Mortality Modelling with Lévy Processes: A Cox Process with Leptokurtic Intensity

Hong-Chih Huang

Chou-Wen Wang*

I-Chien Liu

2011/09/09

Abstract

- When the number of deaths follows a Cox process, this paper provides an iterative fitting algorithm to generate maximum likelihood estimates under the Cox regression model and employs the non-Gaussian distributions to model the residuals of the Renshaw and Haberman (2006) model.
- With mortality data of Finland, France, Italy and the Netherlands over the period 1900–2007, both in-sample model selection criteria and out-of-sample projection errors indicate a preference for modeling the Renshaw and Haberman (2006) model with **non-Gaussian innovations.**

Agenda

- Introduction
- The Stochastic Mortality Models with Cox Error Structures
  - Renshaw and Haberman (2006) Model
  - Normality Test for the RH Model
  - The Heavy-Tailed Distributions
  - A Cox Process with Leptokurtic Intensity
- Empirical Analysis
  - Model Comparison
  - In-Sample Goodness of Fit
  - Mortality Projection
- Conclusions and Suggestions

Introduction

- An improvement to the Lee-Carter model is to model the number of deaths as a Poisson model commonly employed in the literature on mortality modeling

(see, for example, Wilmoth, 1993; Brouhns et al., 2002; Renshaw and Haberman, 2006; Cairns et al., 2009; Haberman and Renshaw, 2009)

Introduction

- Instead of using Poisson model with **deterministic** intensity function, an alternative means of fitting the number of deaths is to specify a *doubly stochastic Poisson* process, also known as *Cox* process (Cox, 1955), to catch the stochastic intensity property.

- This paper provides an iterative fitting algorithm for estimating the Cox regression model under which mortality rates adhere to the RH model with non-Gaussian innovations.

The Stochastic Mortality Models with Cox Error Structures

- Renshaw and Haberman (2006) Model
- Normality Test for the RH Model
- The Heavy-Tailed Distributions
- A Cox Process with Leptokurtic Intensity

Renshaw and Haberman (2006) Model

$$ln\left(m_{x,t}\right) = \alpha_x + \beta_x k_t + \eta_x \gamma_{t-x} + e_{x,t}$$

$$k_t - k_{t-1} = \mu + \varepsilon_t \quad \Leftrightarrow \quad \text{ARIMA(0,1,0)}$$

$$\Delta\gamma_{t-x} = \mu_\gamma + \alpha_\gamma\left(\Delta\gamma_{t-x-1} - \mu_\gamma\right) + \sigma_\gamma z_{t-x} \quad \Leftrightarrow \quad \text{ARIMA(1,1,0)}$$

Normality Test for the RH Model

- Data Source: HMD
- Country:

