Pricing Reverse Mortgages in Japan Using a Multivariate Bayesian Risk-neutral Method

by

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Introduction

- **Reverse mortages involve multiple risks both in insurance and financial sectors (e.g. mortality rates, interest rates, house prices).**
- **We propose a Bayesian multivariate framework to price reverse mortgages by extending the univariate method proposed by Kogure and Kurachi (2010).**
- **We apply the proposed method to Japanese data to examine the possibilities for a successful introduction of reverese mortgage plans into Japan.**

What is a reverse mortgage?

- Mortgage

  Cash(=home loan payment) $\Rightarrow$ Asset (=house)

- Reverse mortgae

  Asset(=house)$\Rightarrow$ Cash(=annuity)

A reverse mortgage is a loan, but the borrower does not have to repay it as long as your live in you home.

Non-recourse Provision

A typical reverse mortgage is non-recourse:

- At the time t of death the borrower pays out the loan L_t, but not more than the house price H_t.
- Thus the payoff of the reverse mortgage provider at time t is

$$\begin{aligned} \min(L_t, H_t) &= \begin{cases} H_t & L_t \geq H_t \\ L_t & L_t \leq H_t \end{cases} \\ &= L_t - \max(L_t - H_t, 0) \end{aligned}$$

payoff of put option on house price *Ht* with strike price *Lt*

Present Value of a Reverse Mortgage

Consider a cohort of age x at year 0. Let I_t denote

$$I_t \equiv \frac{\text{number of deaths in year } t}{\text{population at year } 0}$$

- Then the per capita cash flow of the reverse mortgage is:

$$\{ (L_t - \max(L_t - H_t, 0)) \times I_t, t = 0, 1, \ldots, T\}$$

Maximum life years

- Its present value is

r=discount rate

$$E^* \left[\sum_{t=0}^{T} e^{-rt} \left(L_t - \max\left(L_t - H_t, 0\right)\right) I_t \right]$$

$$= \sum_{t=0}^{T} e^{-rt} L_t E^*[I_t] - \sum_{t=0}^{T} e^{-rt} E^* \left[\max\left(L_t - H_t, 0\right) I_t\right]$$

Risk neutral expectation

We assume that discount rate r is fixed.

Bayesian approach

- **To obtain the present value of the reverese mortgage, we need two steps:**

  1. **Statistical modeling of I_t and H_t**
  2. **Risk-neutralization of I_t and H_t**

- **Recently Bayesian methods have been increasingly used to deal with insurance and finacial risks. However, their use is mainly limited to step 1.**

- **In this paper we attempt to achive steps 1 and 2 in a unified Bayesian framework.**

What is the Risk-neutral Pricing?

- X: a value of risk factor one year from now (e.g. mortality rate, stock price, house price)
- $C(X)$: the payoff of a derivative on X such as
  - Put option: $C(X) = \max(K - X, 0)$; K is a strike price
- A statistical model for X: $\{f(x|\theta), \theta \in \Theta\}$
- The risk neutral version of f: $f^*(x|\theta) = \eta(x) f(x|\theta)$

Price of the derivative

$$\pi(\theta) = e^{-r} \int C(x) f^*(x|\theta) dx$$

r=discount rate

State price density

Risk-neutral Density

Risk-neutral density

A risk neutral verison of f is given as

$$f^*(x|\theta) = f(x|\theta)\eta(x)$$

state price density (weight function)

- The existence of a positive η $\Rightarrow$ "**non-arbitrage**".
- The existence of a unique positive η $\Rightarrow$ "**completeness**".

In the insurance risk theory, a variety of $\eta(x)$ are used such as Esscher transform: $\eta(x) = \dfrac{e^{\lambda x}}{\mathrm{E}[e^{\lambda X}]}$

Bayesian Risk-neutral Pricing

- **frequentist: replace the unknown θ by an estimate $\widehat{\theta}$:**

$$\pi(\widehat{\theta}) = e^{-r} \int C(x) f^*(x|\widehat{\theta}) dx$$

- **Bayesian: Use the posterior density $f(\theta|D)$ given data D**

$$\begin{aligned} \widetilde{\pi} &= \int \pi(\theta) f(\theta|D) d\theta \\ &= e^{-r} \int C(x) \int f^*(x|\theta) f(\theta|D) d\theta dx \\ &= e^{-r} \int C(x) f^*(x|D) dx, \end{aligned}$$

where $f^*(x|D) \equiv \int f^*(x|\theta) f(\theta|D) d\theta$ is the risk-neutral predictive density.

Risk-neutral Predictive Density

risk-neutral predictive density

state price density → $\eta(x)$

$$f^*(x|D) = \eta(x) f(x|D)$$

where $f(x|D) = \int f(x|\theta) f(\theta|D) d\theta$ is the (original) predictive density.

- How to determine $\eta(x)$?
  Candidates : Esscher Transfrom, Wang transform, etc.

- Here we adopt a "nonparametric method" based on the mimimum cross entropy.

Risk-neutral Density based on Cross-entropy (1)

- **Assumption (risk-neutrality): Under the risk neutral predictive density $f^*(x|D)$, a moment condition**

$$\mathrm{E}^*[h(X)] \equiv \int h(x) f^*(x|D)dx = a$$

holds. For example, when X is the stock price one year from now, we have

$$\mathrm{E}^*[e^{-r}X] = x_0 (= \text{the current stock price})$$

- **Then use $f^*(x|D)$ that minimizes the cross-entropy**

$$\int f^*(x) \log\left(\frac{f^*(x|D)}{f(x|D)}\right) dx$$

subject to the moment condition.

