LINEAR INFORMATION DYNAMICS, AGGREGATION, DIVIDENDS AND "DIRTY SURPLUS" ACCOUNTING

by

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(First Draft: March, 2002)
(Second Draft: April, 2002)
(Third Draft: May, 2002)
(Fourth Draft: January, 2003)
(Fifth Draft: June, 2003)
(Sixth Draft: September, 2003)

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ABSTRACT

We generalise the Ashton, Cooke and Tippett (2003) Aggregation Theorem by demonstrating how the market value of equity disaggregates into its recursion and real (adaptation) components when the linear information dynamics incorporate a dirty surplus adjustment and also, when dividends are paid. Our analysis shows that ignoring the dirty surplus adjustment will, in general, induce biases into the functional expressions for the recursion and real (adaptation) values of equity. Furthermore, we show that whilst the recursion value of equity is independent of dividend policy, the real (adaptation) value of equity is affected by the dividend policy invoked by the firm. Tabulated results show that the difference in equity value between a dividend and a non-dividend paying firm is most pronounced at low levels of the recursion value.

KEY WORDS

dirty surplus adjustment, dividends, linear information dynamics, real (adaptation) value, recursion value

1. Introduction

There is now a steadily growing volume of papers that although notionally based on the dividend discount model, express the value of a firm's equity in terms of information variables other than dividends. These information variables typically include earnings and the book value of equity. Furthermore, modelling procedures in this area are invariably based on the clean surplus identity; an equation that links a firm's information variables to the dividend payments which it makes. These information variables are normally assumed to evolve in terms of a first order system of stochastic difference equations which when solved, can be used to determine the expected stream of future dividends and thus, the market value of the underlying equity security. Unfortunately, recent developments in investment theory show that the expected present value rule (on which the dividend discount model is built) is based on some extremely tenuous assumptions and that because of this, is unlikely to give a complete picture of the way equity prices emerge in practice [Dixit and Pindyck (1994), Trigeorgis (1996), Holthausen and Watts (2001, p. 60)]. The principal problem with the expected present value rule is that it assumes a fixed scenario under which firms implement their investment opportunities and then go on to generate a stream of cash flows without any intervening contingencies. Thus contingencies, like a firm's ability to modify or even abandon its existing investment opportunity set in the face of unfavourable market conditions, have no role to play when investment decisions and, by implication equity value, are based exclusively on the expected present value rule. Yet a firm's ability to adapt its investment opportunity set to alternative uses represents a potentially valuable option that will be reflected in the market value of its equity, and this will be over and above the value that emanates from discounting the stream of future dividends it expects to pay [Dixit and Pindyck (1994), Trigeorgis (1996), Holthausen and Watts (2001, p. 60)].

Unfortunately, there are few examples in the literature where these option values are included as components of their equity valuation models. Notable exceptions are the papers by Burgstahler and Dichev (1997), Yee (2000) and Ashton, Cooke and Tippett (2003). In the first of these papers, Burgstahler and Dichev (1997, p. 188) argue that equity value is comprised of two elements. The first is called the recursion value of equity. This is the expected present value of the stream of future dividends computed under the assumption that the system of stochastic difference equations that describe the evolution of the firm's information variables, will remain in force indefinitely. Following Burgstahler and Dichev (1997, p. 192) we shall henceforth define the system of stochastic difference equations which govern the evolution of the firm's information variables and thus of its dividends, as the firm's investment opportunity set or alternatively, its business technology.[2] The second component of value is called the real (adaptation) value of equity. This is the option value that arises out of the fact that the firm can terminate (or amend) its current investment opportunity set by (for example) changing the nature of the capital projects in which it invests. Unfortunately, Burgstahler and Dichev (1997, p. 188) have little to say about how this component of equity value is

[2] This system of stochastic difference equations is also often referred to as the "linear information dynamics" that underpins the model. We shall henceforth use the terms "system of stochastic difference equations", "linear information dynamics", "investment opportunity set" and "business technology" interchangeably.

determined other than to suggest that the book value of a firm's equity might be a reasonable approximation for it.

More rigorous approaches to this issue can be found in the papers of Yee (2000) and Ashton, Cooke and Tippett (2003). Yee (2000) considers a firm with a single capital project whose "excess (book or accounting) earnings" in the first year of its operation are drawn from a probability distribution with compact support. These excess earnings then decay in an exponential and deterministic (or known) manner until (at one of a countably infinite number of points in time) the firm abandons the capital project in favour of a potentially, more profitable project. Yee (2000) shows that under this scenario the real (adaptation) value of equity will be a monotonic decreasing function of the excess earnings figure. Ashton, Cooke and Tippett (2003) draw on ideas contained in both the Burgstahler and Dichev (1997) and Yee (2000) papers. However, Ashton, Cooke and Tippett's (2003) analysis is cast in continuous time and makes full use of the analytical tractability provided by the burgeoning real options literature based on these continuous time methods. Here they focus on the simple capital investment models of Dixit (1989) and Dixit and Pindyck (1994, pp. 185-186), which they adapt to accommodate the first order system of stochastic differential equations on which their analysis is based. In contrast to the Yee (2000) model, Ashton, Cooke and Tippett (2003) use a full set of accounting and information variables all of which evolve stochastically through time. When no dividends are paid and the clean surplus identity holds, Ashton, Cooke and Tippett (2003) demonstrate that the real (adaptation) value of equity will be a monotonic decreasing and proportionate function of its recursion value. Furthermore, they obtain closed form expressions for both the recursion value of equity and the proportionality factor.

Our purpose here is firstly, to generalise the Ashton, Cooke and Tippett (2003) Aggregation Theorem by removing its dependence on the clean surplus identity and then secondly, to allow for the payment of dividends. Here, our analysis shows that ignoring dirty surplus adjustments can lead to systematic biases that in empirical work, will translate to an omitted variables problem [Greene (1997, pp. 402-404)]. Our analysis of this issue is based on two "dirty surplus" propositions. The first of these shows how the recursion value of equity is determined when the clean surplus identity does not hold; that is, when there is a form of dirty surplus accounting. The second proposition outlines how the system of stochastic difference equations on which the valuation models of this area have traditionally been based, is modified so as to account for the dirty surplus adjustment. Through this we assess the nature of the systematic biases that can emerge in both the recursion and real (adaptation) values of equity if the dirty surplus adjustment is ignored. We then move on to assess the impact that dividend payments can have on the Ashton, Cooke and Tippett (2003) equity valuation formula. In common with results reported in the real options literature, our analysis shows that whilst the recursion value of equity is independent of a firm's dividend policy, its real (adaptation) value will in general be affected by its dividend policy [Dixit and Pindyck (1994, p. 154)]. Against this, our analysis also shows that for parsimonious dividend payout assumptions (e.g. dividends proportional to the recursion value of equity), the "structure" of the equity valuation problem is similar (although with some significant differences) to the no-dividend case examined in the Ashton, Cooke and Tippett (2003) paper.

In the U.K. FRS3: Reporting Financial Performance has severely curtailed the practice of dirty surplus accounting. There have been similar developments in relation to the practice of dirty surplus accounting in the U.S. [O'Hanlon and Pope (1999, pp. 460-461)]. Hence, the next section begins by briefly examining why we ought to be concerned with the problem of valuing equity in the presence of dirty surplus adjustments.

2. Dirty Surplus Accounting

Our principal objective is to generalise the Ashton, Cooke and Tippett (2003) analysis by removing the requirement for the clean surplus identity to hold and also, to allow for the payment of dividends. While the extension to dividends requires little justification, removing the requirement for the clean surplus identity to hold is more problematical. Given this, we now briefly assess the potential firms and individual investors have for practising dirty surplus accounting under the accounting standards in force in the U.K. and U.S.

2.1 Dirty Surplus Accounting Practices

We begin by noting that for U.K. firms, FRS3: Reporting Financial Performance requires that with few exceptions, all gains and losses must pass through the profit and loss account [Deloite & Touche (2001, p. 225)]. Despite this, FRS3 still requires reporting entities to make (and disclose) certain "adjustments" to the operating profit (or loss) for the year before determining the headline profit (or loss) disclosed in its financial statements. These include:

(a) profits or losses on the sale or termination of an operation;

(b) costs of a fundamental reorganisation or restructuring having a material effect on the nature and focus of the reporting entity's operations; and

(c) profits and losses on the disposal of fixed assets.

Under SSAP6: Extraordinary items and prior period adjustments, which was withdrawn when FRS3 was issued, adjustments like these would probably have been disclosed as extraordinary items [Deloitte and Touche (2001, pp. 256-257)]. This is clearly implied by FRS3, which requires that the information disclosed about these items must be sufficient to enable an informed reader of the financial statements to be in a position to make their own judgement about how each of these items ought to be treated. In particular, the additional information provided about these items must be such as to allow users to make the necessary adjustments to earnings per share (EPS) if the treatment of any item is different from the one that they prefer [Deloite & Touche (2001, pp. 256-257)]. This effectively invites users of financial statements to exclude items from the published profit or loss about which they feel "uncomfortable", thereby resurrecting a form of "personal" dirty surplus accounting.

There are in addition to these, however, exceptions that FRS3 makes to the general requirement for all components of an entity's financial performance to flow through the profit and loss account. These exceptions must be disclosed in a complementary statement to the profit and loss account called the "statement of total recognised gains and losses". Whilst FRS3 does not discuss what it means by "gains" and "losses", it does provide illustrative examples of the kind of items that may be included in this statement. These include the headline profit or loss before deduction of dividends, plus the following:

(a) adjustments to the valuation of assets; and

(b) differences in the net investment in foreign enterprises arising from fluctuations in exchange rates.

Hence, for these and similar items, FRS3 endorses a form of dirty surplus accounting.

In the U.S., APB Opinion No. 30: Reporting the Results of Operations requires that extraordinary items, prior year adjustments and discontinued operations should be shown separately (net of income tax) after income from continuing operations. Extraordinary items must be unusual in nature and occur infrequently. In effect, unusual in nature suggests that the event or transaction should not be normal and unrelated to the ordinary activities of the business. Consideration should also be given to the nature of the business, extent of operations and policies relating to that business to ensure that the event or transaction should not be expected to recur. In addition to the general guidance provided, Opinion No. 30 states that certain items are extraordinary even though they may not meet the above criteria. These include material gains from the extinguishment of debt (SFAS No. 4), profit or loss from the disposal of a significant part of assets or separate segment within two years of a pooling of interests (APB Opinion No. 16), the write-off of operating rights of motor carriers (SFAS No. 44), and the investor's share of an extraordinary item in an associate when the equity method is applied (APB Opinion No. 18). Finally, SFAS No. 130: Reporting Comprehensive Income requires that any changes in book value of equity which are excluded from Net Income (e.g. prior period adjustments), must be reported as "Other Comprehensive Income" in a statement of total recognised gains and losses similar to that required under the U.K. accounting standard, FRS3.

In practice, a problem that has occurred in the U.S., as well as the U.K., is that exceptional items have become very frequent. For example, companies have not always amortised intangibles but written them off as "impairment charges" and treated them as exceptional items. In the U.S. this has been permissible since the introduction of SFAS No. 142: Goodwill and Other Intangible Assets, in June 2001.[3] An incentive for such an approach is that often bonuses paid to senior management are based on profits before exceptional items. Furthermore, earnings per share (EPS) of publicly listed U.S. companies must be disclosed on the face of the income statement (SFAS No. 128) and be based on income from continuing operations. In addition, EPS should be shown on the basis of income before extraordinary items and the cumulative effects of changes in accounting principles. The difference between the first and second EPS figures is discontinued operations. Once the second earnings figure has been disclosed it is necessary to deduct extraordinary items and the cumulative effect of changes in accounting principles to arrive at net income and EPS on this basis.

