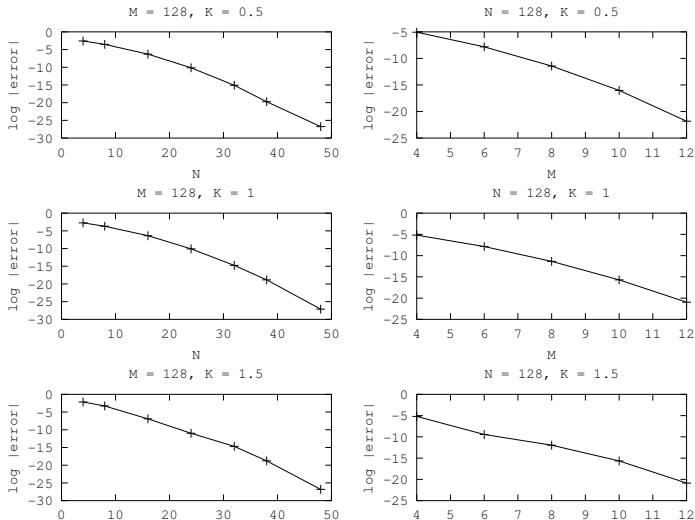


FIGURE: Error convergence for pricing European digital call option with $T = 10$.

NUMERICAL EXPERIMENTS: MERTON JUMP DIFFUSION MODEL

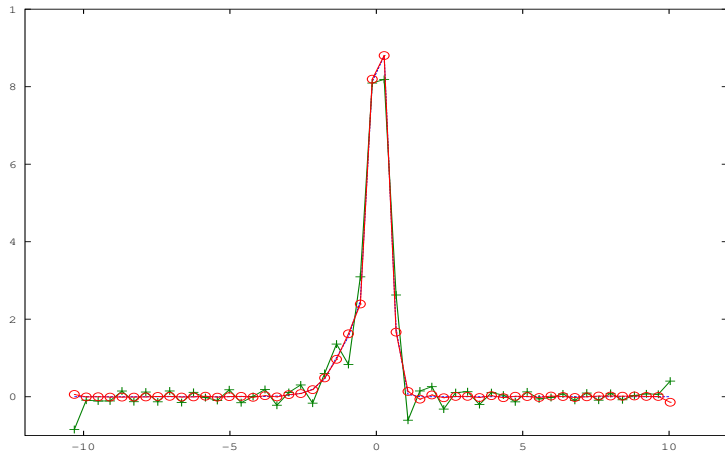


FIGURE: Comparison of the true density function, (solide line) and its approximation based on $N = M = 50$ (solide line with '+' marker) and $N = M = 84$ (solide line with 'o' marker) for maturity $T = 3$.

NUMERICAL EXPERIMENTS: MERTON JUMP DIFFUSION MODEL

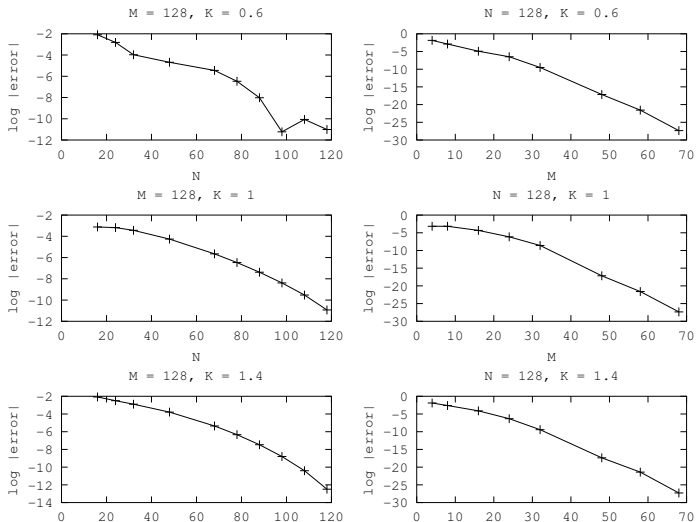


FIGURE: Error convergence for pricing European digital call option with $T = 3$.

NUMERICAL EXPERIMENTS: HESTON STOCHASTIC VOLATILITY MODEL

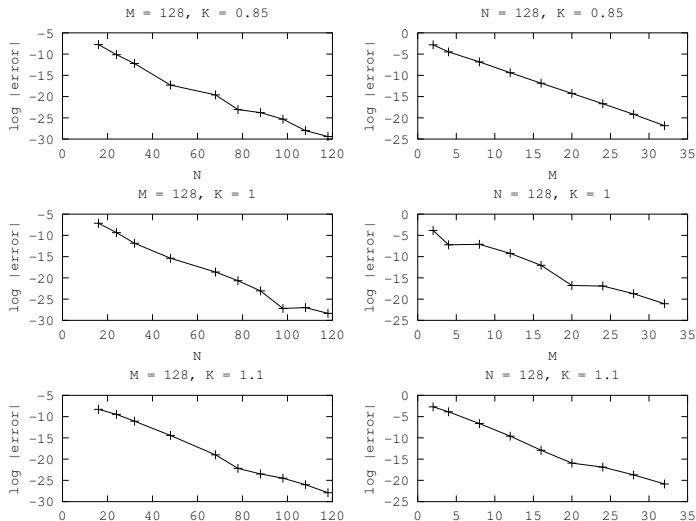


FIGURE: Error convergence for pricing European call option with short maturity $T = 0.1$.

CONCLUSION AND DISCUSSIONS

- New method for pricing European-style options combining Fourier series and generalized Fourier series with Legendre polynomials. It can be used when the characteristic function is available.
- Error analysis: errors bounds have been derived and the study relies on the smoothness properties of the density function of the underlying stochastic models and not of the payoff functions.
- Numerical experiments on call options and digital call options are considered in various models used in quantitative finance. The tests considered, with various strike prices and maturities, show exponential convergence rate.
- The computation of the sensitivity factors w.r.t model parameters are important for the risk management. And it would be interesting to extend the method to cover more exotic contracts like forward start, quanto or spread options. This is let for future research.