Pricing Inflation-Linked Longevity Bonds in the HJM Framework

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1. Introduction

Introduction

- Annuity product is getting more and more important in retirement market.

- However, how to manage longevity risk for annuity provider has become a critical issue when human life expectancy is getting longer.

- Blake and Burrows (2001) first recommend the use of survivor bonds as a hedging tool against longevity risk.

Introduction

- A lot of literatures have contributed on securitization longevity risk.
  - Lin and Cox 2005; Cox et al. 2006; Dowd et al. 2006; Blake et al. 2006; Denuit et al. 2007; Biffis and Blake, 2009; Blake et al. 2010; Dawson et al., 2010

Introduction

- Due to the uncertainty of future inflation which may reduce the purchasing power of the retired people, the use of inflation-linked annuity can serve as the role to hedge inflation risk in retirement.

- Yang and Yeh(2012) price Inflation-linked Annuity using the HJM Model.

- Tiong (2013) prices inflation-linked variable annuities under the HJM model.

Introduction

- Purpose of this study
  - We propose a valuation framework for pricing inflation-linked longevity bond using Heath-Jarrow-Morton (HJM) model.
  - Unlike the standard inflation-indexed derivatives that are based on a single inflation-linked underlying such as inflation bonds or forwards (Jarrow and Yildirim, 2003; Mercurio, 2005; Hinnerich, 2008; Kruse, 2011), the inflation-linked longevity bonds with underlying from more than one risk are typically challenging to price.

Introduction

- Contribution
  - We show that, under a no arbitrage Heath–Jarrow–Morton (HJM) type of framework where interest rate term structures are assumed to be Gaussian, the closed form pricing formula can be obtained for the inflation-linked longevity bond.

2. The Model

Model Formulation 1

The payments of the inflation-linked longevity bond at time t:

$$\text{payment}_t = £50\text{m} \times S(t, x_0) \times P_t$$

P_t: CPI inflation rate at time t;

$S(t, x_0)$: the survival probability at time t aged x_0 at time 0.