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[3] See Zeff (2002, pp. 51-53) for a short history of the "politics" leading up to the promulgation of SFAS No. 130: Reporting Comprehensive Income and SFAS No. 142: Goodwill and Other Intangible Assets. Penman (2003, pp. 86-87) provides some "interesting" examples of how these standards have been applied in practice.

Our brief summary of accounting standards relating to the reporting of results from operations and financial performance in the U.K. and U.S. indicates that there are opportunities, both directly and indirectly, for firms to practise various forms of dirty surplus accounting [Penman (2003, p. 82)]. This, combined with the fact that the presentation of certain items in the income or profit and loss statement are such that users can make their own judgements about how they should be presented, is taken as sufficient justification for extending the Ashton, Cooke and Tippett (2003) analysis to encompass dirty surplus accounting. We begin our analysis with the two crucial theorems that underpin most of our analysis.[4]

2.2 Dirty Surplus Propositions

We define b(t) as the book value of equity, x(t) as the instantaneous accounting (or book) earnings (per unit time) attributable to equity and $\varepsilon(t)$ as the instantaneous "dirty surplus" adjustment (per unit time), all at time t. It then follows that the instantaneous increment in the book value of equity, $db(t) = b(t + dt) - b(t)$, over the interval from time t until time t + dt, will be $db(t) = (x(t) + \varepsilon(t))dt - dD(t)$. Here, D(t) is the function whose value is the accumulated value of dividends paid over the (semi-closed) time interval [0,t). It then follows that $dD(t) = D(t + dt) - D(t)$ represents the dividend payment made at time t. Now, recall that the recursion value of equity, $\eta(t)$, is the expected present value of the stream of future dividends computed under the assumption that the firm's existing investment opportunity set will remain in force indefinitely, or:[5]

$$\eta(t) = E_t[\int_t^{\infty} e^{-i(s-t)} dD(s)]$$

where $E_t(\cdot)$ is the expectations operator taken at time t, conditional on the firm's investment opportunity set remaining in force indefinitely, and i is the cost of capital (per unit time) applicable to equity. Present value expressions like the one formulated here are probably most easily rationalised in terms of the quasi-general equilibrium approaches to valuation theory of which Rubinstein (1974, p. 235) is as good an example as any. Whilst this requires some restrictions on preferences, it does avoid the potentially unrealistic outcome that one encounters with "complete" or "viable" arbitrage economies; namely,

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4 Holthausen and Watts (2001, pp. 47-48) summarise current American practice in this area by noting that in general "… Anglo-American accounting has not been characterized by clean surplus [accounting]. Items other than income and transactions with shareholders are involved in the calculation of the change in the book value of equity …. The magnitude of dirty surplus appears to be material in many cases. Lo and Lys (1999) estimate the amount of dirty surplus as the absolute difference between comprehensive (clean surplus) income and GAAP net income as a percentage of comprehensive net income in the period 1962-1997. They find that while the median deviation is only 0.40 [of one] percent, the mean is 15.71 and 14.4 percent of firm/years hav[ing a] dirty surplus that exceeds 10 percent of comprehensive income." Likewise, Tippett and Warnock (1997, p. 1094) employ a maximum likelihood procedure to show that it is most unlikely the clean surplus relationship holds in the United Kingdom.

5 Recall that the system of stochastic difference equations or equivalently, the linear information dynamics that govern the evolution of the firm's bookkeeping and other information variables, is defined as the firm's "business technology" or its "investment opportunity set".

that in equilibrium "every security [must] earn (in expected value) at the riskless rate …." [Harrison and Kreps (1979, p. 383)].[6] However, in whichever way the above expression for $\eta(t)$ is justified, it can always be used to link the expected present value of a firm's dividends to its bookkeeping and other information variables by using the following result:

Theorem #1

Let $a(t) = x(t) - ib(t)$ be the instantaneous residual income or abnormal earnings (per unit time) and $\varepsilon(t)$ be the instantaneous dirty surplus adjustment (per unit time), both at time t, given that the firm applies its existing investment opportunity set indefinitely into the future. Then the recursion value of equity can be restated as:

$$\eta(t) = b(t) + E_t\left[\int_t^\infty e^{-i(s-t)}a(s)ds\right] + E_t\left[\int_t^\infty e^{-i(s-t)}\varepsilon(s)ds\right]$$

which is the book value of equity plus the sum of the expected present value of the residual income stream and the expected present value of the stream of dirty surplus adjustments, all taken at time t.

Proof: See Appendix.

Note that this result exhibits one crucial difference when compared to the "equivalent" formulation based on the clean surplus identity [Ashton, Cooke and Tippett (2003)]. Under the clean surplus identity there can, by definition, be no dirty surplus adjustments and so, the recursion value of equity will be based on only the first two terms of the expression contained in the above proposition. That is, when the clean surplus identity holds, the expected present value of the stream of dirty surplus adjustments, $E_t\left[\int_t^\infty e^{-i(s-t)}\varepsilon(s)ds\right]$, will be identically equal to zero. The omission of this variable, however, has some important implications for the valuation of a firm's equity and so, we now develop them in further detail.

Probably the most important implication stems from the fact that firms are seldom constrained to use their existing resources in an immutable way; that is, firms normally have the flexibility to change their existing investment opportunity set by employing "liquidations, sell-offs, spin-offs, divestitures, CEO changes, mergers, takeovers, bankruptcies, restructurings, and new capital investments …." As previously noted, the potential to make changes like these gives rise to what is known as the real (adaptation) value of equity [Burgstahler and Dichev (1997, p. 188)]. However, whilst it is usually a relatively simple matter to determine an analytical expression for the recursion value of a firm's equity, determining its real (adaptation) value has traditionally been a far more

[6] If one dislikes the restrictions imposed by these quasi-general equilibrium models, one can always follow Dixit and Pindyck (1994, p. 185) in using "dynamic programming with an exogenously specified discount rate … although we will [then] not be able to relate this discount rate to the riskless rate and the market price of risk using the CAPM."

troublesome exercise. Fortunately, Ashton, Cooke and Tippett (2003) show that under reasonable assumptions and a continuous time generalisation of the system of stochastic difference equations on which most models in this area are based, the real (adaptation) value of equity is a monotonic decreasing and proportionate function of its recursion value. Furthermore, they obtain closed form expressions for both the recursion value of equity and the proportionality factor. They note that once the recursion value and the proportionality factor are known, then computing the real (adaptation) value of equity is a relatively simple exercise and they illustrate the procedures involved by determining the real (adaptation) value of equity for both the Ohlson (1995) and a more realistic "square root", linear information dynamics interpretation of their general model.

A principal limitation of the Ashton, Cooke and Tippett (2003) analysis, however, is that it assumes no dividends are paid and that the firm's accounting procedures will satisfy the clean surplus identity. We now develop a series of theorems that generalise the Ashton, Cooke and Tippett (2003) Aggregation Theorem in the sense that they accommodate dividend payments and also, remove the requirement for the accounting system to satisfy the clean surplus identity. The hypothesis (important definitions and assumptions) on which the first of these theorems is based is stated in part (i) of the theorem; the conclusions are contained in part (ii) of the theorem.

Theorem #2

(i) Suppose, the firm's existing investment opportunity set can be summarised in terms of the abnormal earnings variable, $a(t)$, the dirty surplus adjustment, $\varepsilon(t)$, and an "information variable", $\nu(t)$, which evolve in accordance with the following system of stochastic differential equations:

$$d\underset{\sim}{y}(t) = C\underset{\sim}{y}(t)dt + \eta^{\delta}(t)d\underset{\sim}{z}(t)$$

where $0 \le \delta \le \frac{1}{2}$ is a parameter[7], $\underset{\sim}{y}(t) = \begin{pmatrix} a(t) \\ \nu(t) \\ \varepsilon(t) \end{pmatrix}$ and $d\underset{\sim}{y}(t) = \begin{pmatrix} da(t) \\ d\nu(t) \\ d\varepsilon(t) \end{pmatrix}$ are vectors containing the levels and instantaneous changes of these variables, and $C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$ is the matrix of "structural coefficients". The vector of stochastic terms, $d\underset{\sim}{z}(t) = \begin{pmatrix} k_1 dz_1(t) \\ k_2 dz_2(t) \\ k_3 dz_3(t) \end{pmatrix}$, is composed of the Wiener processes, $dz_j(t)$, and

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[7] Ashton, Cooke and Tippett (2003, pp. 417-419) impose the assumption that $0 \le \delta \le \frac{1}{2}$ in order to ensure that the adaptation value of equity converges. However, if there are no adaptation options available to the firm this assumption may be removed and more general stochastic processes can be used to model the evolution of equity value.

"normalising" constants, k_j.[8] The Wiener processes have variance parameters σ_1^2, σ_2^2 and σ_3^2 and correlation parameters, ρ_{12}, ρ_{13} and ρ_{23}, respectively.

(ii) Under the investment opportunity set summarised in (i), the recursion value of equity will be:

$$\eta(t) = b(t) + (1,0,1)(iI - C)^{-1}\underset{\sim}{y}(t)$$

or:

$$\eta(t) = b(t) + \frac{[(i - c_{22})(i - c_{33} + c_{31}) + c_{32}(c_{21} - c_{23})]a(t)}{\Delta} +$$

$$\frac{[c_{12}(i - c_{33} + c_{31}) + c_{32}(i - c_{11} + c_{13})]\nu(t) + [(i - c_{22})(i - c_{11} + c_{13}) + c_{12}(c_{23} - c_{21})]\varepsilon(t)}{\Delta}$$

where $\Delta = \det(iI - C)$ is the determinant of the matrix $(iI - C)$ and I is the 3×3 identity matrix. Furthermore, the recursion value of equity will evolve in accordance with the following stochastic differential equation:

$$d\eta(t) = i\eta(t)dt - dD(t) + \eta^{\delta}(t)dq(t)$$

where $dD(t)$ represents the instantaneous dividend payment at time t and $dq(t) = (dz_1(t) + dz_2(t) + dz_3(t))$ is a Wiener process with variance parameter $\zeta = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\rho_{12}\sigma_1\sigma_2 + 2\rho_{13}\sigma_1\sigma_3 + 2\rho_{23}\sigma_2\sigma_3$.

Proof: See Appendix.

There are several points about this theorem that require an airing. The first relates to the fact that we can obtain a more detailed understanding of the system of stochastic differential equations on which this theorem is based by noting that:

$$da(t) = (c_{11}a(t) + c_{12}\nu(t) + c_{13}\varepsilon(t))dt + k_1\eta^{\delta}(t)dz_1(t)$$

is the stochastic process which describes the evolution of the firm's abnormal earnings through time. This means that instantaneous changes in the abnormal earnings attributable to equity will be normally distributed with a mean (per unit time) of $\frac{E_t[da(t)]}{dt} = c_{11}a(t) + c_{12}\nu(t) + c_{13}\varepsilon(t)$ and a variance (per unit time) of

8 The Appendix demonstrates how these "normalising" constants may be computed.

$\frac{Var_t[da(t)]}{dt} = k_1^2\sigma_1^2\eta^{2\delta}(t)$, where $Var_t(\cdot)$ is the variance operator, taken at time t. This stochastic process can be restated in the alternative form:

$$da(t) = -c_{11}\left[\frac{-c_{12}\nu(t) - c_{13}\varepsilon(t)}{c_{11}} - a(t)\right]dt + k_1\eta^{\delta}(t)dz_1(t)$$

Thus, the system of stochastic differential equations employed here implies that apart from a stochastic component, the firm's current abnormal earnings, a(t), will gravitate towards a long run mean of $\frac{-c_{12}\nu(t) - c_{13}\varepsilon(t)}{c_{11}}$. The force with which it will do so is proportional to the difference between this long run mean and the current instantaneous abnormal earnings, a(t), where the constant of proportionality, or speed of adjustment coefficient, is given by $c_{11} < 0$.

