Fast Change Detection on Proportional Two-Population Hazard Rates

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Motivation

- Mainly concerned with mortality evolution which may change over a projection period
- Mortality profile/level/trend
  - Heterogeneity and other factors
  - Underwriting strategy
  - Adverse selection
- Interest in experienced mortality
  - Deaths can be observed sequentially
- Model risk, parameter uncertainty
  - Long-term projection
- Monitoring and surveillance of mortality dynamics
  - Sequential information on death occurrences
  - Updating mortality assumptions

Mathematical settings

We consider a portfolio of insured population:

- Let $N = (N_t)_{t\geq 0}$ be a counting process indicating the deaths of policyholders and $\lambda = (\lambda_t)_{t\geq 0}$ its intensity.
- The counting process N_t, is available sequentially through the filtration $\mathcal{F}_t = \sigma\{N_s, 0 < s \leq t\}$.
- We suppose that the insurance company relies on a Cox-like model to project her own experienced mortality:

$$\lambda_t = \underline{\rho}\lambda_t^0,$$

  - λ_t^0 is a reference intensity and $\underline{\rho}$ is a positive parameter.
  - λ^0 is considered deterministic and may refer whether to a projection of national population/best estimate...

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Model risk/parameter uncertainty: **Change-point**

$$\lambda_t = \mathbf{1}_{\{t < \boldsymbol{\theta}\}} \underline{\rho} \lambda_t^0 + \mathbf{1}_{\{t \geq \boldsymbol{\theta}\}} \overline{\rho} \lambda_t^0.$$

Without loss of generality we can assume that $\underline{\rho} = 1$ and let $\boldsymbol{\rho} = \overline{\rho} > 1$.

Probabilistic formulation

Let $\mathbb{P}_\theta$ (resp. $\mathbb{E}_\theta[\cdot]$) be the probability measure (resp. expectation) induced when the change takes place at time θ

Example

- For $\theta = 0$, the process is *out-of-control*
- For $\theta = \infty$, the process is *in-control*

Detect the change-point θ as quick as possible while avoiding false alarms

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Optimality Criteria, Lorden (1971)-like

- The detection delay $\mathbb{E}_\theta\left[(N_\tau - N_\theta)^+ \middle| \mathcal{F}_\theta\right]$
- The frequency of false alarm $\mathbb{E}_\infty[N_\tau]$

Optimization Problem

Optimization Problem

$$\text{Find } \tau^* \text{ such that } C(\tau^*) = \inf_\tau \mathbf{sup}_{\theta\in[0,\infty]} \mathbf{ess\,sup} \mathbb{E}_\theta \left[(N_\tau - N_\theta)^+ \middle| \mathcal{F}_\theta \right]$$
$$\text{subject to } \mathbb{E}_\infty[N_\tau] = \omega.$$

Assumption

1. $\int_0^t \lambda_s ds < \infty, \quad \mathbb{P}_\infty, \mathbb{P}_0$-a.s.
2. $N_\infty = \infty \qquad \mathbb{P}_\infty, \mathbb{P}_0$-a.s.

Optimality of the Cusum procedure (1/7)

Let the Radon-Nikodym density of $\mathbb{P}_0$ with respect to $\mathbb{P}_\infty$ be defined as

$$\left.\frac{d\mathbb{P}_0}{d\mathbb{P}_\infty}\right|_{\mathcal{F}_t} = \exp U_t,$$

where $U_t = \log(\rho)N_t + (1-\rho)\int_0^t \lambda_s^0 ds$ is the log-likelihood ratio.
Let $V(x)$ be the CUSUM process; with head-start $0 \le x < m$; defined as

$$V_t(x) = U_t - (-x) \wedge \underline{U}_t \tag{1}$$

where $\underline{U}_t$ is the running infimum of U, i.e. $\underline{U}_t = \inf_{s\le t} U_s$.
The process $V(x)$ measures the size of the drawup, comparing the present value of the process U to its historical infimum $\underline{U}$.
Let $\tau_m(x)$ be the fist hitting time of $V(x)$ of the barrier m, i.e.