Risk Neutral Density based on Cross-entropy (2)

Risk-neutral Density based on cross-entropy

$$f^*(x|D) = f(x|D)\exp\{\gamma_0 + \gamma_1 h(x)\}$$

- **If $h(x) = x$, then it reduces to the Esscher transform.**
- **If threre are m moment conditions, then**

$$f^*(x|D) = f(x|D)\exp\{\gamma_0 + \gamma_1 h_1(x) + \cdots + \gamma_m h_m(x)\}$$

Cf. Stutzer (1996)

Bivariate Risk-neutral Density based on Cross-entropy (1)

- **Assumption (risk neutrality): Under the bivariate risk neutral predictive density $f^*(x,y|D)$, we have moment conditions:**

$$\mathbf{E}^*[h_i(X,Y)] \equiv \int\int h(x,y) f^*(x,y|D)\,dx\,dy = a_i$$

$$i = 1, 2, \cdots, m$$

- **Then, subject to the moment conditions and the condition that $f^*(x,y|D)$ integrates to 1, we use $f^*(x,y|D)$ which minimizes the cross entropy**

$$\int\int f^*(x,y|D) \log\left(\frac{f^*(x,y|D)}{f(x,y|D)}\right) dx\,dy.$$

Bivariate Risk-neutral Density based on Cross-entropy (2)

Bivariate Risk-neutral density based on cross-entropy

$$f^*(x,y|D) = f(x,y|D)\exp\{\gamma_0+\gamma_1 h_1(x,y)+\cdots+\gamma_m h_m(x,y)\}$$

- **In general, the independence of X and Y under f does not imply the same under f^*.**

- **However, this will be true if each h_i is additive:**

$$h_i(x,y) = h_{i1}(x) + h_{i2}(y),\ i = 1,2,\ldots,m$$

Present value of reverse mortgage

We assume that

- the death ratio I_t and the house price H_t are independent under f.
- there is no financial product which depends on both H_t and I_t.

Then I_t and H_t are also independent under f^*, and thus:

The present value of reverese mortgages is

$$= \sum_{t=0}^{T} e^{-rt} L_t \mathbf{E}^*[I_t] - \sum_{t=0}^{T} e^{-rt} \mathbf{E}^*[I_t] \mathbf{E}^*[\max(L_t - H_t, 0)],$$

where E^* is the expectation under f^*

$\Rightarrow$ It suffices to calculate $\mathbf{E}^*[I_t]$ and $\mathbf{E}^*[\max(L_t - H_t, 0)]$ seperately

Evaluation of $E^*[I_t]$: Modeling Mortality Risk

Hereafter I_t is taken to be:

$$\begin{aligned} I_t &= \text{the probability that an individual aged } x \text{ at year 0} \\ &\quad \text{dies in year } t \\ &= {}_tp_x q_{x+t}(t) = {}_tp_x - {}_{t+1}p_x, \end{aligned}$$

where

- ${}_tp_x$ is the probability he/she is still alive at year t:

  $${}_tp_x = (1 - q_x(0)) \times \cdots \times (1 - q_{x+t-1}(t-1))$$

- $q_x(t)$ is an individual aged x at year t will die within one year.

$\Rightarrow$ **It suffices to model $q_x(t)$.**

Lee-Carter method

Lee-Carter model

Force of mortality of an individual aged x at year t:

$$\mu_x(t) = \exp\left(\alpha_x + \sum_{i=1}^{p} \beta_{ix}\kappa_{it}\right)$$

- α_x: age parameter
- κ_{it}: ith year factor
- β_{ix}: sensitivity parameter in response to κ_{it}

- Typically, one-factor L-C model ($p = 1$) is used.
- Here two-factor L-C model ($p = 2$) is also considered. Cf. Renshaw and Haberman (2005) and Lazer and Denuit (2009)

One-factor Lee-Carter model

Statistical model for central death rates:

$$m_{xt} \equiv \frac{D_{xt}}{E_{xt}} = \frac{\#\ \text{of dead aged } x \text{ at year } t}{\text{mid-year population of age } x \text{ at year } t}$$

State-space modeling of one-factor L-C model

- **observation equation:**

$$\ln m_{xt} = \underbrace{\alpha_x + \beta_x \kappa_t}_{\ln \mu_x(t)} + \varepsilon_{xt}$$

$$\varepsilon_{xt} \overset{\text{iid}}{\sim} N(0, \sigma_\varepsilon^2)$$

- **state equation:**

$$\kappa_t = \lambda + \kappa_{t-1} + \omega_t$$

$$\omega_t \overset{\text{iid}}{\sim} N(0, \sigma_\omega^2)$$

temporal changes (κ_t)

Bayesian estimation of Lee-Carter model

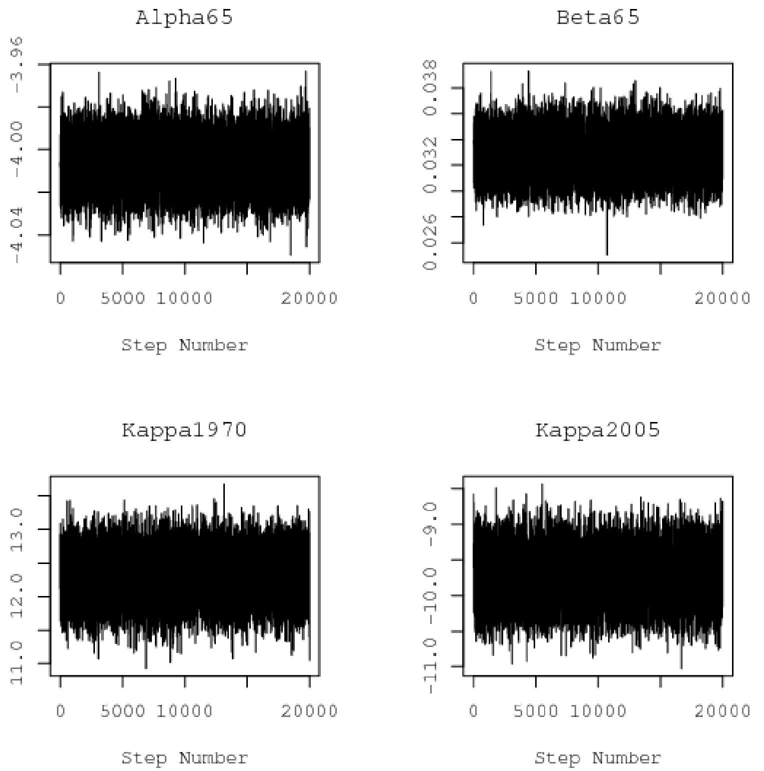
- **The parameters are**
  - α_x, β_x, σ_ε^2, λ, σ_ω^2
  - **red=Normal distribution**
  - **blue=INverse Gamma distribution**
- **We use the MCMC method to derive predictive distribution.**
- **Data sets**