Model Formulation 2

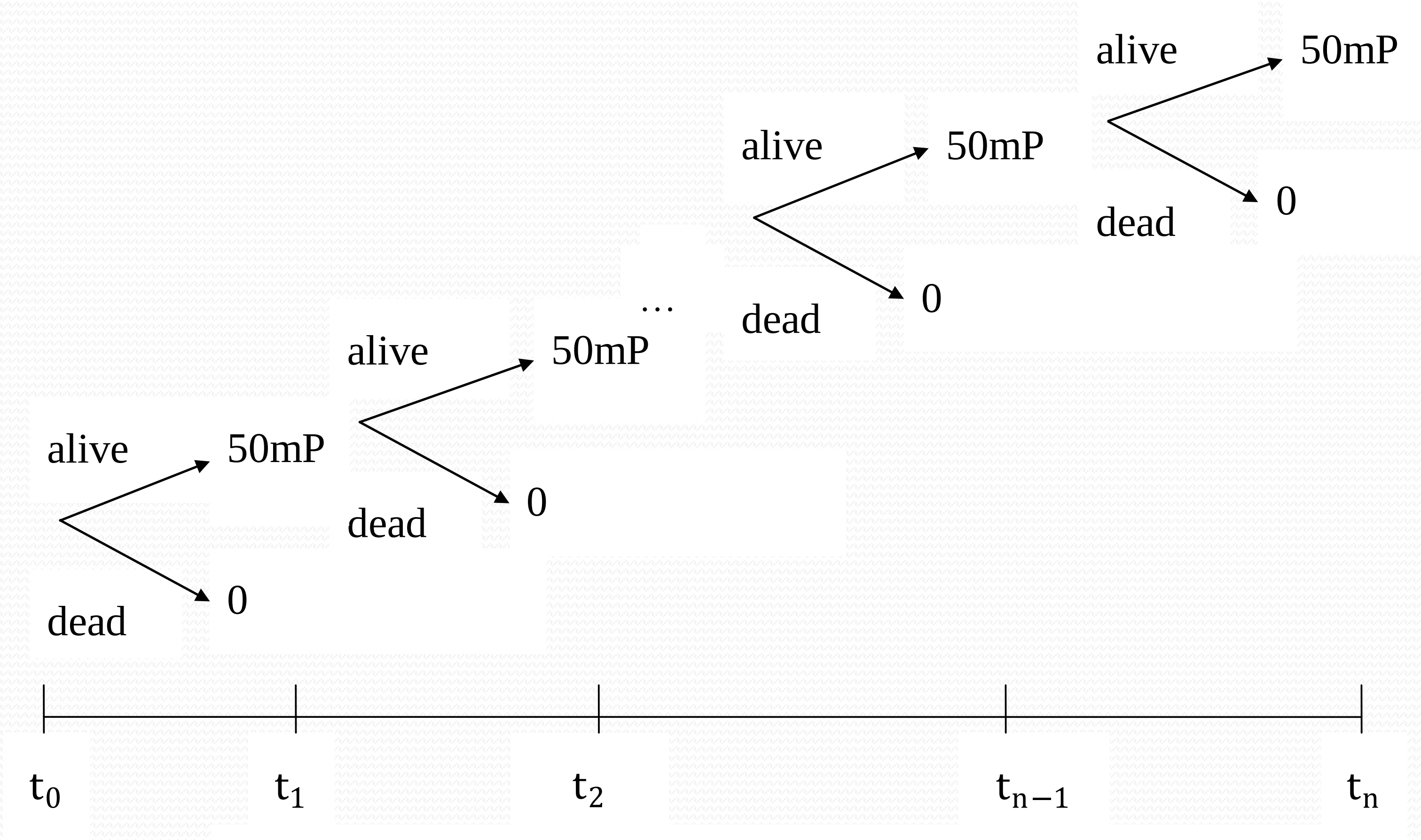
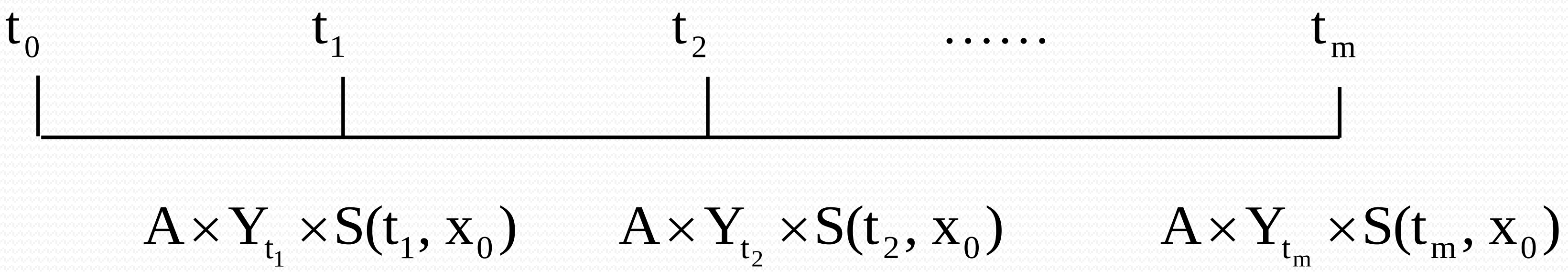


Figure 2 Payoffs of Inflation-Linked Longevity Bond

Model Formulation 3: Payoff of Inflation-linked Longevity Bond



Model Formulation 4

The cash flow of the bond at each period is:

$$LB^{CF}_{t_{j-1},\, t_j}(t_j) = \begin{bmatrix} £50m * P_{t_j}, & \text{if} \quad \text{the holder is alive} \\ 0, & \text{if} \quad \text{the holder is dead.} \end{bmatrix}, \qquad (3)$$

for all $j=1, 2, ..., n, \quad t_0 = 0$, and $t_n = T$.

Financial Dynamics

The dynamic processes of forward interest rate, inflation index, survivor probability and zero coupon bond under original probability measure are:

$$df(t,T)=\mu(t,T)dt+\sigma(t,T)dW_t \ , \qquad (4)$$

$$\frac{dP(t)}{P(t)}=\mu_P dt+\sigma_P dW_t \, , \qquad (5)$$

$$\frac{dS(t,\, x_0)}{S(t,\, x_0)}=\mu_s dt+\sigma_s dW_{s,t} \, , \qquad (6)$$

$$\frac{dB(t,T)}{B(t,T)}=r_t dt+b(t,T)dW_t \, , \qquad (7)$$

Financial Dynamics

$f(t,T)$: forward interest rate at time t with a maturity of T;

$\mu(t,T)$: an instantaneous drift term of forward interest rate;

$\sigma(t,T)$: volatility of forward interest rate;

$P(t)$: inflation index at time t;

μ_P: instantaneous drift term of inflation index;

σ_P: volatility of inflation index;

μ_s :instantaneous drift term of the survivor probability;

σ_s: volatility of the survivor probability;

Financial Dynamics

$W_{s,t}, W_t$: one-dimensional Brownian motion at time t under the original probability measure, P

$\rho_{1,2} = \text{corr}(dW_t, dW_{s,t})$.