The long run mean of the abnormal earnings variable can be thought of as a weighted average of the current information variable, $\nu(t)$, and the dirty surplus variable, $\varepsilon(t)$. The weights applied to these variables to determine the long run mean are $\frac{-c_{12}}{c_{11}}$ and $\frac{-c_{13}}{c_{11}}$, respectively. Furthermore, whilst we might normally expect the first of these weights, $\frac{-c_{12}}{c_{11}}$, to be positive, reflecting the fact that favourable information will normally imply larger abnormal profits in future, there are conflicting forces determining the sign of the weight associated with the dirty surplus variable. Intuition suggests that positive dirty surplus adjustments would normally have favourable implications for future profitability, in which case $\frac{-c_{13}}{c_{11}}$ ought to be positive. However, this argument ignores a myriad of other factors, such as the potential political and regulatory costs associated with a continuing history of "excess" abnormal profits and the fact that excessive profits will normally dissipate under the influence of competitive pressures. These factors suggest that management will have incentives to use whatever flexibility is available in accounting standards to conceal the magnitude of current and future abnormal earnings. This points to the possibility of a negative value for $\frac{-c_{13}}{c_{11}}$.[9] The sum result of these

[9] Here we should note that the "self correcting" property of the double entry bookkeeping system places limits on the extent to which management can manipulate a firm's (abnormal) earnings. Whilst this might mean the coefficient $\frac{-c_{13}}{c_{11}}$ will be relatively "small" there are nonetheless numerous examples in the literature of how firms do manipulate their earnings measures in ways that are consistent with the linear information dynamics assumed here. Watts and Zimmerman (1986, pp. 208-209) note, for example, that "estimating the future costs of restoring land after strip mining involves sufficient uncertainty to give

considerations is that it is probably reasonable to assume c_{12} is positive, thereby implying that the weight associated with the information variable, $\frac{-c_{12}}{c_{11}}$, will also be positive. It is less clear, however, what sign the coefficient c_{13} will take, in which case it follows that the weight, $\frac{-c_{13}}{c_{11}}$, associated with the dirty surplus variable could either be positive or negative.

The hypothesis of Theorem #2 also implies that the information variable, $\nu(t)$, evolves in terms of the following stochastic process:

$$d\nu(t) = (c_{21}a(t) + c_{22}\nu(t) + c_{23}\varepsilon(t))dt + k_2\eta^{\delta}(t)dz_2(t)$$

This means that instantaneous increments in the information variable are normally distributed with a mean (per unit time) of $\frac{E_t[d\nu(t)]}{dt} = c_{21}a(t) + c_{22}\nu(t) + c_{23}\varepsilon(t)$ and a variance (per unit time) of $\frac{Var_t[d\nu(t)]}{dt} = k_2^2\sigma_2^2\eta^{2\delta}(t)$. This implies that apart from a stochastic component, the information variable gravitates towards a long run mean of $\frac{-c_{21}a(t) - c_{23}\varepsilon(t)}{c_{22}}$ with a speed of adjustment coefficient equal to $c_{22} < 0$. Here, Myers (1999, p. 11) argues that $\nu(t)$ provides "a structure for adding non-accounting information into the analysis." As such, $\nu(t)$ captures "information that will affect future residual income, either directly or indirectly … [such as] new patents, regulatory approval of a new drug for pharmaceutical companies, new long lived contracts and order backlog." It thus follows that $\nu(t)$ is prospective in nature and that because of this, neither $a(t)$ nor $\varepsilon(t)$, both of which are normally retrospective in nature, can adequately reflect or capture movements in the long term mean of the information variable, $\nu(t)$. These arguments suggest that both c_{21} and c_{23} will be (close to) zero.

Finally, we have the stochastic process for the dirty surplus variable, $\varepsilon(t)$. Theorem #2 assumes this variable evolves in accordance with the following process:

$$d\varepsilon(t) = (c_{31}a(t) + c_{32}\nu(t) + c_{33}\varepsilon(t))dt + k_3\eta^{\delta}(t)dz_3(t)$$

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management substantial latitude in determining the cost to be charged off in any given year …. [Likewise], if a firm has a loss, managers increase the loss by including all possible future losses that they can write off - take a 'big bath' - so that future periods' earnings are higher …." Other provisions and accruals, such as those for future maintenance expenditures, doubtful debts etc. can also be manipulated within broad limits under existing accounting standards, especially in an environment where dirty surplus accounting is permitted. Given this, it is not hard to envisage how firms can find the "flexibility" they need to implement linear information dynamics of the kind assumed here. Penman (2003) contains further examples of the manipulations that can be applied to the profit and loss statement.

Hence, instantaneous changes in the dirty surplus variable have a mean and variance (per unit time) of $\frac{E_t[d\varepsilon(t)]}{dt} = c_{31}a(t) + c_{32}\nu(t) + c_{33}\varepsilon(t)$ and $\frac{Var_t[d\varepsilon(t)]}{dt} = k_3^2\sigma_3^2\eta^{2\delta}(t)$, respectively. Furthermore, this means that the dirty surplus variable gravitates towards a long run mean of $\frac{- c_{31}a(t) - c_{32}\nu(t)}{c_{33}}$ with a speed of adjustment coefficient given by $c_{33} < 0$. Now, we have previously noted that potential political, regulatory and competitive costs will mean that managers have incentives to adopt accounting policies which reduce abnormal earnings. This suggests that the weight associated with the abnormal earnings variable, $\frac{- c_{31}}{c_{33}}$, will be positive since managers can use the dirty surplus variable to reduce the firm's (headline) abnormal earnings. Furthermore, since favourable information, ν(t), will normally have positive implications for future profitability (and therefore the manipulation of profits through the dirty surplus variable), it appears reasonable to assume that the weight associated with the information variable, $\frac{- c_{32}}{c_{33}}$, is also likely to be positive. This will mean that both the structural coefficients, c_{31} and c_{32}, are likely to be positive.[10]

Now, Theorem #2 also implies that the instantaneous growth rate in the recursion value of equity evolves in terms of a process whose mean and variance (per unit time) are

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[10] Letting sign(·) signify the signs of the coefficients of the affected matrix means that we can summarise our discussion of the probable signs of the structural coefficients in the following terms:

$$\text{sign}\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} - & + & ? \\ 0 & - & 0 \\ + & + & - \end{pmatrix}$$

The eigenvalues for the matrix with these structural coefficients will thus be $\lambda = c_{22} < 0$ and:

$$\lambda = \frac{(c_{11} + c_{33}) \pm \sqrt{(c_{11} - c_{33})^2 + 4c_{13}c_{31}}}{2}$$

Now, all three eigenvalues will be negative if $c_{13}c_{31} < c_{11}c_{33}$. This condition is satisfied if the matrix is diagonally dominant, or under the slightly weaker conditions $|c_{11}| > |c_{13}|$ and $|c_{33}| > |c_{31}|$. Intuitively, such conditions imply that the main factor influencing changes to a variable are its value in the previous period with the values of the other variables in the previous period playing a relatively minor role. Under such a scenario, we can expect that a(t), ν(t) and ε(t) will asymptotically converge towards zero. If, however, $c_{13}c_{31} > c_{11}c_{33}$ then one of the eigenvalues will be positive and a(t), ν(t) and ε(t) may exhibit explosive properties. A much more detailed consideration of this issue is to be found in O'Neil (1987, pp. 373-386), Ashton (1997) and Tippett and Warnock (1997, pp. 1076-1084).

$\frac{1}{\eta(t)} \cdot \frac{E_t[d\eta(t)]}{dt} = i - \frac{1}{\eta(t)} \cdot \frac{dD(t)}{dt}$ and $\frac{Var_t[\frac{d\eta(t)}{\eta(t)}]}{dt} = \frac{\zeta}{\eta^{2(1-\delta)}(t)}$, respectively. This means that on average instantaneous increments in the recursion value of equity (gross of any dividends paid) will be just sufficient to yield a return (on recursion value) equal to the cost of the firm's equity capital, i. Here it is also worth emphasising that since the convergence criteria outlined in Ashton, Cooke and Tippett (2003, pp. 417-419) require $\delta \leq \frac{1}{2}$, the uncertainty surrounding the rate at which the recursion value of equity grows, as measured by its variance, declines as the recursion value becomes larger. This provides a potential explanation of, and certainly a basis for future research into, firm size as the significant explanatory variable in empirical research that it has proved to be [Fama and French (1992, 1995)].

Theorem #2 also shows that the recursion value of equity is a weighted sum of the book value of equity, the instantaneous abnormal earnings, the instantaneous information variable and the instantaneous dirty surplus adjustment. However, the weights, or valuation coefficients hinge on the system of stochastic differential equations that underscore the evolution of each of these variables. We can demonstrate this by assuming that the structural coefficients c_{31} and c_{32} are both zero, in which case substitution into part (ii) of Theorem #2 shows the recursion value of equity will be:[11]

$$\eta(t) = b(t) + \frac{(i - c_{22})a(t)}{(i - c_{11})(i - c_{22}) - c_{21}c_{12}} + \frac{c_{12}\nu(t)}{(i - c_{11})(i - c_{22}) - c_{21}c_{12}}$$
$$+ \frac{[(i - c_{22})(i - c_{11} + c_{13}) + c_{12}(c_{23} - c_{21})]\varepsilon(t)}{(i - c_{33})[(i - c_{11})(i - c_{22}) - c_{21}c_{12}]}$$

Now, the coefficients associated with the abnormal earnings and information variables in this formula for the recursion value of equity are the same as those obtained for the more restricted linear information dynamics based on the clean surplus identity used in the Ashton, Cooke and Tippett (2003, pp. 420-425) analysis. Thus, if this parsimonious interpretation of dirty surplus accounting applies in practice, it follows that the last term in this formula will be the valuation bias in the recursion value of equity as a result of

[11] Setting c_{31} and c_{32} to zero implies that the dirty surplus variable evolves in terms of the parsimonious process:

$$d\varepsilon(t) = c_{33}\varepsilon(t)dt + k_3\eta^{\delta}(t)dz_3(t)$$

This process implies that the dirty surplus adjustment has an unconditional mean of zero, independent of the current magnitudes of the abnormal earnings and information variables. Whilst it is unlikely that such a simple model would apply in practice, it does provide a useful benchmark through which to compare models based on abnormal earnings and the information variable alone, with models that also incorporate a dirty surplus adjustment.

falsely assuming the validity of the clean surplus identity.[12] For more realistic versions of dirty surplus accounting under which $c_{31} \neq 0$ and $c_{32} \neq 0$, however, there will be biases in the valuation coefficients associated with the abnormal earnings and the information variables as well. Furthermore, since Ashton, Cooke and Tippett (2003, pp. 417-419) show that the overall market value of equity is parameterised in terms of its recursion value, it necessarily follows that any biases in recursion value will also be reflected as biases in the market value of equity itself. This is an issue that we now explore in further detail.