Finland, France, Italy and the Netherlands

- Age: 60 to 89
- Period: 1900–1999

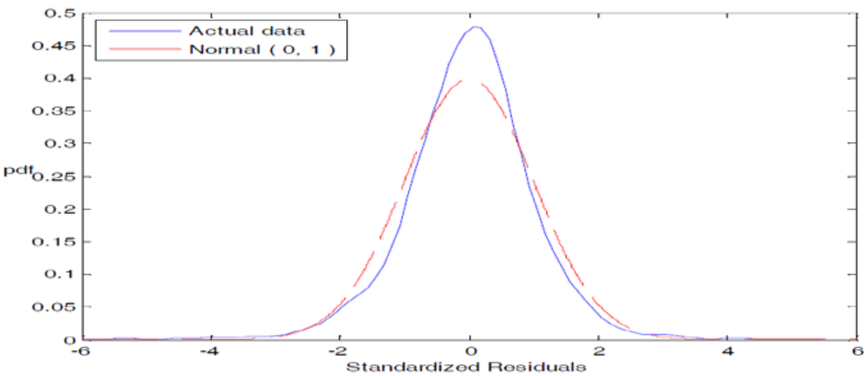
Normality Test

$$ln\left(m_{x,t}\right)=\alpha_x+\beta_x k_t+\eta_x \gamma_{t-x}+e_{x,t}$$

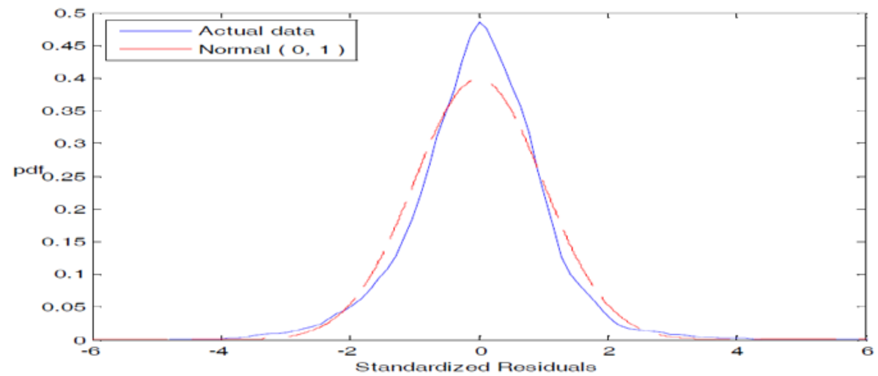
$$k_t-k_{t-1}=\mu+\varepsilon_t$$

$$\Delta\gamma_{t-x}=\mu_\gamma+\alpha_\gamma\left(\Delta\gamma_{t-x-1}-\mu_\gamma\right)+\sigma_\gamma z_{t-x}$$

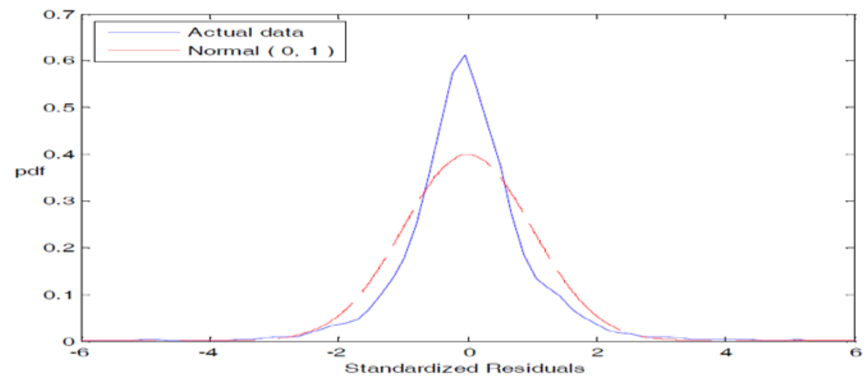
Normality Test for the RH Model



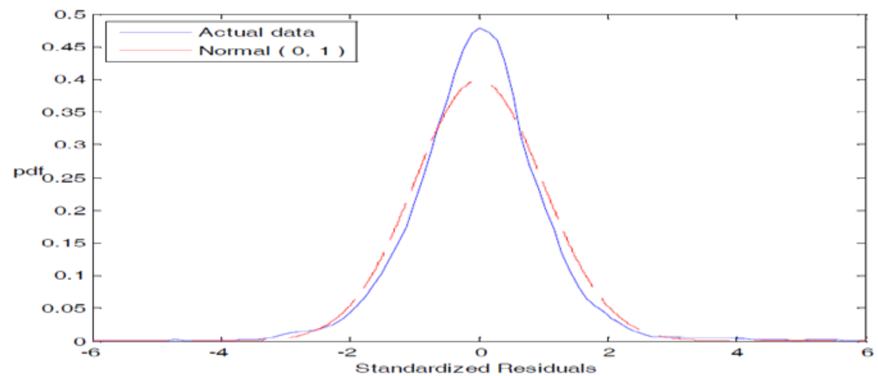
Finland



France



Italy



The Netherlands

Normality Test for the RH Model - JB test

Panel A: the Residuals of the RH Model

Country	Finland	France	Italy	Netherlands
Skewness	-0.408	0.087	0.236	0.243
	(0.045)	(0.045)	(0.045)	(0.045)
Excess Kurtosis	3.407	2.623	9.531	3.015
	(0.089)	(0.089)	(0.089)	(0.089)
JB Test	1533.667	863.566	11383.204	1165.756
	[< 0.001]	[< 0.001]	[< 0.001]	[< 0.001]

The table presents the skewness and excess kurtosis of the residuals of the RH model. Standard errors of the skewness and excess kurtosis given in the parentheses are calculated as $\sqrt{6/n}$ and $\sqrt{24/n}$, respectively. n denotes the number of observations. The p-values of Jarque-Bera (JB) test are given in bracket.