  Japan's population Data for male and female

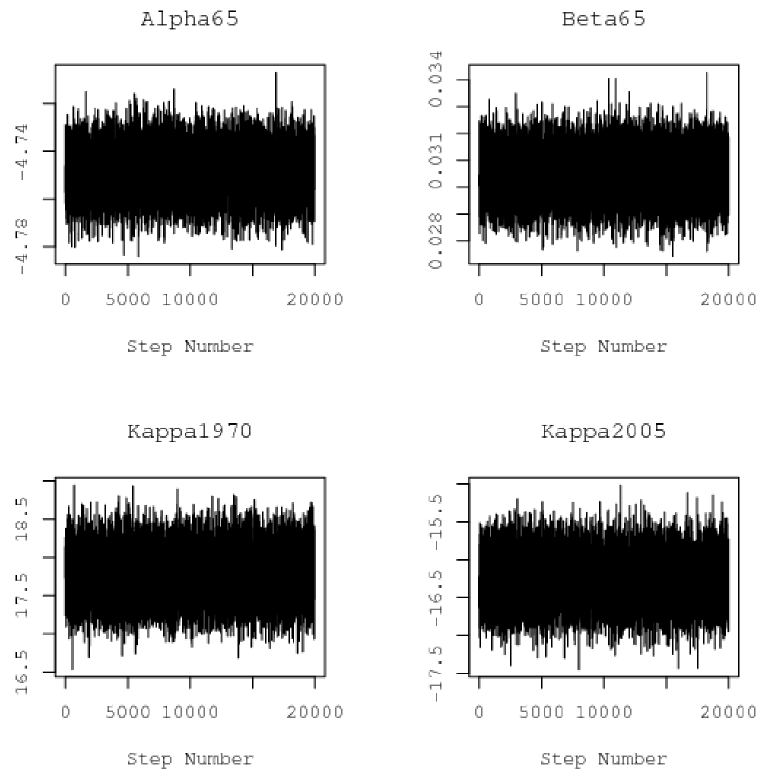
  ages : 65-98; years : 1970-2005

Evaluation of $E^*[I_t]$: MCMC sampling results

male



female



MCMC sampling results（basic statistics）

male

	Posterior mean	Posterior sd	95%HPD	Geweke
α_{65}	-4.0067	0.0105	(-4.0270, -3.9858)	0.97
β_{65}	0.0329	0.0015	(0.0299, 0.0358)	0.42
κ_{1970}	12.302	0.3299	(11.6891, 12.9763)	0.09
κ_{2005}	-9.7325	0.3253	(-10.3521, -9.0798)	0.58

female

	Posterior mean	Posterior sd	95%HPD	Geweke
α_{65}	-4.7484	0.0092	(-4.7669, -4.7310)	0.18
β_{65}	0.0305	0.0008	(0.0289, 0.0321)	0.14
κ_{1970}	17.7812	0.2966	(17.1937, 18.3509)	0.13
κ_{2005}	-16.2597	0.2938	(-16.8414, -15.6887)	0.42

Risk-neutralization of Predictive Distribution of ${}_tp_x$

- Generate N independent sample paths distributed as $({}_1p_x, {}_2p_x, \ldots, {}_Tp_x)$:

$$p_x^{(j)} = \left({}_1p_x^{(j)}, {}_2p_x^{(j)}, \ldots, {}_Tp_x^{(j)}\right), \quad j = 1, 2, \ldots, N$$

- Consider a simple annuity that pays \$1 for $1 \le t \le T$ and set the moment condition on the risk neutral distribution π^* over $\{p_x^{(j)}\}$ to be :

$$\sum_{j=1}^{N} \pi_j^* a_x^{(j)} = a_x^{\text{market}}$$

market value of the annuity (given)

expectation of $a_x^{(j)}$ under π^*

$$a_x^{(j)} \equiv \sum_{t=1}^{T} e^{-rt}\, {}_tp_x^{(j)}$$

$$= \text{present value of the annuity given } {}_tp_x^{(j)}$$

Bayesian Risk-neutral Density of $_tp_x$

Bayesian risk neutral predictive distribution

$$\pi_j^* = \frac{\exp\left\{\gamma a_x^{(j)}\right\}}{\sum_{j=1}^{N} \exp\left\{\gamma a_x^{(j)}\right\}},$$

$$\text{for } j = 1, 2, \ldots, N$$

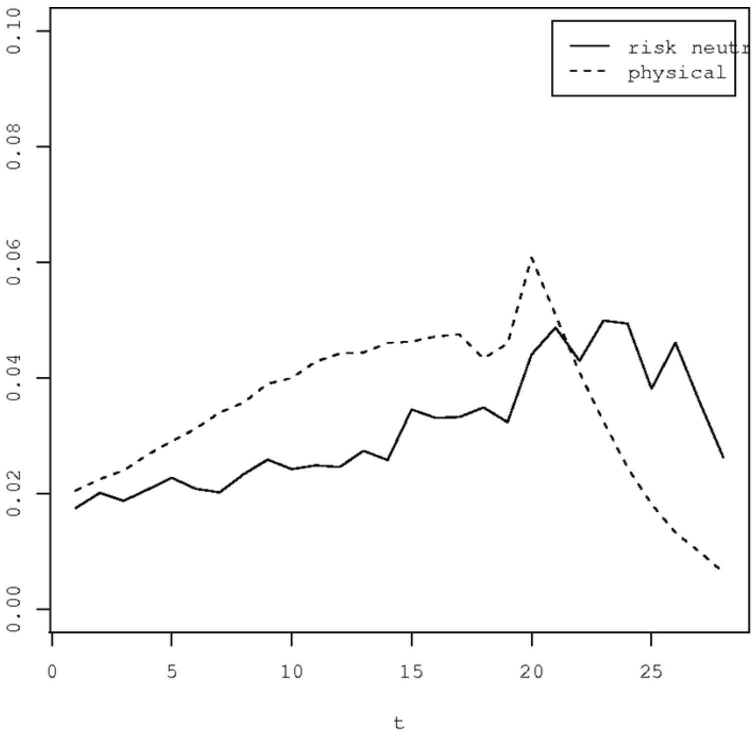
where γ is determined to satisfy:

$$a_x^{\text{market}} = \frac{\sum_{j=1}^{N} a_x^{(j)} \exp\left\{\gamma a_x^{(j)}\right\}}{\sum_{j=1}^{N} \exp\left\{\gamma a_x^{(j)}\right\}}$$