3. Equity Valuation, Dividends and Dirty Surplus Accounting

The previous section develops and analyses some important theorems relating to the recursion value of equity when there is dirty surplus accounting. We now develop some related theorems for the real (adaptation) value of equity and by implication, the overall market value of equity. Our analysis assumes that the firm practises dirty surplus accounting and importantly, that it also pays dividends. The relationship between the recursion value of equity and its real (adaptation) and overall market values is stated in the first of these theorems:

Theorem #3

If instantaneous dividends, $dD(t)$, are a deterministic function of the recursion value of equity, then the market value of equity, $P(\eta)$, will be:

$$P(\eta) = \eta + c_0 P_1(\eta) + c_1 P_2(\eta)$$

where $P_1(\eta)$ and $P_2(\eta)$ are linearly independent solutions of the auxiliary equation:

$$\frac{1}{2}\zeta\eta^{2\delta}.\frac{d^2P}{d\eta^2} + (i\eta - \frac{dD}{dt}).\frac{dP}{d\eta} - iP(\eta) = 0$$

The constants c_0 and c_1 are determined by boundary conditions which represent managerial options that can be exercised as the firm proves successful or alternatively, approaches bankruptcy. Thus, we shall require that the market value of equity, $P(\eta)$, is exclusively made up of its real (adaptation) value as $\eta \to 0$, in which case management make appropriate decisions to avoid bankruptcy and exclusively made up of its recursion value as the firm proves successful; that is, as $\eta \to \infty$.

Proof: See Appendix.

Now, here it will be recalled that η is the recursion value of the firm's equity and so, this result says that the market value of equity, $P(\eta)$, is equal to its recursion value

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12 At an empirical level, this will translate into an omitted variables problem. This will mean that regression models that ignore the dirty surplus component of changes in book value will be both inconsistent and inefficient [Greene (1997, pp. 402-404)].

plus a linear sum of the two functions, $P_1(\eta)$ and $P_2(\eta)$, which satisfy the auxiliary equation and the associated boundary conditions. However, since equity value is exclusively composed of its recursion and real (adaptation) values, it necessarily follows that $P_1(\eta)$ and $P_2(\eta)$ must capture the real (adaptation) value of the firm's equity.

We can demonstrate the economic intuition behind this result by supposing that dividends are proportional to the recursion value of the firm's equity. It then follows that $\frac{dD}{dt} = \alpha\eta$, where $0 \leq \alpha < i$ is the constant of proportionality.[13] Furthermore, if we assume that the recursion value of equity evolves in accordance with an Ashton, Cooke and Tippett (2003) "square root" process ($\delta = \frac{1}{2}$), it follows that the real (adaptation) value of equity will be captured by the two solutions of the following auxiliary equation:

$$\frac{1}{2}\zeta\eta\frac{d^2P}{d\eta^2} + (i - \alpha)\eta\frac{dP}{d\eta} - iP(\eta) = 0$$

Unfortunately, there are no closed form solutions for this differential equation. However, in the Appendix we show that there are analytic solutions in the form of infinite power series expansions; namely:[14]

$$P_1(\eta) = \eta + \frac{\alpha}{\zeta}\eta^2 + \frac{\alpha(2\alpha - i)}{3\zeta^2}\eta^3 + \frac{\alpha(2\alpha - i)(3\alpha - 2i)}{18\zeta^3}\eta^4 + \_\_\_\_$$

and:

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[13] Theorem #2 shows that the recursion value of equity is a weighted sum of book value, abnormal earnings, the information variable and the dirty surplus adjustment. Hence, this assumption means that dividends can also be interpreted as being a weighted average of these four variables.

[14] The terms in this series expansion for $P_1(\eta)$ may be developed from the formulae contained in the Appendix and are obtained by letting j vary over all the positive integers in the following expression:

$$\frac{2^j\alpha(2\alpha - i)\_\_\_\_(j\alpha - (j - 1)i)}{(1.2)(2.3)(3.4)\_\_\_\_j(j + 1)\zeta^j}.\eta^{j+1}$$

Letting j = 3, for example, shows that the third order term in the expansion for $P_1(\eta)$ will be:

$$\frac{2^3\alpha(2\alpha - i)(3\alpha - 2i)}{(1.2)(2.3)(3.4)\zeta^3}.\eta^4 = \frac{\alpha(2\alpha - i)(3\alpha - 2i)}{18\zeta^3}.\eta^4$$

The other terms in the series expansion are determined in the same way.

$$P_2(\eta) = \int_{\eta}^{\infty} \exp[\frac{2(\alpha - i)y}{\zeta}] \frac{P_1(\eta)}{P_1^2(y)} dy$$

Now, in the Appendix we also show that the second of these solutions, $P_2(\eta)$, incorporates the two important properties specified in the boundary conditions stated in Theorem #3; namely, $\lim_{\eta \to 0} P_2(\eta) = 1$ and $\lim_{\eta \to \infty} P_2(\eta) = 0$. These conditions capture the fact that when recursion value is "small" ($\eta \to 0$), then equity value is composed mainly of its real (adaptation) value. Likewise, when recursion value is large ($\eta \to \infty$), then equity value is principally composed of its recursion value. This means that when dividends are proportional to the recursion value of equity, the overall market value of equity, $P(\eta)$, will be:

$$P(\eta) = \eta + P(0) \int_{\eta}^{\infty} \exp[\frac{2(\alpha - i)y}{\zeta}] \frac{P_1(\eta)}{P_1^2(y)} dy$$

where P(0) is the "dormant value" of the firm's equity; that is, the market value of equity when the recursion value of equity approaches zero ($\eta \to 0$). The non-zero value that we assume for P(0) implies that the firm in effect exercises an option to change its investment opportunity set when the prospective dividend stream from its existing portfolio of capital projects has all but dried up. Furthermore, comparing this solution with the one obtained by Ashton, Cooke and Tippett (2003, pp. 420-425) using the clean surplus and no dividend assumptions shows that the payment of dividends accentuates the non-linear nature of the equity valuation relationships. This is of particular significance given the penchant amongst empirical researchers for applying purely linear methodologies to estimate valuation relationships in this area. It is not hard to show, however, that if, as the empirical evidence seems to suggest, there is a non-linear relationship between equity prices and their determining variables [Lev (1989), Burgstahler and Dichev (1997), Ashton, Cooke and Tippett (2003)], then the purely linear methodologies that characterise empirical research in this are will be an unreliable basis for estimating the relevant valuation relationships.[15]

[15] Freeman (1963, p. 50) provides a simple example that demonstrates the importance of the point being made here. Consider a standardised variable, x, which is symmetric about its mean. Let x and y be functionally related according to the formula $y = x^2$. Now, the covariance between x and y will be $E[(x - E(x))(y - E(y))] = E(x^3 - x) = 0$ and so, a naïve interpretation of the linear regression of y on x is that these two variables are completely unrelated. Admittedly this is a simple example concocted to provide a parsimonious illustration of the point at issue. However, it is not hard to construct more realistic (but less parsimonious) examples of linear methodologies that are unable to detect (or alternatively, distort) the complicated non-linear relationships that appear to exist in this area. Miller (1986, p. S461) and Holthausen and Watts (2001, pp. 52-63) contain very useful discussions and extensions of some of the issues raised here.

There is, however, a second issue that emerges out of the non-linear pricing relationships that our analysis suggests ought to exist in this area. And this relates to the role that dividends play in the equity valuation formula, $P(\eta)$. Here, it will be recalled that the Modigliani and Miller (1961) theorem says that if, in a perfect capital market without taxation a firm's existing investment opportunity set (or equivalently, its linear information dynamics) is applied indefinitely into the future, then "the division" of a firm's earnings "between cash dividends and retained earnings in any period is a mere detail ...." In particular, "the dividend payout ... determine[s] merely how a given return to stockholders ... split[s] as between current dividends and current capital gains and [does] not affect either the size of the total return or the current value of the shares." [Graham, Dodd and Cottle (1962, pp. 487-488)]. Despite this, empirical research has consistently shown that dividend payments do appear to affect the value firms' shares in ways that are contrary to the Modigliani and Miller (1961) theorem [Black (1972), Miller (1986)]. It is not hard to show, however, that dividend payments can affect a firm's real (adaptation) option value and so, this provides a further reason as to why variations in dividend payout rates might affect equity values. The important result can be summarised in the following terms:

Corollary

The functional expression for the recursion value of equity given in Theorem #2 *does not* depend on the dividend function, $D(t)$. However, the functional expression for the real (adaptation) value of equity *does* depend on the dividend function, $D(t)$.

Proof: See Appendix.

The intuition behind this result arises out of the fact that the Miller and Modigliani (1961, p. 414) theorem shows that a firm's value is exclusively determined by its investment opportunity set; that is, its existing business technology. Since the recursion value of equity is computed on the assumption that the firm's existing investment opportunity set will be applied indefinitely into the future, it follows from the Miller and Modigliani (1961) theorem that the dividend function, $D(t)$, cannot enter into the (functional) expression for the recursion value of a firm's equity. However, the real (adaptation) value of equity depends on the potential changes a firm can make to its existing investment opportunity set and, consistent with the real options literature [Dixit and Pindyck (1994, p.154)], our analysis shows that the dividend function, $D(t)$, will enter into the (functional) expression for this component of equity value. In other words, dividend payments reduce the resources available to the firm and thereby adversely affect its capacity to "ride out" unfavourable economic circumstances. This in turn increases the probability that the firm will have to exercise its real (adaptation) options and so, these options become more valuable as a consequence.

We can demonstrate the importance of the point we are making here by comparing the way equity value evolves as a function of its recursion value for a dividend paying and a non-dividend paying firm. We begin by considering a firm with a dormant value of $P(0) = 1$ and whose dividend payout rate (as a proportion of its recursion value) is equal to one half the cost of its equity capital. This means the dividend payout rate will be $\alpha = \frac{i}{2}$ and so from previous analysis, the overall market value of the firm will be:

$$P(\eta) = \eta + \left(\eta + \frac{i}{2\zeta}.\eta^2\right) \int_{\eta}^{\infty} \frac{\exp(\frac{-i}{\zeta}.y)}{(y + \frac{i}{2\zeta}.y^2)^2} dy$$

In Figure I we plot the above pricing relationship when the parameter $\frac{i}{\zeta} = \frac{2\alpha}{\zeta}$ assumes the values 0.25, 0.375 and 0.5. Note that as $\frac{i}{\zeta}$ becomes larger, these graphs show that the real (adaptation) value of equity decays quickly away. Thus, when $\frac{i}{\zeta} = 0.25$ (or equivalently, $\frac{\alpha}{\zeta} = 0.125$) and the recursion value of equity is $\eta = 1$ then the real (adaptation) value of

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FIGURE ONE ABOUT HERE

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equity will be 0.3612. When, however, $\frac{i}{\zeta} = 0.375$ (so that $\frac{\alpha}{\zeta} = 0.1875$) and the recursion value of equity is $\eta = 1$ then the real (adaptation) value of equity falls to 0.2693. Finally, when $\frac{i}{\zeta} = 0.50$ (or $\frac{\alpha}{\zeta} = 0.25$) and the recursion value of equity is $\eta = 1$ then the real (adaptation) value of equity falls again to 0.2101. This continuous fall in the real (adaptation) value equity occurs because as $\frac{i}{\zeta}$ grows, the uncertainty associated with the evolution of the recursion value of equity, declines. If, for example, the variance parameter, $\zeta = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\rho_{12}\sigma_1\sigma_2 + 2\rho_{13}\sigma_1\sigma_3 + 2\rho_{23}\sigma_2\sigma_3$, is very close to zero then there is virtually no uncertainty associated with the evolution of the recursion value of equity. This will also mean that $\frac{i}{\zeta}$ will have a relatively large value. If, however, there is a great deal of uncertainty about the evolution of the recursion value of equity then ζ will be large relative to i and this means that $\frac{i}{\zeta}$ will have a relatively small value. Now, it is well known that option value is an increasing function of the volatility of its determining variables. This in turn will mean that when $\frac{i}{\zeta}$ has a relatively low value (or equivalently, when the volatility, ζ, has a relatively large value) the real (adaptation) value of equity will have to be relatively large. When, however, $\frac{i}{\zeta}$ assumes a relatively large value (so that the volatility, ζ, has a comparatively small value) then the real (adaptation) value of equity will have to be relatively small.