$b(t,T) = -\int_t^T \sigma(u,T)du$: the volatility of zero coupon bond.

Pricing Inflation-Linked Longevity Bond

Proposition 1: The dynamic processes of $P(t)\times S(t,x_0)$, and $\frac{P(t)\times S(t,x_0)}{B(t,T)}$ can be obtained under spot original probability measure, Q, respectively.

$$\frac{dG(t)}{G(t)} = r_t dt + \sigma_G(t) dW_{G,t}^{Q}, \qquad (8)$$

$$\frac{dX(t)}{X(t)} = \sigma_X dW_{X,t}^{PT}, \qquad (9)$$

with $\sigma_G(t) = \sqrt{\sigma_s^2 + \sigma_P^2 + 2\rho_{1,2}\sigma_s\sigma_P}$,

$$\sigma_X(t) = \sqrt{\sigma_G^2 + b_t^2 - 2\rho_{1,2}\sigma_G b_t},$$

and $\rho_{1,2} = corr(dW_t^Q, dW_{s,t}^Q)$.

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Thus, the time- t_{n-1} and time- t_{n-2} value of the inflation-linked longevity bond under a forward risk neutral probability, PT:

$$LB_{t_{n-1},t_n}(t_{n-1}) = 50m \times B(t_{n-1}, t_n) \times X(t_{n-1}),$$

$$LB_{t_{n-2},t_{n-1}}(t_{n-2}) = 50m \times B(t_{n-2}, t_{n-1}) \times \left[X(t_{n-2}) + E_{t_{n-2}}^{PT}\left[G(t_{n-1})F_S(t_{n-1}, t_{n-1}, x_0)\right]\right]$$

(13)

Pricing Inflation-Linked Longevity Bond

in which $F_S(t_{n-1}, t_{n-1}, x_0) = \dfrac{S(t_{n-1}, x_0)}{B(t_{n-1}, t_{n-1})}$.

$$LB_{t_{n-3}, t_{n-2}}(t_{n-3}) = 50m \times B(t_{n-3}, t_{n-2}) \times \begin{bmatrix} X(t_{n-3}) + E_{t_{n-3}}^{PT}\left[G(t_{n-2})F_S(t_{n-2}, t_{n-2}, x_0)\right] + \\ E_{t_{n-3}}^{PT}\left[B(t_{n-2}, t_{n-1})G(t_{n-1})F_S(t_{n-2}, t_{n-2}, x_0)F_S(t_{n-1}, t_{n-1}, x_0)\right] \end{bmatrix},$$

$$LB_{t_{n-4}, t_{n-3}}(t_{n-4}) = 50m \times B(t_{n-4}, t_{n-3}) \times \begin{bmatrix} X(t_{n-4}) + E_{t_{n-4}}^{PT}\left[G(t_{n-3})F_S(t_{n-3}, t_{n-3}, x_0)\right] \\ + E_{t_{n-4}}^{PT}\begin{bmatrix} B(t_{n-3}, t_{n-2})G(t_{n-2}) \\ F_S(t_{n-3}, t_{n-3}, x_0)F_S(t_{n-2}, t_{n-2}, x_0) \end{bmatrix} \\ + E_{t_{n-4}}^{PT}\begin{bmatrix} B(t_{n-3}, t_{n-2})B(t_{n-2}, t_{n-1})G(t_{n-1}) \\ F_S(t_{n-3}, t_{n-3}, x_0)F_S(t_{n-2}, t_{n-2}, x_0)F_S(t_{n-1}, t_{n-1}, x_0) \end{bmatrix} \end{bmatrix}$$