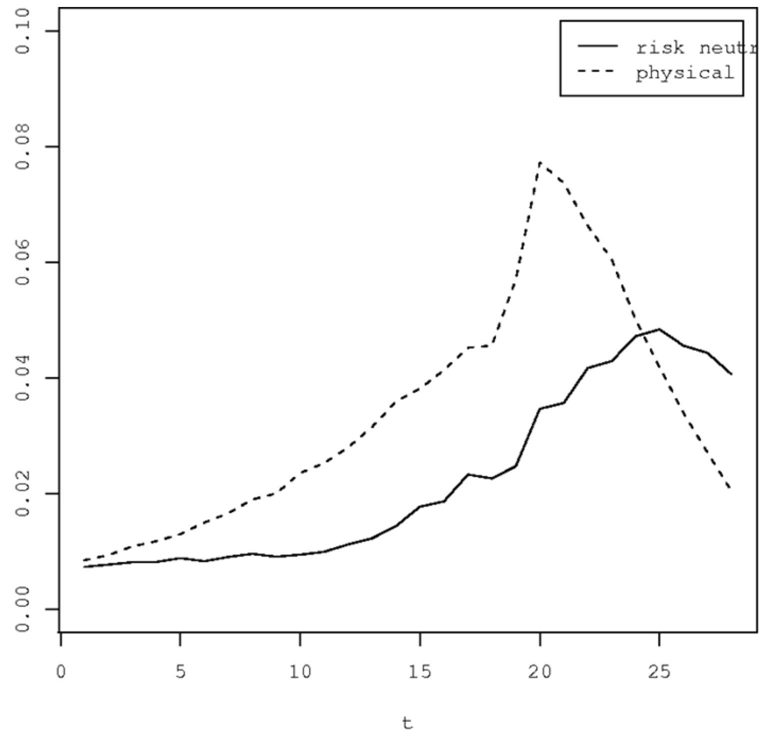
market value of the annuity (given)

$E^*[I_t]$: cohort of age 65

male



female



$\mathbf{E}^*[\boldsymbol{I_t}]$ (solid line) and $\mathbf{E}[\boldsymbol{I_t}]$ (dotted line)

Evaluation of of $E^*[\max(L_t-H_t, 0)]$

The PV of the reverese mortgage cash flow

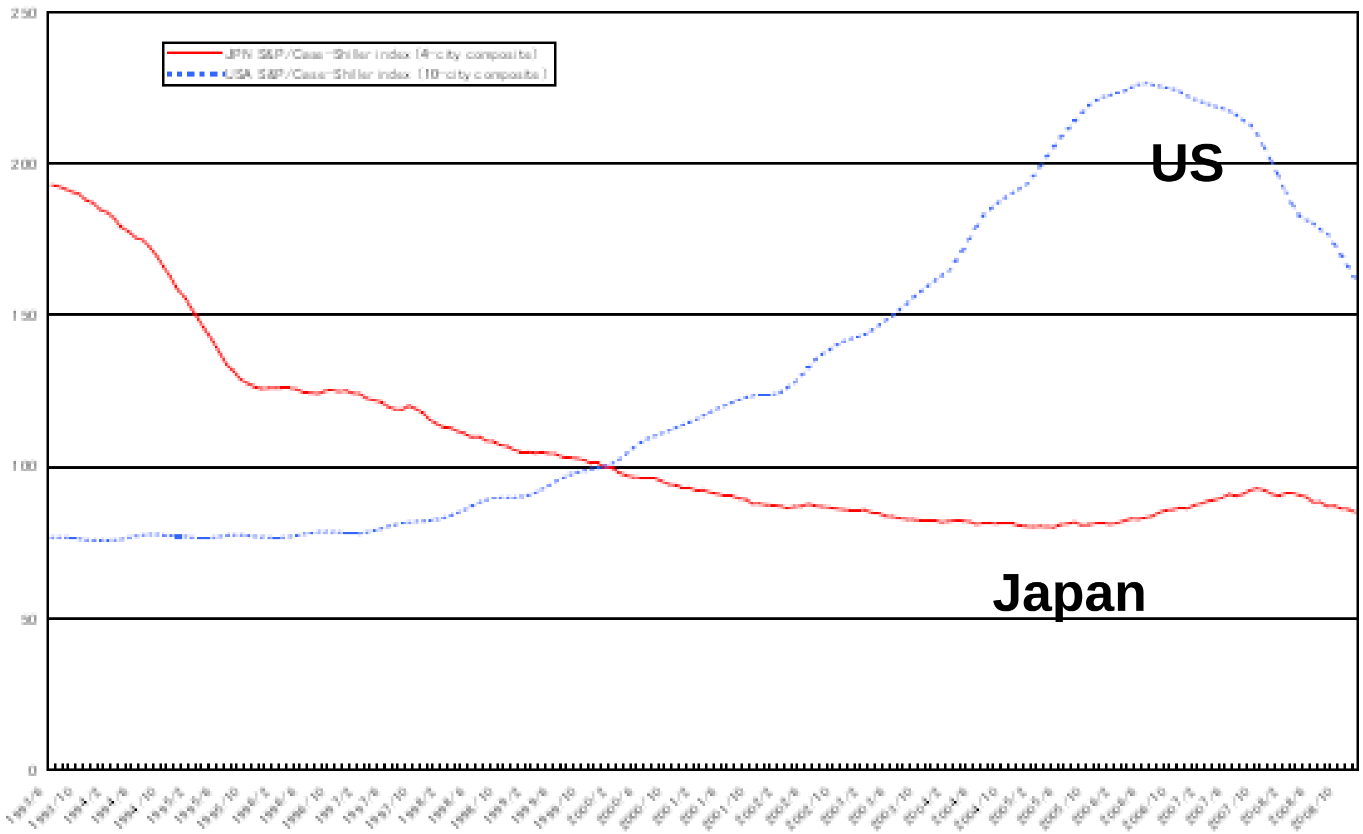
$$\sum_{t=0}^{T} e^{-rt} L_t \mathbf{E}^*[I_t] - \sum_{t=0}^{T} e^{-rt} \mathbf{E}^*[I_t] \mathbf{E}^*[\max(L_t - H_t, 0)]$$