There is one further characteristic of the graphs appearing in Figure I that requires an airing. Note that as the parameter $\frac{i}{\zeta}$ becomes larger, a "cusp" gradually appears in the overall value of equity when the recursion value of equity is "small". This cusp arises out of the fact that for small recursion values the decline in the real (adaptation) value of equity is much larger than is the increase in the recursion value itself. As a consequence, as $\frac{i}{\zeta}$ becomes larger there is an increasingly large domain over which the aggregate sum of the real (adaptation) and recursion values of equity and hence, the overall market value of equity, will decline. Interestingly, both Burgstahler (1998, pp. 338-339) and Ashton, Cooke and Tippett (2003, pp. 428-430) find evidence for the existence of such a cusp in the samples of publicly listed companies on which their empirical work is based.

In contrast to the dividend paying firm considered in previous paragraphs, suppose now, we consider a non-dividend paying firm which also has a dormant value of P(0) = 1. Then Ashton, Cooke and Tippett (2003, p. 434) show that the market value of this non-dividend paying firm's equity will be:

$$P(\eta) = \eta + \eta \int_{\eta}^{\infty} \frac{\exp(\frac{-2i}{\zeta}.y)}{y^2} dy$$

In Figure II we plot the above pricing relationship when the parameter $\frac{i}{\zeta}$ assumes the values 0.25, 0.375 and 0.5. Note that on the surface at least, these graphs appear to be almost identical to the graphs for the dividend-paying firm with the same parameter

FIGURE TWO ABOUT HERE

values. Thus, as $\frac{i}{\zeta}$ becomes larger, the real (adaptation) value of the firm decays more quickly - reflecting the fact that for large values of $\frac{i}{\zeta}$ there is relatively less volatility in the evolution of recursion value and option values must be lower as a consequence. Likewise, as $\frac{i}{\zeta}$ becomes larger the rate of decline in the real (adaptation) value of equity is much greater than the rate of increase in the recursion value and so once again a cusp emerges in the overall market value of the firm's equity. Yet, despite the similarities in these graphs there are, in fact, significant differences in the values of the dividend and non-dividend paying firms, something that can be confirmed from the graphs appearing in Figure III.

This Figure graphs the difference between the equity value of a firm that pays no dividends at all and a firm whose dividend payout rate (as a proportion of its recursion value) is equal to one half the cost of its equity capital. The equity value for a dividend-paying firm is summarised in Figure I whilst the equity value for a non-dividend paying firm is summarised in Figure II. Thus, Figure III summarises the differences between these two sets of graphs when the parameter $\frac{i}{\zeta}$ assumes the values 0.25, 0.375 and 0.5. Note that for large recursion values there is very little difference between the overall

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FIGURE THREE ABOUT HERE

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equity value of a dividend paying firm and the equity value of a non-dividend paying firm. This reflects the fact that when the recursion value of equity is large there is only a small probability the firm will have to exercise its real (adaptation) options and this will be so irrespective of whether the firm pays dividends or not. Hence, the value of the real (adaptation) options will be small for both the dividend and the non-dividend paying firms. For small recursion values, however, the graphs in Figure III show that there can be quite significant differences between the equity values of the dividend and non-dividend paying firms. When the recursion value of equity is already small dividends will reduce recursion value even more and so, there is a significantly higher probability that a dividend-paying firm will have to exercise its real (adaptation) options when compared to a non-dividend paying firm. This means, of course, that the real (adaptation) options will be much more valuable to the dividend paying firm and so, for a given (common) recursion value the dividend paying firm will have a higher equity value than the non-dividend paying firm. The reader will confirm that this story is born out by the graphs appearing in Figure III. Thus, in all three graphs the difference between the equity value of a dividend paying firm and the equity value of a non-dividend paying firm moves steadily towards a maximum and then gradually decays away until there is very little difference between the equity values of the two firms.

5. Summary Conclusions

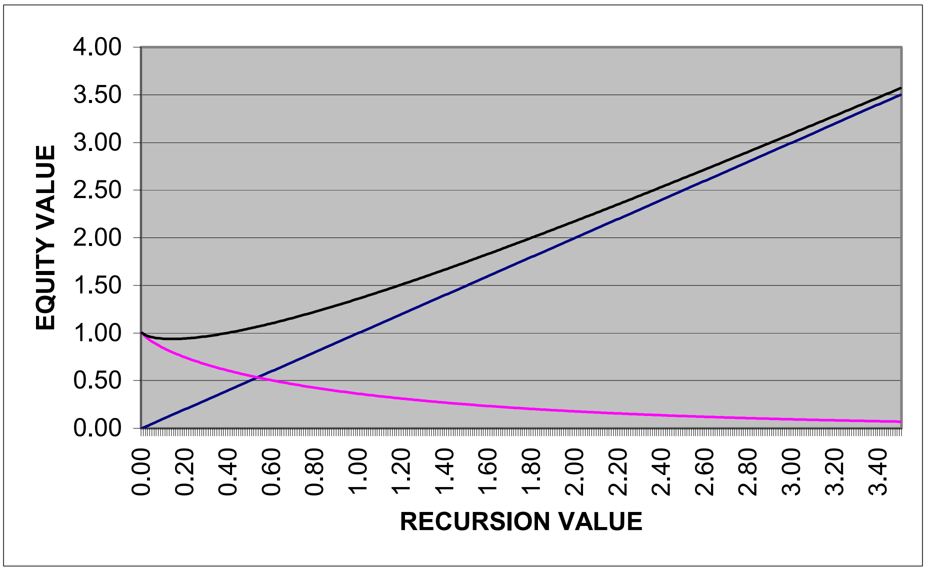
Our purpose here is to respond to Burgstahler and Dichev (1997, p. 212) and Penman's (2001, p. 692) call for the development of more refined equity valuation models. We do so by determining an analytical expression for the value of a firm's equity under linear information dynamics that encompasses dirty surplus accounting and also, where dividends are paid by the firm. Our analysis is based on two "dirty surplus" propositions. The first of these shows how the recursion value of equity is determined when there is dirty surplus accounting; that is, when the clean surplus identity does not hold. The second shows that the recursion value of equity will be a weighted sum of the book value of equity, abnormal earnings, the information variable and the dirty surplus adjustment. Furthermore, this theorem also shows that ignoring the dirty surplus variable will, in general, induce biases in the functional expression for the recursion value of equity.

Our analysis also shows that whilst the Modigliani and Miller (1961) dividend irrelevance theorem applies to the recursion value of equity, it will not, in general, apply to its real (adaptation) value. Recursion value is computed under the assumption that the firm's investment opportunity set will not change and since Miller and Modigliani (1961) show that in a perfect capital market without taxation, it is the investment opportunity set and not dividends that determines value, it is clear that recursion value will have to be independent of the firm's dividend policy. However, real (adaptation) value is determined by the potential changes a firm can make to its existing investment opportunity set and our analysis shows that dividend payments can have an impact on this component of equity value. For parsimonious dividend payout assumptions (e.g. dividends proportional to the recursion value of equity), however, the "structure" of the equity valuation problem is essentially unchanged (when compared to the no-dividend case) and leads to similar valuation formulae to those contained in the ACT (2003, pp. 5-7) Aggregation Theorem.

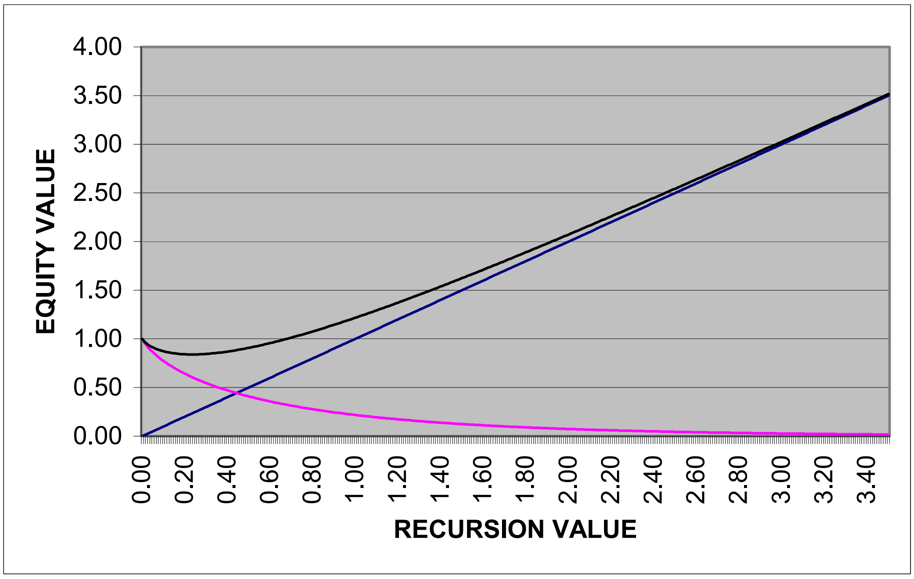
Our study also raises issues of some significance for future research in the area. A more sophisticated model of equity pricing based on finer dis-aggregations of changes in the book value of equity would enable researchers to explore the value implications of other components of a firm's financial statements [Stark (1997)]. Furthermore, since most empirical work reported in this area is predicated on the assumption of a linear relationship between equity value and the information variables, there is the important issue of the potential biases which arise when the non-linearities induced by a firm's real (adaptation) options are ignored. Finally, there is the crucial issue of how alternative dividend policies impact on the real (adaptation) value and overall market value of a firm's equity.