$$\tau_m(x) = \inf\{t \ge 0, V_t(x) \ge m\}.$$

Theorem

If $\mathbb{E}_\infty[N_{\tau_m(0)}] = \omega$ *then* $\tau_m(0)$ *is optimal, i.e.* $\inf_\tau C(\tau) = C(\tau_m(0))$

Optimality of the Cusum procedure (2/7)

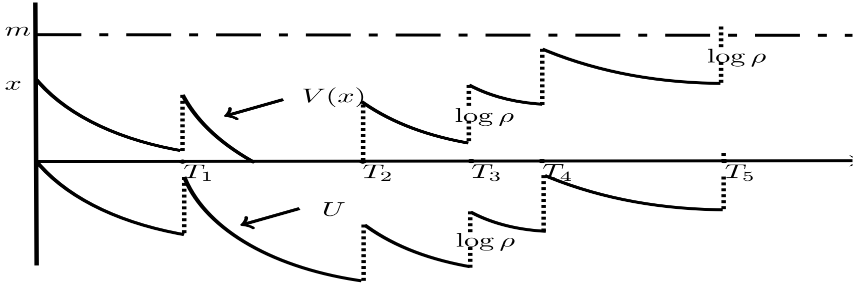

Figure : Example of the path of the processes U and V. The process V coincides with U before hitting the level 0.

Optimality of the Cusum procedure (2/7)

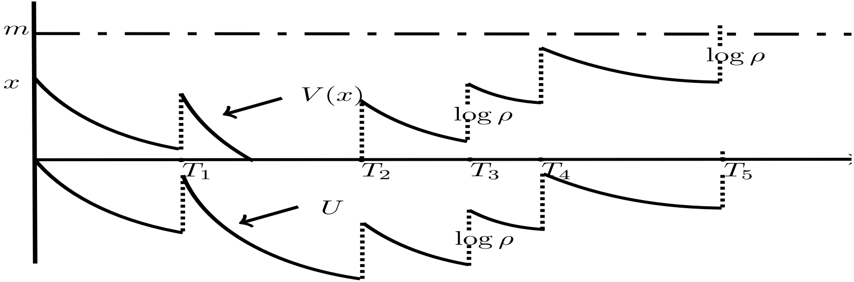

Figure : Example of the path of the processes U and V. The process V coincides with U before hitting the level 0.

The process $V(x)$ can be rewritten in the following form

$$V_t(x) \quad = \quad x + U_t + (-\underline{U}_t - x)^+$$

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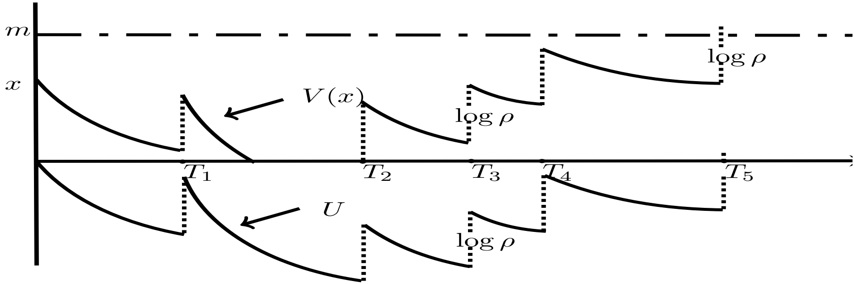

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The process $V(x)$ can be rewritten in the following form

$$V_t(x) \;=\; x + U_t + (-\underline{U}_t - x)^+ = x + \log(\rho) N_t + (1-\rho)\int^t \lambda_s^0 \mathbf{1}_{\{V_s(x)>0\}}$$

Optimality of the Cusum procedure (3/7)

Connection to Ruin Theory

Observe that

- $m - V_t(x)$ is nothing but the surplus process, with
  - initial wealth $m - x$
  - premium rate $(1-\rho)\lambda_t$
  - constant size of claims $\log \rho$
- $\tau_m(x)$ is the ruin time

Optimality of the Cusum procedure (4/7)

Lemma

$\tau_m(x)$ is $\mathbb{P}_0$ and $\mathbb{P}_\infty$-a.s. finite.