$\Rightarrow$

Now we evaluate $\mathbf{E}^*[\max(L_t - H_t, 0)]$

House Price data: Case-Shiller Indices

S&P/Case-Shiller Home Price Indices for Japan as well as US (1993.06-2008.12)



Modeling house prices

Let h_t be the logarithm of the house price H_t and assume that h_t follows a local level model:

local level model

$$
\begin{aligned}
h_t &= \mu_t + v_t, \quad v_t \sim N(0, V) \\
\mu_t &= \mu_{t-1} + w_t, \quad w_t \sim N(0, W)
\end{aligned}
$$

where V and W are distributed as inverse Gamma distributions.

- $V \sim IG(\alpha_h, \beta_h)$
- $W \sim IG(\alpha_\mu, \beta_\mu)$

Here we set $\alpha_h = \beta_h = 0.001$, $\alpha_\mu = 0.1, \beta_\mu = 0.01$

Risk-neutralization of predictive distribution for Ht

- **Generate N independent sample paths distributed as $(h_1, h_2, \ldots, h_T)$.**

$$h^{(j)} = \left(h_1^{(j)}, h_2^{(j)}, \ldots, h_t^{(j)}\right), j = 1, 2, \ldots, N$$

- **For π^* on $\{h_t^{(j)}\}$ to be risk neutral we impose the momnet condition**

$$d(t) \sum_{j=1}^{N} \pi_j^* H_t^{(j)} = H_0, t = 1, 2, \ldots, T$$

with $H_t^{(j)} \equiv e^{h_t^{(j)}}$

the current house price

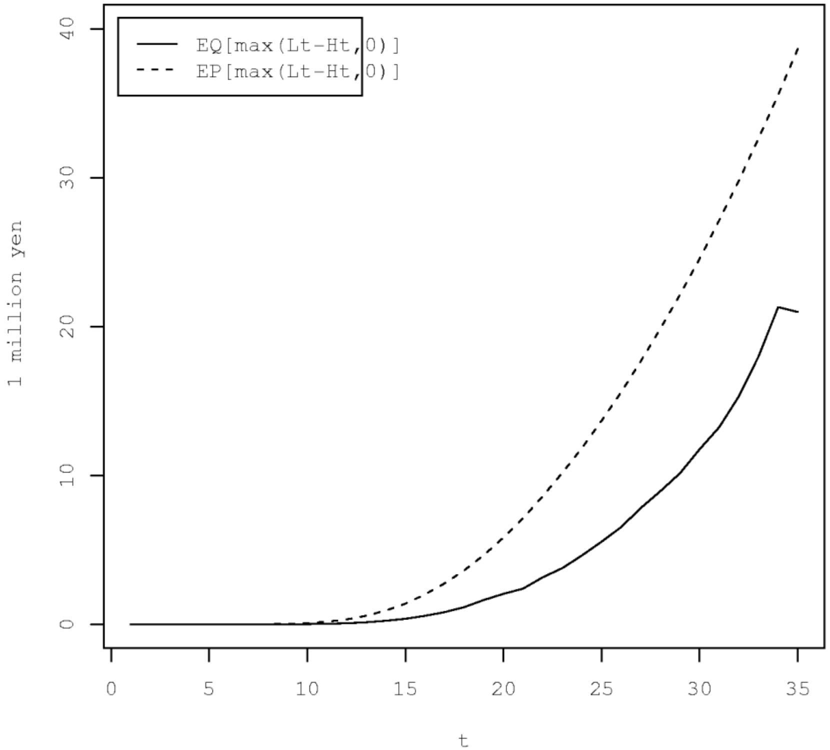
Evaluation of E*[max(Lt-Ht,0)]

We set

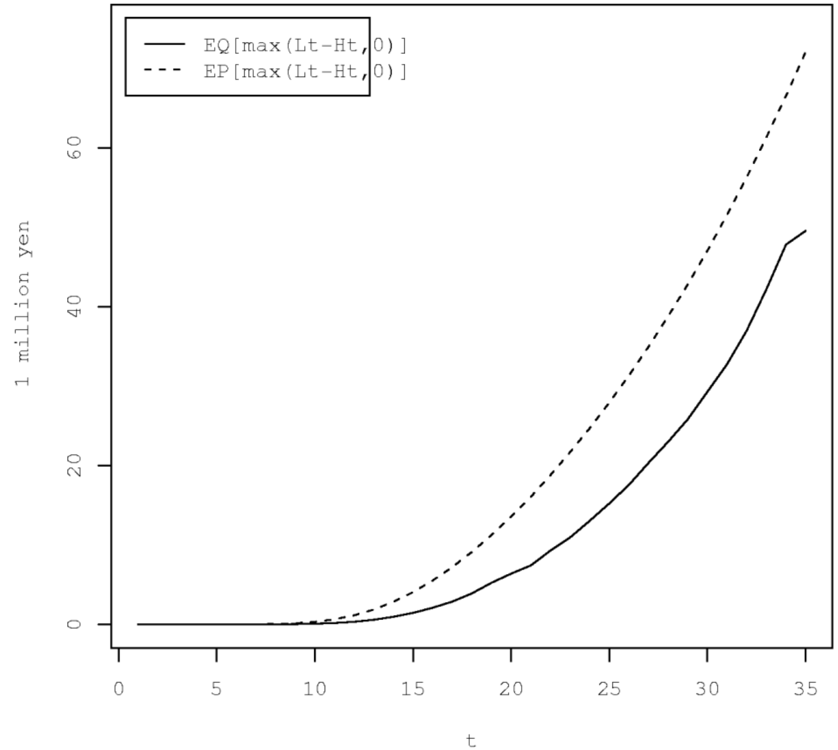
- **the curent house price :** $H_0 = 40$**(million yen)**
- **the loan at year 0 :** $L_0 = 20$**(million yen)**
- **the loan at year** t **:** $L_t = L_0 e^{ut}$
- **loan rates :** $u = 0.04, 0.05, 0.06$
- **discount rate :** $r = 0.015$