FIGURE I

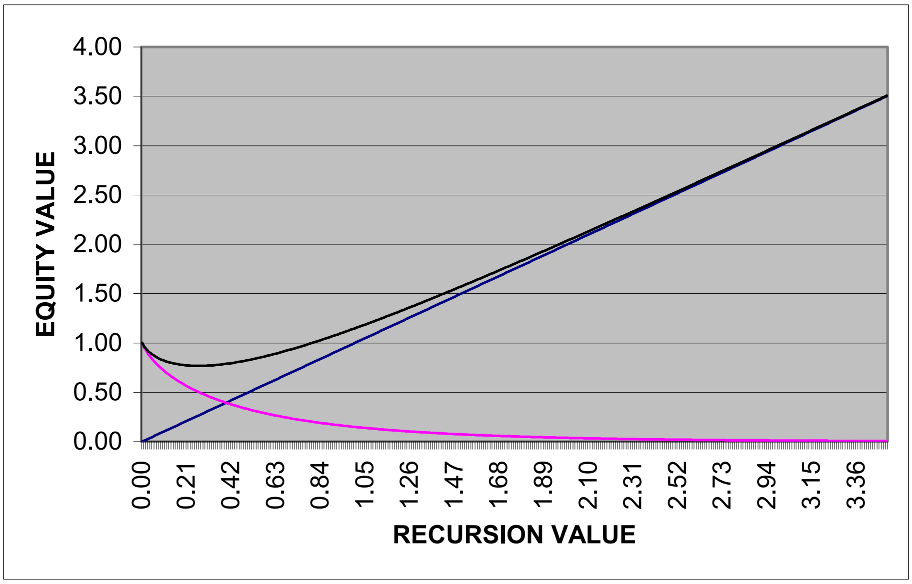
EQUITY VALUE FOR A FIRM THAT PAYS AN INSTANTANEOUS PROPORTIONATE DIVIDEND EQUAL TO $\alpha = \frac{i}{2} = 0.125\zeta$ OF ITS RECURSION VALUE AND WHERE ζ IS THE VARIANCE (PER UNIT TIME) OF THE STOCHASTIC TERM, dq(t), ASSOCIATED WITH INSTANTANEOUS INCREMENTS IN RECURSION VALUE



EQUITY VALUE FOR A FIRM THAT PAYS AN INSTANTANEOUS PROPORTIONATE DIVIDEND EQUAL TO $\alpha = \frac{i}{2} = 0.1875\zeta$ OF ITS RECURSION VALUE



EQUITY VALUE FOR A FIRM THAT PAYS AN INSTANTANEOUS PROPORTIONATE DIVIDEND EQUAL TO $\alpha = \frac{i}{2} = 0.25\zeta$ OF ITS RECURSION VALUE



The downward sloping curve in each of the above graphs is the real (adaptation) value of equity; namely:

$$(\eta + \frac{i}{2\zeta}.\eta^2)\int_{\eta}^{\infty}\frac{\exp(\frac{-i}{\zeta}.y)}{(y + \frac{i}{2\zeta}.y^2)^2}dy = \frac{1}{2}\int_{-1}^{1}\frac{\exp\left[1 - \sqrt{1 + \frac{\frac{4i}{\zeta}(\eta + \frac{i}{2\zeta}.\eta^2)}{1+z}}\right]}{\sqrt{1 + \frac{\frac{4i}{\zeta}(\eta + \frac{i}{2\zeta}.\eta^2)}{1+z}}}dz$$

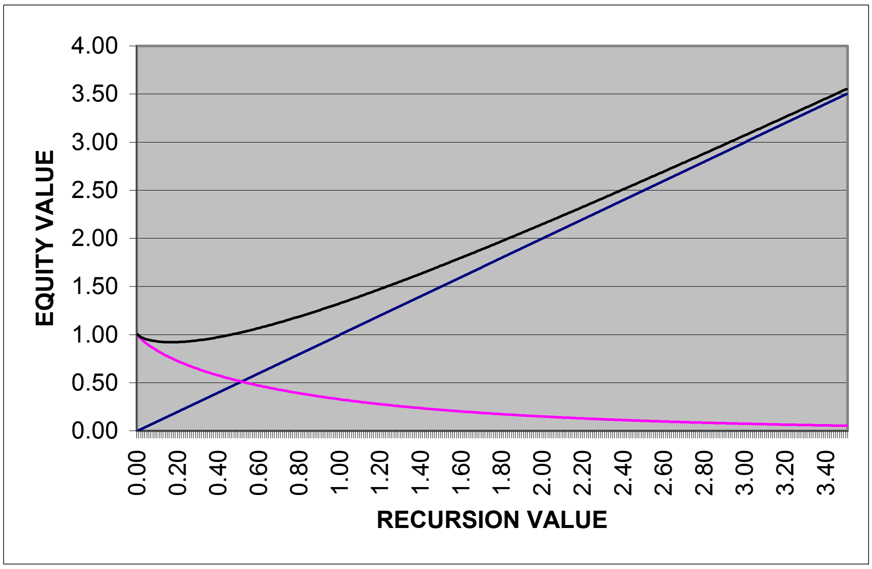
The upward sloping line is the recursion value of equity, η. The upward sloping curve is the overall value of equity which is the sum of its recursion value and its real (adaptation) value, or:

$$P(\eta) = \eta + (\eta + \frac{i}{2\zeta}.\eta^2)\int_{\eta}^{\infty}\frac{\exp(\frac{-i}{\zeta}.y)}{(y + \frac{i}{2\zeta}.y^2)^2}dy$$

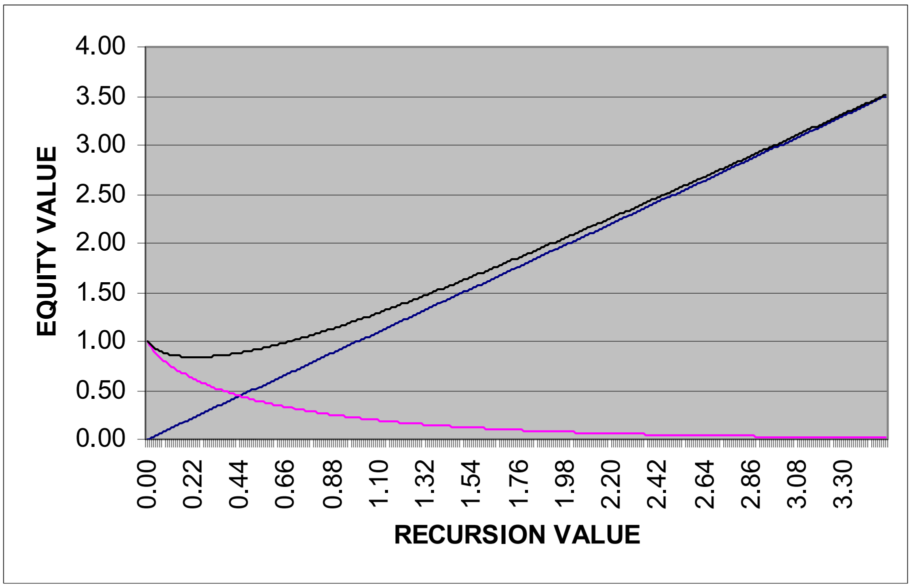
The integrals in the above expressions were estimated using 15 point Gauss-Legendre quadrature [Carnahan, Luther and Wilkes (1969, pp. 101-105)].

TABLE II

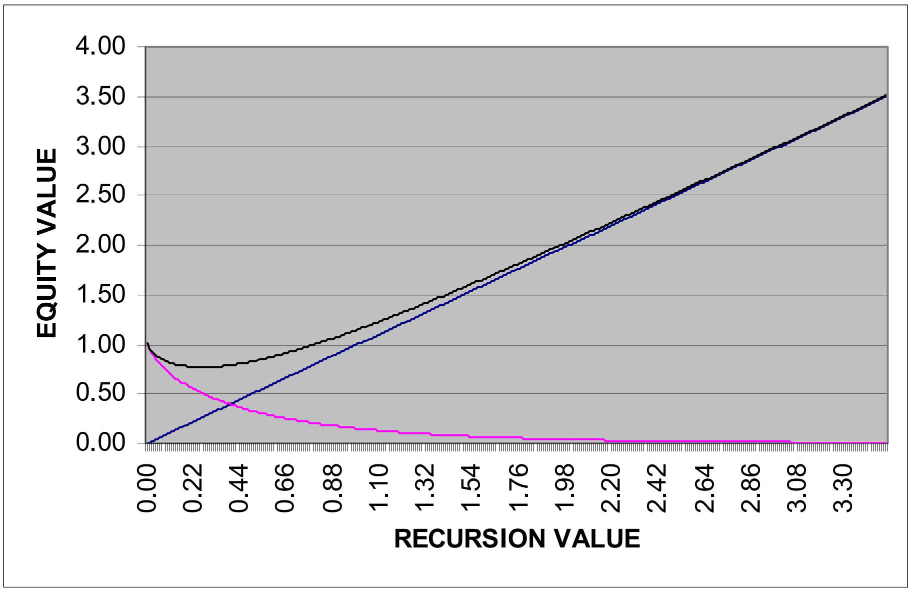
EQUITY VALUE FOR A NON-DIVIDEND PAYING FIRM WHEN i = 0.25ζ AND WHERE ζ IS THE VARIANCE (PER UNIT TIME) OF THE STOCHASTIC TERM, dq(t), ASSOCIATED WITH INSTANTANEOUS INCREMENTS IN RECURSION VALUE



EQUITY VALUE FOR A NON-DIVIDEND PAYING FIRM WHEN i = 0.375ζ



EQUITY VALUE FOR A NON-DIVIDEND PAYING FIRM WHEN i = 0.50ζ



The downward sloping curve in each of the above graphs is the real (adaptation) value of equity; namely:

$$\eta \int_{\eta}^{\infty} \frac{\exp(\frac{-2i}{\zeta}.y)}{y^2} dy = \frac{1}{2} \int_{-1}^{1} \exp\left(\frac{\frac{-4i}{\zeta}.\eta}{1+z}\right) dz$$

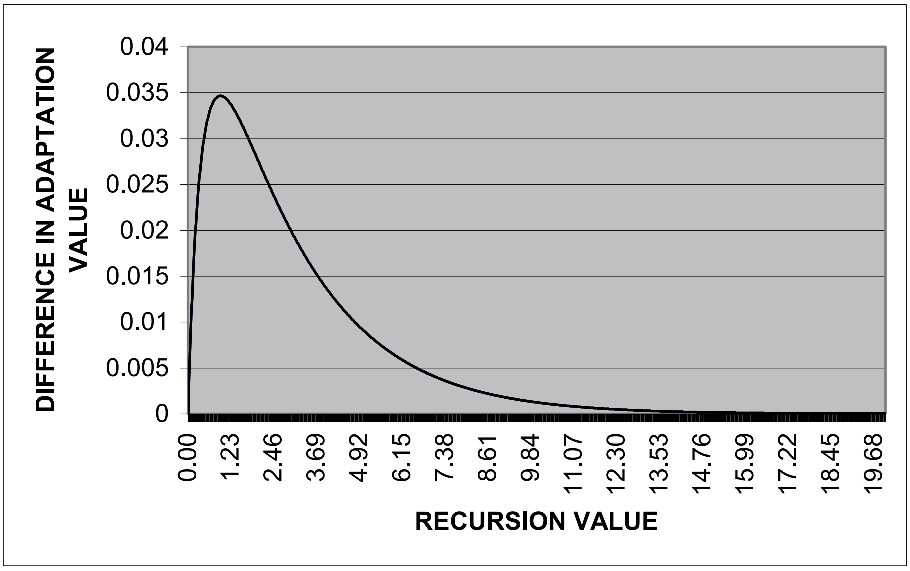
The upward sloping line is the recursion value of equity, η. The upward sloping curve is the overall value of equity which is the sum of its recursion value and its real (adaptation) value, or:

$$P(\eta) = \eta + \eta \int_{\eta}^{\infty} \frac{\exp(\frac{-2i}{\zeta}.y)}{y^2} dy$$

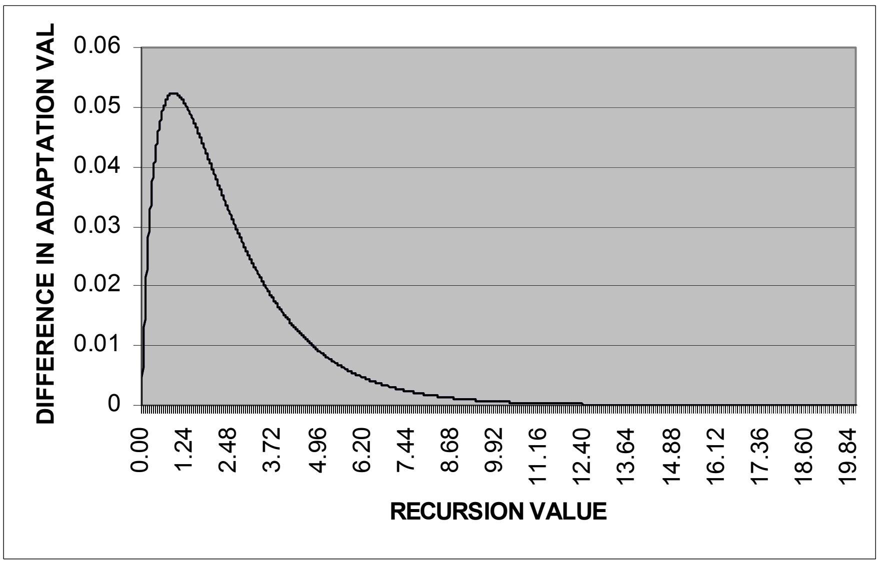
The integrals in the above expressions were estimated using 15 point Gauss-Legendre quadrature [Carnahan, Luther and Wilkes (1969, pp. 101-105)].