Let $h_0(x) = \mathbb{E}_0[N_{\tau_m(x)} | V_0 = x]$ and $h_\infty(x) = \mathbb{E}_\infty[N_{\tau_m(x)} | V_0 = x]$.

Theorem

1. *h_0 and h_∞ are decreasing in x and uniformly bounded on $[0, m]$*
2. *h_0 and h_∞ are continuously differentiable and satisfy the following backward delay differential equations (d.d.e.), $0 \le x < m$,*

$$\begin{aligned} h_0'(x) &= \frac{1}{1-\rho}\Big(\rho\big(h_0(x) - h_0(x + \log\rho)\big) - 1\Big), \\ h_\infty'(x) &= \frac{1}{1-\rho}\Big(h_\infty(x) - h_\infty(x + \log\rho) - 1\Big). \end{aligned}$$

Optimality of the Cusum procedure (5/7)

Lemma

The worst detection delay for the stopping time $\tau_m(0)$ is given by

$$\mathbb{E}_\theta[(N_{\tau_m(0)} - N_\theta)^+ | \mathcal{F}_\theta] = h_0(V_\theta(0)).$$

Proof.

Applying the Itô rule to $h_0(V_t)$, using the backward p.d.e. and optional stopping. □

1. The *cusum detection delay* is the function only of the value of $V_\theta(0)$
2. The worst-case *cusum detection delay* does not depend on θ, i.e. $C(\tau_m(0)) = h_0(0)$.

Optimality of the Cusum procedure (6/7)

Theorem

Let τ be an $\mathbb{F}$-stopping time, then

$$C(\tau) \geq \rho \frac{\mathbb{E}_\infty \left[\int_0^\tau \exp(V_{t-}) dN_t\right]}{\mathbb{E}_\infty[\exp(V_\tau)]}$$

Proof.

Follows the same reasoning as Moustakides (2004) and Shiryaev (1998). □

Finally, to show the optimality we need to show that for each stopping τ satisfying the false alarm constraint such that $\mathbb{E}_\infty[N_\tau] \geq \mathbb{E}_\infty[N_{\tau_m(0)}] = h_\infty(0)$ we have

$$\rho \frac{\mathbb{E}_\infty \left[\int_0^\tau \exp(V_{t-}) dN_t\right]}{\mathbb{E}_\infty[\exp(V_\tau)]} \geq h_0(0)$$

Optimality of the Cusum procedure (7/7)

Applying Itô formula yields:

$$\mathbb{E}_\infty\left[\rho\int_0^\tau \exp(V_{t-})dN_t - \exp(V_\tau)h_0(0)\right] = \mathbb{E}_\infty\Big[\rho\underbrace{\Big(\mathbb{E}_\infty[N_\tau] - h_\infty(0)\Big)}_{\geq 0}\Big]$$
$$+\mathbb{E}_\infty\Big[\Big(\rho h_\infty(V_\tau) - \exp(V_\tau)h_0(V_\tau)\Big)\Big]$$

Thus to show the optimality we only need the following lemma.

Lemma

For any stopping time τ we have

$$\mathbb{E}_\infty\Big[\rho h_\infty(V_\tau) - \exp(V_\tau)h_0(V_\tau)\Big] \geq 0$$

.

DETECTION PROCEDURE – ALGORITHM

Step 1: Fix the input parameters: The post-change intensity through the specification of ρ and the false alarm constraint ω.

Step 2: Determine the threshold m as the solution of the equation $\mathbb{E}_\infty[N_{\tau_m}] = \omega$.