E*[max(Lt-Ht,0)]:

$L_0 = 20, u = 0.04$



$L_0 = 20, u = 0.05$



Solid line: risk neutral and dotted line: original

Evaluation of Reverse Mortgages in Japan

In addition to

- **the curent house price** : $H_0 = 40$**(million yen)**
- **the loan at year 0** : $L_0 = 20$**(million yen)**
- **the loan at year** t : $L_t = L_0 e^{ut}$
- **loan rates** : $u = 0.04, 0.05, 0.06$
- **discount rate** : $r = 0.015$

we set

- **age at year 0:**$x = 65, 70, 75, 80,$
- **maximum life years** : $T = 98 - x - 1$

Evaluation of Reverse Mortgages in Japan (u=4%)

PV of reverse mortgage:

$$\sum_{t=0}^{T} e^{-rt} L_t \mathbf{E}^*[I_t] - \sum_{t=0}^{T} e^{-rt} \mathbf{E}^*[I_t]\mathbf{E}^*[\max(L_t - H_t, 0)],$$

whose value should be no less than $L_0(=20)$

$H_0 = 40, L_0 = 20, \quad r = 0.015, u = 0.04$

x	male		female	
	Risk-neutral	Physical	Risk-neutral	Physical
65	27.60459	26.28193	20.18924	25.90555
70	25.34564	25.63689	19.12708	25.89425
75	24.26313	24.34514	20.07780	25.02251
80	21.93222	22.49217	19.11508	23.23801

Evaluation of Reverse Mortgages in Japan (u=5%)

PV of reverse mortgage:

$$H_0 = 40, L_0 = 20, r = 0.015, u = 0.05$$

	male		female	
x	Risk-neutral	Physical	Risk-neutral	Physical
65	30.56589	27.68048	22.32114	26.92601
70	28.13248	27.31018	21.3044	27.35481
75	26.88519	26.11497	22.40625	26.85644
80	23.85832	24.07022	21.01858	25.07722

PV of reverse mortgage should be no less than $L_0 = 20$

Evaluation of Reverse Mortgages in Japan (u=6%)

PV of reverse mortgage:

$$H_0 = 40, L_0 = 20, r = 0.015, u = 0.06$$

	male		female	
x	Risk-neutral	Physical	Risk-neutral	Physical
65	32.43857	28.50989	23.54640	27.4126
70	30.10885	28.43035	22.71018	28.15926
75	29.03342	27.47149	24.18518	28.08092
80	25.70015	25.48779	22.78073	26.61531

PV of reverse mortgage should be no less than $L_0 = 20$.

Two-factor Lee-Carter model

Two-factor Lee-Carter model

- **Observation equation**

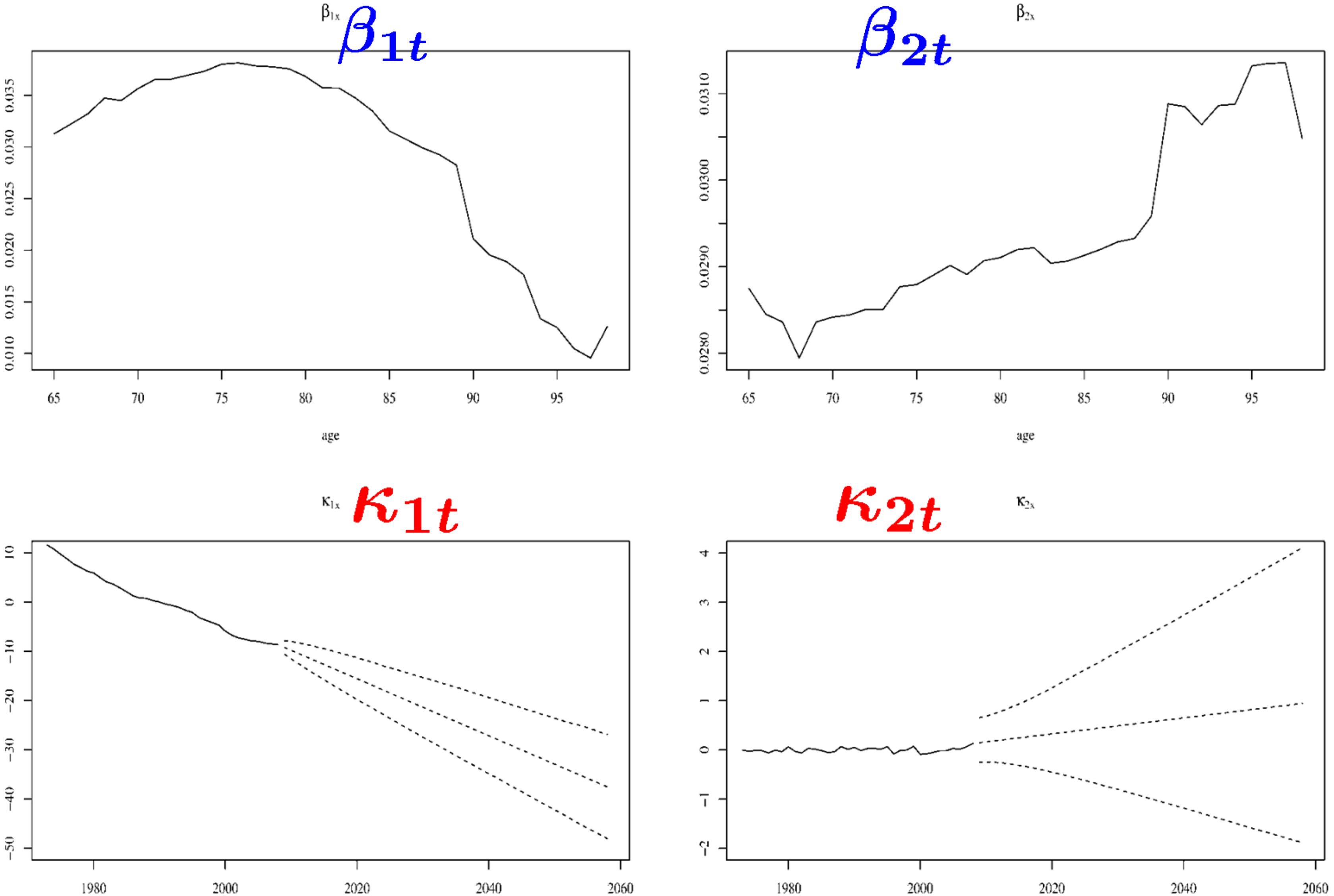
$$\ln m_{xt} = \alpha_x + \beta_{1x}\kappa_{1t} + \beta_{2x}\kappa_{2t} + \epsilon_{xt},$$
$$\varepsilon_{xt} \overset{\text{iid}}{\sim} N(0, \sigma_\varepsilon^2)$$