Normality Test for the RH Model - JB test

Panel B: the First Difference in Mortality Indices

Country	Finland	France	Italy	Netherlands
Skewness	-0.039	0.246	-0.183	-1.994
	(0.246)	(0.246)	(0.246)	(0.246)
Excess Kurtosis	0.308	0.747	1.174	12.991
	(0.492)	(0.492)	(0.492)	(0.492)
JB Test	0.417	3.301	6.241	761.788
	[0.500]	[0.121]	[0.039]	[< 0.001]

The table presents the skewness and excess kurtosis of the first difference in mortality indices. Standard errors of the skewness and excess kurtosis given in the parentheses are calculated as $\sqrt{6/n}$ and $\sqrt{24/n}$, respectively. n denotes the number of observations. The p-values of Jarque-Bera (JB) test are given in bracket.

Normality Test for the RH Model - JB test

Panel C: the Residuals of Cohort Effects

Country	Finland	France	Italy	Netherlands
Skewness	-0.301	-0.264	2.287	-0.605
	(0.217)	(0.217)	(0.217)	(0.217)
Excess Kurtosis	2.322	5.905	11.997	3.915
	(0.435)	(0.435)	(0.435)	(0.435)
JB Test	30.438	186.004	872.381	88.850
	[< 0.001]	[< 0.001]	[< 0.001]	[< 0.001]

The table presents the skewness and excess kurtosis of the residuals of cohort effects. Standard errors of the skewness and excess kurtosis given in the parentheses are calculated as $\sqrt{6/n}$ and $\sqrt{24/n}$, respectively. n denotes the number of observations. The p-values of Jarque-Bera (JB) test are given in bracket.

The Heavy-Tailed Distributions

- JD

$$f_{e_{x,t}}^{JD}\left(y\middle|\sigma,\lambda_N,\mu_Y,\delta_Y\right)=\sum_{n=0}^{\infty}\frac{\lambda_N^n e^{-\lambda_N}}{n!}\Phi\left(y\middle|(n-\lambda_N)\mu_Y,\sigma^2+n\delta_Y^2\right)$$

- NIG

$$f_{e_{x,t}}^{NIG}\left(y\middle|\alpha,\beta,\delta,\theta\right)=\frac{\alpha\delta}{\pi}\exp\left(\delta\sqrt{\alpha^2-\beta^2}+\beta\left(y-\theta\right)\right)\frac{K_1\left(\alpha\sqrt{\delta^2+\left(y-\theta\right)^2}\right)}{\sqrt{\delta^2+\left(y-\theta\right)^2}}$$

- VG

$$f_{e_{x,t}}^{VG}\left(y\middle|\alpha,\beta,\gamma,\theta\right)=\frac{\left(\alpha^2-\beta^2\right)^{\gamma}\left|y-\theta\right|^{\gamma-0.5}K_{\gamma-0.5}\left(\alpha\left|y-\theta\right|\right)}{\sqrt{\pi}\left(2\alpha\right)^{\gamma-0.5}\Gamma\left(\gamma\right)}\exp\left(\beta\left(y-\theta\right)\right)$$

A Cox Process with Leptokurtic Intensity

$$D_{x,t} \sim Cox\left(\lambda_{x,t}\right)$$

$$\lambda_{x,t} = E_{x,t} m_{x,t} = E_{x,t} \exp\left(\alpha_x + \beta_x k_t + \eta_x \gamma_{t-x} + e_{x,t}\right)$$

$$LLF = \sum_{x,t} \int_{-\infty}^{\infty} \log f\left(D_{x,t} = d_{x,t} \mid e_{x,t} = y\right) f_{e_{x,t}}\left(y\right) dy$$

where

$$\log f\left(D_{x,t} = d_{x,t} \mid e_{x,t} = y\right)$$

$$= d_{x,t} \log\left(E_{x,t} \exp\left(\alpha_x + \beta_x k_t + \eta_x \gamma_{t-x} + y\right)\right) - E_{x,t} \exp\left(\alpha_x + \beta_x k_t + \eta_x \gamma_{t-x} + y\right) - \log\left(d_{x,t}!\right)$$