Pricing Inflation-Linked Longevity Bond

Thus, the time-0 value of the bond with maturity 25 is below

$$
\begin{aligned}
& LB_{t_0,\,t_1}(t_0) \\
& = 50m \times B(t_0,\,t_1) \\
& \times \begin{bmatrix}
X(t_0) + E_0^{PT}\left[G(t_1)F_S(t_1,t_1,\ x_0)\right] \\
+E_{t_0}^{PT}\begin{bmatrix} B(t_1,\,t_2)G(t_2) \\ F_S(t_1,t_1,\ x_0)F_S(t_2,t_2,\ x_0) \end{bmatrix} \\
+E_{t_0}^{PT}\begin{bmatrix} B(t_1,\,t_2)B(t_2,\,t_3)G(t_3) \\ F_S(t_1,t_1,\ x_0)F_S(t_2,t_2,\ x_0)F_S(t_3,t_3,\ x_0) \end{bmatrix} \\
+\cdots + E_{t_0}^{PT}\begin{bmatrix} B(t_1,\,t_2)B(t_2,\,t_3)\ldots B(t_{23},\,t_{24})G(t_{24}) \\ F_S(t_1,t_1,\ x_0)F_S(t_2,t_2,\ x_0)F_S(t_3,t_3,\ x_0)\ldots F_S(t_{24},t_{24},\ x_0) \end{bmatrix}
\end{bmatrix},
\end{aligned}
$$

Pricing Inflation-Linked Longevity Bond

in which

$$E_0^{PT}\left[G(t_1)F_S(t_1,t_1,\ x_0)\right]$$

$$= X_G(t_0,t_1)F_S(t_0,t_1,\ x_0)e^{\rho_{1,2}\sigma_X\sigma_t(t_1-t_0)},$$

$$E_{t_0}^{PT}\begin{bmatrix} B(t_1,\ t_2)G(t_2) \\ F_S(t_1,t_1,\ x_0)F_S(t_2,t_2,\ x_0) \end{bmatrix}$$

$$=B(t_0,\ t_1,\ t_2)X_G(t_0,t_2)\times F_S(t_0,t_1,\ x_0)F_S(t_0,t_2,\ x_0)$$

$$\times e^{\rho_{1,2}\left[-b(t,T_2)+b(t,T_1)\ \right]\sigma_t(t_2-t_0)}$$

3. Numerical Study

Assumption

1. Correlation coefficient between inflation index and survivor probability is zero.
2. Risk free rate is 2%.
3. For simplicity, the prices of zero coupon bonds of $B(t_0, t_1, t_2)$, $B(t_0, t_2, t_3)$, $B(t_0, t_3, t_4)$, $B(t_0, t_4, t_5)$,……and $B(t_0, t_{23}, t_{24})$ are equal to 0.9.
4. Volatilities of survivor probability, inflation index and zero coupon bond are 0.02, 0.2 and 0.1 respectively.

Numerical Results

Table 1 Sensitivity of Model Parameters on LB

Panel A: Impact of Volatility of Inflation rate on LB Price					
σ_p	0.1	0.2	0.3	0.4	0.5
Fair price of LB	16.8751	17.0153	17.3359	17.8156	18.0123
Panel B: Impact of Volatility of Survivor Probability on LB Price					
σ_s	0.01	0.02	0.03	0.04	0.05
Fair price of LB	15.9812	16.0159	16.1189	16.3358	16.5971
Panel C: Impact of Volatility of Zero Coupon Bond on LB Price					
b	0.1	0.2	0.3	0.4	0.5
Fair price of LB	16.7569	16.8956	16.9023	17.0808	17.2523

Note that σ_p and σ_s denote volatilities of inflation index and survivor probability, respectively. b stands for volatility of zero coupon bond. Suppose the maturity of the bond is 25 years.

Numerical Results

Suppose the maturity of the inflation-linked longevity bond is 25 years. When volatilities of zero coupon bond, survivor probability, and inflation index are 0.1, 0.02 and 0.3, respectively, the fair price of inflation-linked longevity bond is 16.6917. If inflation risk does not exist, the fair price is 15.2153.

4. Conclusion

Conclusion

1.This paper develops an inflation-linked longevity bond to transfer both longevity and inflation risk in the capital market.

2. Considering inflation risk and interest risk, we obtain a closed-form solution of the fair price of the bond in HJM model. This can efficiently help investors to hedge.

Thank you for your attention.