- **State equation**

$$\begin{cases} \kappa_{1t} = \lambda_1 + \kappa_{1,t-1} + \omega_{1t}, & \omega_{1t} \sim N(0, \sigma_{\omega_1}^2) \\ \kappa_{2t} = \lambda_2 + \phi\kappa_{2,t-1} + \omega_{2t}, & \omega_{2t} \sim N(0, \sigma_{\omega_2}^2) \end{cases}$$

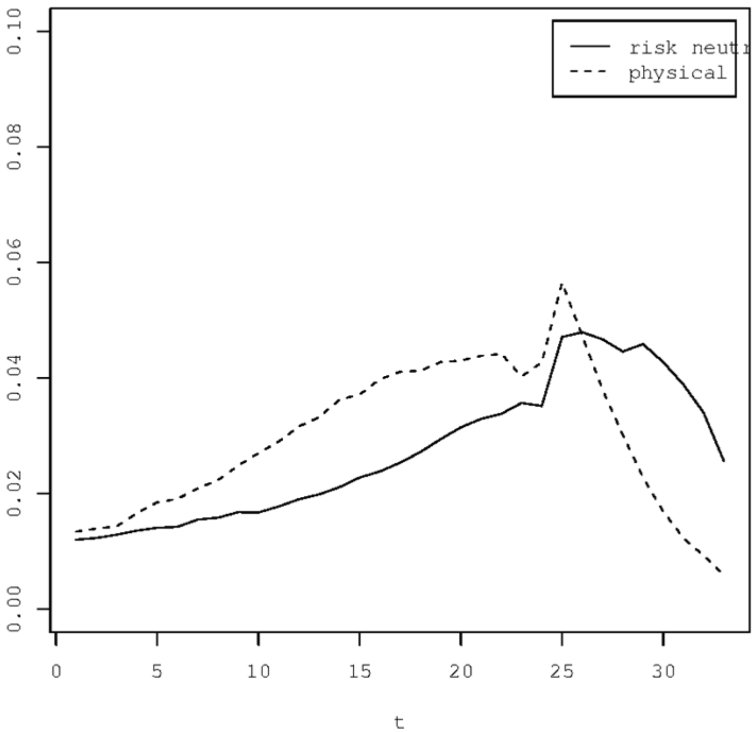
cyclical changes （$|\Phi|<1$)

Estimated Parameters of Two-factor Lee-Carter Model (male)