TABLE III

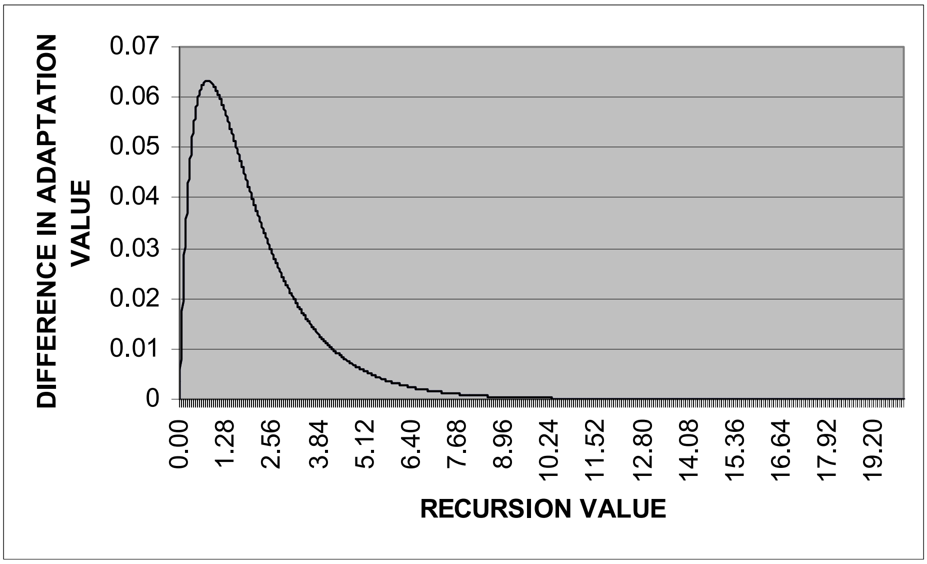
DIFFERENCE BETWEEN THE REAL (ADAPTATION) VALUE OF EQUITY FOR A FIRM THAT PAYS AN INSTANTANEOUS PROPORTIONATE DIVIDEND EQUAL TO $\alpha = \frac{i}{2} = 0.125\zeta$ OF ITS RECURSION VALUE AND THE REAL (ADAPTATION) VALUE OF EQUITY FOR A FIRM THAT DOES NOT PAY DIVIDENDS



DIFFERENCE BETWEEN THE REAL (ADAPTATION) VALUE OF EQUITY FOR A FIRM THAT PAYS AN INSTANTANEOUS PROPORTIONATE DIVIDEND EQUAL TO $\alpha = \frac{i}{2} = 0.1875\zeta$ OF ITS RECURSION VALUE AND THE REAL (ADAPTATION) VALUE OF EQUITY FOR A FIRM THAT DOES NOT PAY DIVIDENDS



DIFFERENCE BETWEEN THE REAL (ADAPTATION) VALUE OF EQUITY FOR A FIRM THAT PAYS AN INSTANTANEOUS PROPORTIONATE DIVIDEND EQUAL TO $\alpha = \frac{i}{2} = 0.25\zeta$ OF ITS RECURSION VALUE AND THE REAL (ADAPTATION) VALUE OF EQUITY FOR A FIRM THAT DOES NOT PAY DIVIDENDS



The above graphs summarise the difference between the equity value of a dividend-paying firm and the equity value of a non-dividend paying firm, or:

$$\left(\eta + \frac{i}{2\zeta}.\eta^2\right)\int_{\eta}^{\infty}\frac{\exp\left(\frac{-i}{\zeta}.y\right)}{\left(y + \frac{i}{2\zeta}.y^2\right)^2}dy - \eta\int_{\eta}^{\infty}\frac{\exp\left(\frac{-2i}{\zeta}.y\right)}{y^2}dy$$

The value of equity for the dividend-paying firm is graphed in Figure I. The value of equity for the non-dividend paying firm is graphed in Figure II. The integrals in the above expressions were estimated using 15 point Gauss-Legendre quadrature [Carnahan, Luther and Wilkes (1969, pp. 101-105)].

APPENDIX

Dirty Surplus Accounting and the Recursion Value of Equity

It will be recalled from the text that D(t) is the function whose value is the accumulated dividends paid over the (semi-closed) time interval [0,t). It then follows that dD(t) = D(t + dt) - D(t) represents the instantaneously known dividend payment which is made at time t. Hence, the expected present value of the future dividend stream will be:

$$\eta(t) = E_t\left[\int_t^{\infty} e^{-i(s-t)} dD(s)\right]$$

where $E_t(\cdot)$ is the expectations operator taken at time t and i is the cost of capital (per unit time) applicable to equity. Now, changes in the book value of equity, db(t), are related to the dividends paid through the instantaneous dirty surplus identity, which is:

$$db(t) = (x(t) + \varepsilon(t))dt - dD(t)$$

where x(t) is the instantaneous accounting (or book) earnings (per unit time) and ε(t) is the instantaneous dirty surplus adjustment (per unit time), both at time t. It thus follows that the expected present value of dividends can be restated as:

$$\eta(t) = E_t\left[\int_t^{\infty} e^{-i(s-t)}(x(s) + \varepsilon(s))ds\right] - E_t\left[\int_t^{\infty} e^{-i(s-t)} db(s)\right]$$

Hence, if we impose the transversality condition:

$$\underset{s \to \infty}{\text{Limit}}\ e^{-i(s-t)} E_t[b(s)] = 0$$

and integrate the last term by parts, then we have:

$$\eta(t) = b(t) + E_t\left[\int_t^{\infty} e^{-i(s-t)} a(s)ds\right] + E_t\left[\int_t^{\infty} e^{-i(s-t)} \varepsilon(s)ds\right]$$

where a(t) = x(t) - ib(t) is the instantaneous residual income (per unit time) applicable to equity. Thus, the expected present value of the future dividend stream is equivalent to the book value of equity plus the sum of the expected present value of the residual income stream and the expected present value of the dirty surplus adjustment, both at time t.

Dirty Surplus Accounting and the Recursion Value of Equity

Part (i) of Theorem #2 assumes that the stochastic process that generates abnormal earnings is:

$$da(t) = (c_{11}a(t) + c_{12}\nu(t) + c_{13}\varepsilon(t))dt + k_1\eta^{\delta}(t)dz_1(t)$$

The expected present value of the stream of abnormal earnings is $E_t[\int_t^{\infty} e^{-i(s-t)}a(s)ds]$, where $E_t(\cdot)$ is the expectations operator taken at time t. Integrating by parts and invoking the transversality condition:

$$\underset{s \rightarrow \infty}{\text{Limit}}\ e^{-i(s-t)}E_t[a(s)] = 0$$

implies:

$$(i - c_{11})E_t[\int_t^{\infty} e^{-i(s-t)}a(s)ds] = a(t) + c_{12}E_t[\int_t^{\infty} e^{-i(s-t)}\nu(s)ds] + c_{13}E_t[\int_t^{\infty} e^{-i(s-t)}\varepsilon(s)ds]$$

Furthermore, part (i) of Theorem #2 also assumes that the stochastic process which describes the evolution of the information variable is:

$$d\nu(t) = (c_{21}a(t) + c_{22}\nu(t) + c_{23}\varepsilon(t))dt + k_2\eta^{\delta}(t)dz_2(t)$$

We can use this assumption and similar procedures to those applied to the abnormal earnings variable to show that:

$$(i - c_{22})E_t[\int_t^{\infty} e^{-i(s-t)}\nu(s)ds] = \nu(t) + c_{21}E_t[\int_t^{\infty} e^{-i(s-t)}a(s)ds + c_{23}E_t[\int_t^{\infty} e^{-i(s-t)}\varepsilon(s)ds]$$

Finally, part (i) of Theorem #2 also assumes that the stochastic process which describes the evolution of the dirty surplus variable is:

$$d\varepsilon(t) = (c_{31}a(t) + c_{32}\nu(t) + c_{33}\varepsilon(t))dt + k_3\eta^{\delta}(t)dz_3(t)$$

We can again use this assumption and similar procedures to those applied to the abnormal earnings variable to show that:

$$(i - c_{33})\,{}_t[\int_t^{\infty} e^{-i(s-t)}\varepsilon(s)ds] = \varepsilon(t) + c_{31}E_t[\int_t^{\infty} e^{-i(s-t)}a(s)ds + c_{32}E_t[\int_t^{\infty} e^{-i(s-t)}\nu(s)ds]$$

Collecting these results into matrix form shows:

$$\begin{pmatrix} (i - c_{11}) & -c_{12} & -c_{13} \\ -c_{21} & (i - c_{22}) & -c_{23} \\ -c_{31} & -c_{32} & (i - c_{33}) \end{pmatrix} \begin{pmatrix} E_t[\int_t^{\infty} e^{-i(s-t)} a(s)ds] \\ E_t[\int_t^{\infty} e^{-i(s-t)} \nu(s)ds] \\ E_t[\int_t^{\infty} e^{-i(s-t)} \varepsilon(s)ds] \end{pmatrix} = \begin{pmatrix} a(t) \\ \nu(t) \\ \varepsilon(t) \end{pmatrix}$$

or, in more compact notation:

$$(iI - C)\underset{\sim}{E}(t) = \underset{\sim}{y}(t)$$

where I is the 3×3 identity matrix, $C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$ is the matrix of structural coefficients, $\underset{\sim}{E}(t) = \begin{pmatrix} E_t[\int_t^{\infty} e^{-i(s-t)} a(s)ds] \\ E_t[\int_t^{\infty} e^{-i(s-t)} \nu(s)ds] \\ E_t[\int_t^{\infty} e^{-i(s-t)} \varepsilon(s)ds] \end{pmatrix}$ and $\underset{\sim}{y}(t) = \begin{pmatrix} a(t) \\ \nu(t) \\ \varepsilon(t) \end{pmatrix}$. Simple matrix operations then show that the recursion value of equity will be:

$$\eta(t) = b(t) + E_t[\int_t^{\infty} e^{-i(s-t)} a(s)ds] + E_t[\int_t^{\infty} e^{-i(s-t)} \varepsilon(s)ds] = b(t) + (1,0,1)(iI - C)^{-1}\underset{\sim}{y}(t)$$

or:

$$\eta(t) = b(t) + \frac{[(i - c_{22})(i - c_{33} + c_{31}) + c_{32}(c_{21} - c_{23})]a(t)}{\Delta} +$$

$$\frac{[c_{12}(i - c_{33} + c_{31}) + c_{32}(i - c_{11} + c_{13})]\nu(t) + [(i - c_{22})(i - c_{11} + c_{13}) + c_{12}(c_{23} - c_{21})]\varepsilon(t)}{\Delta}$$

where $\Delta = \det(iI - C)$ is the determinant of the matrix $(iI - C)$.