Step 3: For each new observation at time t compute the value of the CUSUM process V given by the iterative relation $V_{t+1} = (V_{t-1} + U_t)^+$.

Step 4: Compare the current value of V to the threshold m and stop the procedure once $V_t \geq m$ and sound an alarm. Hence $\tau_m(0) = t$.

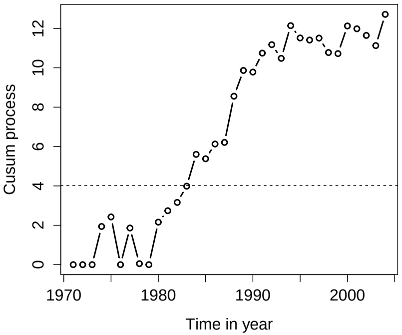
Detection Procedure – Real World (1/4)

We consider the Continuous Mortality Investigation assured lives dataset and England & Wales national population. We split data into two periods:

- We consider the period 1947-1969 as a training period.
- The Cox model is estimated over this period using the MLE.

Hence we monitor the sequentially the dataset over the period 1970-2005 and look for changes on the mortality of assured lives.

DETECTION PROCEDURE – REAL WORLD (2/4)



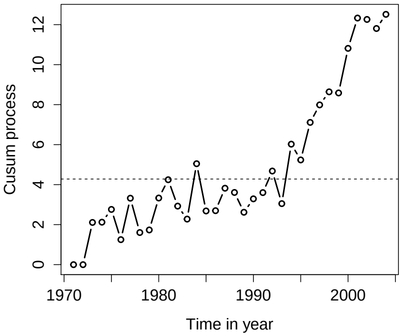


Figure : Detection scheme for age groups 50 – 59 (right) and 80 – 89 (left). The post-change is set to $\rho = 15\%$ and the false alarm constraint to $\omega = 100\bar{\lambda}$.

Detection Procedure – Real World (3/4)

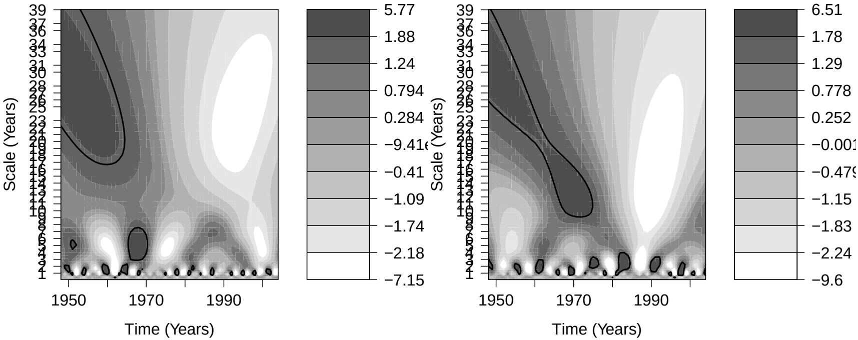


Figure : Wavelet transform of the process λ_t/λ_t^0 with a Gaussian-wavelet. The solid curve shows the regions where the null hypothesis of no change in the parameter ρ is rejected, for two age groups: 50-59 (left) and 80-89 (right)

Detection Procedure – Real World (4/4)

	τ_m		
Age	$\rho = 1.50$	$\rho = 1.15$	Observed
$50 - 59$	1984	1978	1970
$60 - 69$	1991	1985	1974
$70 - 79$	1988	1984	1974
$80 - 89$	1983	1978	1973

Table : Detection of mortality change with a post-change ratio of $\rho = 1.15$ and an average run length (false alarm) constraint of 100. The right column reports the detected change-point using an off-line procedure.

Conclusion

- The CUSUM detection rule is optimal in the case of non-homogeneous Poisson process with a modified Lorden criterion
- The considered methodology can be extended to the case where the mortality intensity is stochastic
- The "quickness" depends on the size of the change and the initial population