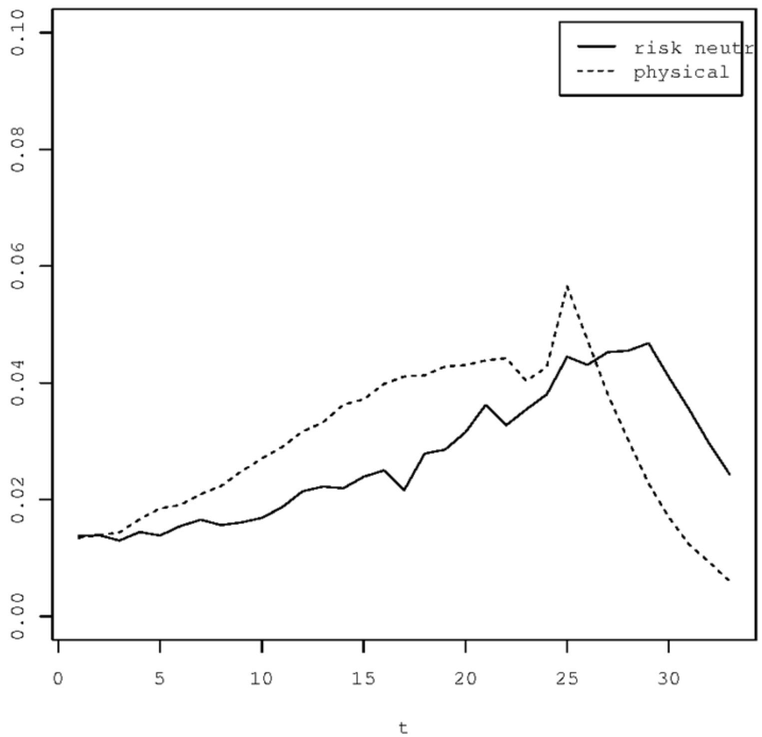


Evaluation of $E^*[I_t]$: age 65 male

one-factor model



two-factor model

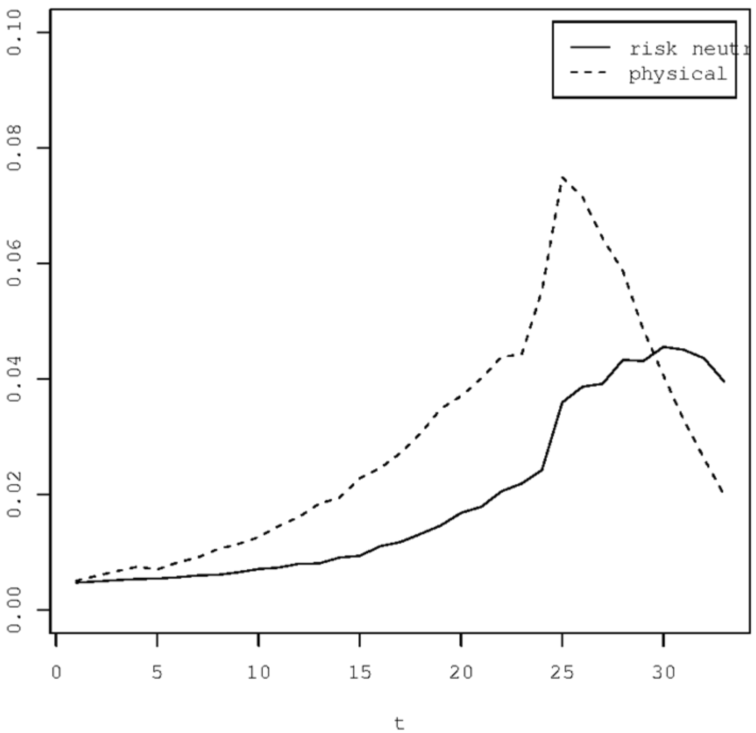


$\mathbf{E}^*[\boldsymbol{I_t}]$ (solid line) and $\mathbf{E}[\boldsymbol{I_t}]$ (dotted line)

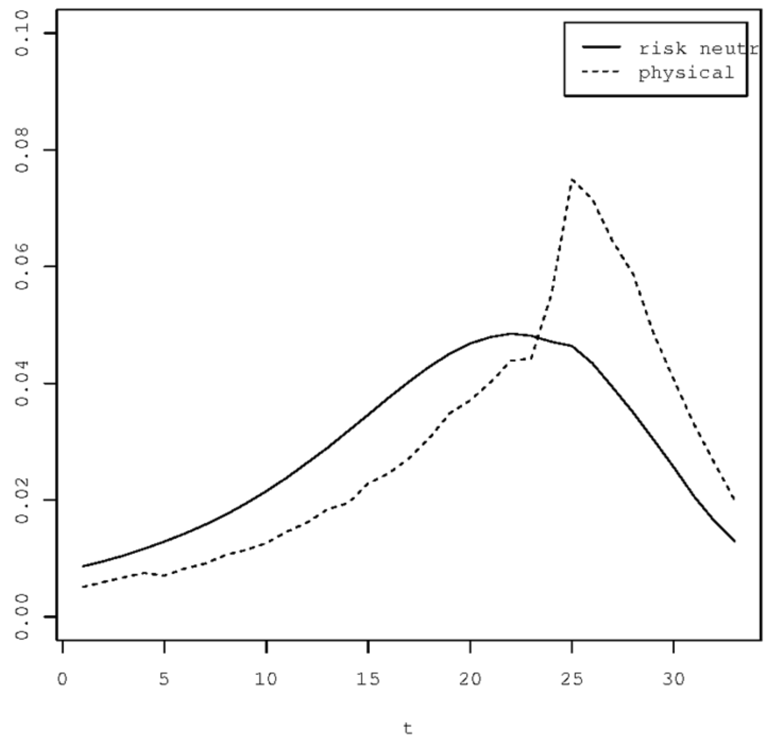
Little difference between the models

Evaluation of $E^*[I_t]$: age 65 female

one-factor model



two-factor model



$\mathbf{E}^*[I_t]$ (solid line) and $\mathbf{E}[I_t]$ (dotted line)

Some difference between the models !

One-factor vs. Two-factor (u=4%)

PV of reverse mortgage:

$$\sum_{t=0}^{T} e^{-rt} L_t \mathbf{E}^*[I_t] - \sum_{t=0}^{T} e^{-rt} \mathbf{E}^*[I_t] \mathbf{E}^*[\max(L_t - H_t, 0)],$$

whose value should be no less than $L_0(=20)$

$H_0 = 40, L_0 = 20, \quad r = 0.015, u = 0.04$

	male		female	
x	one-factor	two-factor	one-factor	two-factor
65	27.60459	27.37908	20.18924	29.28988
70	25.34564	26.21647	19.12708	27.30318
75	24.26313	24.16739	20.07780	25.09417
80	21.93222	21.84386	19.11508	22.75158

no change

large increase

One-factor vs. Two-factor (u=5%)

PV of reverse mortgage

$H_0 = 40, L_0 = 20, r = 0.015, u = 0.05$

x	male One-factor	male Two-factor	female One-factor	female Two-factor
65	30.56589	30.30559	22.32114	32.58761
70	28.13248	29.08291	21.30440	30.36296
75	26.88519	26.61771	22.40625	27.66275
80	23.85832	23.63987	21.01858	24.65523

no change

large increase

One-factor vs. Two-factor (u=6%)

PV of reverse mortgage:

$$H_0 = 40, L_0 = 20, r = 0.015, u = 0.06$$

	male		female	
x	One-factor	Two-factor	One-factor	Two-factor
65	32.43857	32.17907	23.54640	34.80077
70	30.10885	31.19492	22.71018	28.15926
75	29.03342	28.70905	24.18518	29.90626
80	25.70015	25.37896	22.78073	26.50773

no change

large increase

Conclusions

- We proposed a multivariate Bayesian risk neutral method to evaluate reverse mortgages involving mortality risk and house price risk.
- We used one-factor and two-factor Lee-Carter models for mortality risk and a local level model for house price risk to derive present values of reverse mortgage plans.
- The results indicate
  - siginificant possibilities for a successuful introduction of reverse mortgages in Japan.
  - the use of two-factor Lee-Carter model increases the possibilities for female cohorts.

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