To determine the stochastic process that generates the recursion value of equity, differentiate through the expression for $\eta(t)$ to give:

$$d\eta(t) = db(t) + (1,0,1)(iI - C)^{-1}d\underset{\sim}{y}(t)$$

Now since $db(t) = (a(t) + ib(t) + \varepsilon(t))dt - dD(t) = ib(t)dt + (1,0,1)\underset{\sim}{y}(t)dt - dD(t)$ and $d\underset{\sim}{y}(t) = C\underset{\sim}{y}(t)dt + \eta^{\delta}(t)d\underset{\sim}{z}(t)$ it follows that:

$$d\eta(t) = ib(t)dt + (1,0,1)\underset{\sim}{y}(t)dt - dD(t) + (1,0,1)(iI - C)^{-1}(C\underset{\sim}{y}(t)dt + \eta^{\delta}(t)d\underset{\sim}{z}(t))$$

or:

$$d\eta(t) = ib(t)dt - dD(t) + (1,0,1)[I + (iI - C)^{-1}C]\underset{\sim}{y}(t)dt + \eta^{\delta}(t)(1,0,1)(iI - C)^{-1}d\underset{\sim}{z}(t)$$

Note here, however, that $(iI - C)I + C = iI$ or $(iI - C)^{-1}(iI - C)I + (iI - C)^{-1}C = i(iI - C)^{-1}I$. Since $(iI - C)^{-1}(iI - C) = I$, we therefore have that $I + (iI - C)^{-1}C = i(iI - C)^{-1}$. It thus follows that $ib(t) + (1,0,1)[I + (iI - C)^{-1}C]\underset{\sim}{y}(t) = ib(t) + i(1,0,1)(iI - C)^{-1}\underset{\sim}{y}(t) = i\eta(t)$ in which case we have:

$$d\eta(t) = i\eta(t)dt - dD(t) + \eta^{\delta}(t)(1,0,1)(iI - C)^{-1}d\underset{\sim}{z}(t)$$

If we now choose the normalising constants so that $(iI - C)^{T}\begin{pmatrix} k_1^{-1} \\ k_2^{-1} \\ k_3^{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, where $(iI - C)^{T}$ is the transpose of the matrix $(iI - C)$, then it follows that the recursion value of equity evolves in accordance with the stochastic differential equation:

$$d\eta(t) = i\eta(t)dt - dD(t) + \eta^{\delta}(t)dq(t)$$

where $dq(t) = (dz_1(t) + dz_2(t)) + dz_3(t))$ is a Wiener process with variance parameter $\zeta = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\rho_{12}\sigma_1\sigma_2 + 2\rho_{13}\sigma_1\sigma_3 + 2\rho_{23}\sigma_2\sigma_3$.

Dirty Surplus Accounting and the Real (Adaptation) Value of Equity

Now, equity value satisfies the recursion relationship [Miller and Modigliani (1961, pp. 412-413), Dixit and Pindyck (1994, pp. 122-123)]:

$$P(\eta(t)) = dD(t) + e^{-idt}E_t[P(\eta(t + dt))]$$

where $P(\eta(t))$ is the market value of a unit investment in equity and $dD(t)$ is the instantaneously known dividend payment at time t. If we expand $P(\eta(t + dt))$ as a Taylor series about the point $\eta(t)$, use the fact that $e^{-idt} = 1 - idt + \frac{1}{2}(idt)^2 + \_\_\_\_$, take expectations across the right hand side of the above recursive relationship, substitute the expressions for $E_t[d\eta(t)] = i\eta(t)dt - dD(t)$ and $Var_t[d\eta(t)] = \zeta\eta^{2\delta}dt$, divide both sides by dt and then take limits in such a way as to let $dt \to 0$, then the recursive relationship implies that equity value will also have to satisfy the fundamental valuation equation:

$$\frac{1}{2}\zeta\eta^{2\delta}.\frac{d^2P}{d\eta^2} + (i\eta - \frac{dD}{dt}).\frac{dP}{d\eta} + (\frac{dD}{dt} - iP(\eta)) = 0$$

Determining the general solution of this equation is comprised of two steps [Boyce and DiPrima (1969, p. 115)]. First, a (particular) solution must be found. Here, direct substitution shows that the recursion value of equity, $P(\eta) = \eta$, satisfies the fundamental valuation equation irrespective of the form of the dividend function, D(t), and is, therefore, a (particular) solution of the equation.[16] Second, we must determine the general solution of the auxiliary (or homogeneous) form of the fundamental valuation equation:

$$\frac{1}{2}\zeta\eta^{2\delta}.\frac{d^2P}{d\eta^2} + (i\eta - \frac{dD}{dt}).\frac{dP}{d\eta} - iP(\eta) = 0$$

Since the auxiliary equation is a second order ordinary differential equation, all of its solutions will take the form $c_0P_1(\eta) + c_1P_2(\eta)$, where c_0 and c_1 are constants and $P_1(\eta)$ and $P_2(\eta)$ are linearly independent functions [Boyce and DiPrima (1969, p. 93)]. The general solution of the fundamental valuation equation will then consist of the particular solution plus the general solution of the auxiliary equation [Boyce and DiPrima (1969, p. 115)]. Since the particular solution is the recursion value of equity, η, the general solution of the auxiliary equation must capture the real (adaptation) value of equity. It thus follows that the market value of equity, $P(\eta)$, will be the sum of its recursion value and its real (adaptation) value, or:

$$P(\eta) = \eta + c_0P_1(\eta) + c_1P_2(\eta)$$

[16] The importance of this result cannot be overemphasised since it shows that the functional form of the recursion value of equity does not depend on the dividend function, D(t). Since recursion value is computed on the assumption that the firm's existing investment opportunity set will be applied indefinitely into the future, this is compatible with the Miller and Modigliani (1961) theorem which shows that the recursion value of equity must be independent of the firm's dividend policy. However, the real (adaptation) value of equity hinges on the potential changes a firm can make to its investment opportunity set and our analysis shows that this does, in general, depend on the dividend function, D(t).

The constants, c_0 and c_1, are determined by boundary conditions which require that equity value is exclusively made up of its real (adaptation) value as $\eta \to 0$ and exclusively made up of its recursion value as $\eta \to \infty$.

Dividends and the Value of Equity

Now, suppose the recursion value of equity evolves in accordance with an Ashton, Cooke and Tippett (2003) "square root" process ($\delta = \frac{1}{2}$) and that dividends are proportional to the recursion value of equity. It then follows that the real (adaptation) value of equity will be captured by the solutions of the following auxiliary equation:

$$\frac{1}{2}\zeta\eta\frac{d^2P}{d\eta^2} + (i - \alpha)\eta\frac{dP}{d\eta} - iP(\eta) = 0$$

where $\frac{dD}{dt} = \alpha\eta$ is the instantaneous dividend and $0 \le \alpha < i$ is the constant of proportionality. Since there are no closed form solutions to this equation, we determine analytic solutions in the form of the power series expansion, $P(\eta) = \sum_{j=0}^{\infty} a_j\eta^{j+r}$, where a_j are coefficients and r is the exponent of singularity. Furthermore, since this expression also implies $\frac{dP}{d\eta} = \sum_{j=0}^{\infty} (j+r)a_j\eta^{j+r-1}$ and $\frac{d^2P}{d\eta^2} = \sum_{j=0}^{\infty} (j+r)(j+r-1)a_j\eta^{j+r-2}$, then substitution into the auxiliary equation shows:

$$\frac{1}{2}\zeta\sum_{j=0}^{\infty} (j+r)(j+r-1)a_j\eta^{j+r-1} + (i - \alpha)\sum_{j=0}^{\infty} (j+r)a_j\eta^{j+r} - i\sum_{j=0}^{\infty} a_j\eta^{j+r} \quad = 0$$

However, if we use the fact that $\sum_{j=0}^{\infty} a_j\eta^{j+r} = \sum_{j=1}^{\infty} a_{j-1}\eta^{j+r-1}$ and

$\sum_{j=0}^{\infty} (j+r)a_j\eta^{j+r} = \sum_{j=1}^{\infty} (j+r-1)a_{j-1}\eta^{j+r-1}$, then the above power series expansion for the auxiliary equation may be restated as:

$$\frac{1}{2}\zeta\sum_{j=0}^{\infty} (j+r)(j+r-1)a_j\eta^{j+r-1} + (i - \alpha)\sum_{j=1}^{\infty} (j+r-1)a_j\eta^{j+r-1} - i\sum_{j=1}^{\infty} a_{j-1}\eta^{j+r-1} \quad = 0$$

Expanding out the term for j = 0 shows that the indicial equation is $\frac{1}{2}\zeta a_0 r(r - 1)\eta^{r-1} = 0$, or that the exponents of singularity are r = 0 and r = 1. Now, letting r = 1 in the series expansion for the auxiliary equation shows:

$$\sum_{j=1}^{\infty} [\tfrac{1}{2}\zeta j(j+1)a_j + (i - \alpha)ja_{j-1} - ia_{j-1}]\eta^j = 0$$

This leads to the following recursion formula for the relationship between the coefficients:

$$a_j = \frac{2(i + (\alpha - i)j)}{j(j + 1)\zeta}.a_{j-1}$$

Thus, letting j = 1 shows that $a_1 = \frac{\alpha}{\zeta}.a_0$. Furthermore, letting j = 2 implies $a_2 = \frac{(2\alpha - i)}{3\zeta}.a_1$ or $a_2 = \frac{(2\alpha - i)}{3\zeta}.\frac{\alpha}{\zeta}.a_0 = \frac{\alpha(2\alpha - i)}{3\zeta^2}.a_0$. Letting $a_0 = 1$ for convenience and expanding the recursion formula in this way shows that the following power series expansion is a formal solution of the auxiliary equation:

$$P_1(\eta) = \eta + \frac{\alpha}{\zeta}\eta^2 + \frac{\alpha(2\alpha - i)}{3\zeta^2}\eta^3 + \frac{\alpha(2\alpha - i)(3\alpha - 2i)}{18\zeta^3}\eta^4 + \_\_\_\_$$

Furthermore, by reduction of order, a second linearly independent solution will be [Boyce and DiPrima (1969, pp. 103-104)]:

$$P_2(\eta) = P_1(\eta)\int_{\eta}^{\infty}\frac{\exp[\frac{2(\alpha - i)y}{\zeta}]}{P_1^2(y)}dy$$

Now, for this second solution it is easily shown that the indefinite integral, $\int_{0}^{\infty}\frac{\exp[\frac{2(\alpha - i)y}{\zeta}]}{P_1^2(y)}dy$, is non-convergent [Spiegel (1974, p. 264)]. However, applying L'Hôpital's Rule shows that [Spiegel (1974, p. 62)]:

$$\lim_{\eta \to 0} P_2(\eta) = \lim_{\eta \to 0} \frac{\exp[\frac{2(\alpha - i)\eta}{\zeta}]}{\frac{dP_1(\eta)}{d\eta}} = 1$$

Likewise, L'Hôpital's Rule also shows that:

$$\underset{\eta \to \infty}{\text{Limit}} \; P_2(\eta) = \underset{\eta \to \infty}{\text{Limit}} \frac{\exp[\frac{2(\alpha - i)\eta}{\zeta}]}{\frac{dP_1(\eta)}{d\eta}} = 0$$

Hence, this second solution satisfies the boundary conditions laid down in Theorem #3 and thereby captures the real (adaptation) value of equity. Furthermore, since the overall market value of equity, P(η), is the sum of its recursion value and its real (adaptation) value, it follows that:

$$P(\eta) = \eta + P(0) \int_{\eta}^{\infty} \exp[\frac{2(\alpha - i)y}{\zeta}] \frac{P_1(\eta)}{P_1^2(y)} dy$$

where P(0) is the “dormant value” of the firm’s equity; that is, the market value of equity when the recursion value of equity approaches zero ($\eta \to 0$).

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