A Cox Process with Leptokurtic Intensity

Theorem 1: When the death rates follows the RH model, the closed-form solution of the log-likelihood function in Equation (15) is derived as follows:

$$LLF = \sum_{x.t}\left[d_{x,t}\left(\alpha_x + \beta_x k_t + \eta_x \gamma_{t\text{-}x}\right) \text{-} \left(E_{x,t}\exp\left(\alpha_x + \beta_x k_t + \eta_x \gamma_{t\text{-}x}\right)\right) M_{e_{x,t}}\right] + C,$$

where $M_{e_{x,t}}$ *is the moment generating function of* $e_{x,t}$; *C represents a constant term equal to* $\sum_{x.t}\left[d_{x,t}\log E_{x,t} - \log\left(d_{x,t}!\right)\right]$.

Iterating Procedure

$$LLF_{e_{x,t}} = \sum_{x,t} \log\left(f\left(e_{xt} \middle| \alpha_x, \eta_x, \gamma_{t\text{-}x}, \beta_x, k_t \right)\right).$$

$$update(\theta) = u(\theta) = \theta - \frac{\partial LLF / \partial \theta}{\partial^2 LLF / \partial \theta^2}.$$

$$u\left(\alpha_x\right) = \alpha_x + \frac{\sum_t \left[d_{x,t} - E_{x,t} \exp\left(\alpha_x + \beta_x k_t + \eta_x \gamma_{t-x}\right) M_{e_{x,t}} \right]}{\sum_t \left[E_{x,t} \exp\left(\alpha_x + \beta_x k_t + \eta_x \gamma_{t-x}\right) M_{e_{x,t}} \right]}$$

$$u\left(\gamma_z\right) = \gamma_z + \frac{\sum_{\substack{x,t \\ z=t-x}} \left[d_{x,t}\eta_x - E_{x,t}\eta_x \exp\left(\alpha_x + \beta_x k_t + \eta_x \gamma_z\right) M_{e_{x,t}} \right]}{\sum_{\substack{x,t \\ z=t-x}} \left[E_{x,t} \left(\eta_x\right)^2 \exp\left(\alpha_x + \beta_x k_t + \eta_x \gamma_z\right) M_{e_{x,t}} \right]}$$

Iterating Procedure

$$u(\eta_x) = \eta_x + \frac{\sum_t \left[d_{x,t}\gamma_{t-x} - E_{x,t}\gamma_{t-x} \exp(\alpha_x + \beta_x k_t + \eta_x \gamma_{t-x}) M_{e_{x,t}} \right]}{\sum_t \left[E_{x,t}\gamma_{t-x}^2 \exp(\alpha_x + \beta_x k_t + \eta_x \gamma_{t-x}) M_{e_{x,t}} \right]}$$

$$u(k_t) = k_t + \frac{\sum_x \left[d_{x,t}\beta_x - E_{x,t}\beta_x \exp(\alpha_x + \beta_x k_t + \eta_x \gamma_{t-x}) M_{e_{x,t}} \right]}{\sum_x \left[E_{x,t}\beta_x^2 \exp(\alpha_x + \beta_x k_t + \eta_x \gamma_{t-x}) M_{e_{x,t}} \right]}$$

$$u(\beta_x) = \beta_x + \frac{\sum_t \left[d_{x,t}k_t - E_{x,t}k_t \exp(\alpha_x + \beta_x k_t + \eta_x \gamma_{t-x}) M_{e_{x,t}} \right]}{\sum_t \left[E_{x,t}k_t^2 \exp(\alpha_x + \beta_x k_t + \eta_x \gamma_{t-x}) M_{e_{x,t}} \right]}$$

A Cox Process with Leptokurtic Intensity

- Constraints:

$$\sum_t k_t = 0, \ \sum_x \beta_x = 1, \ \sum_x \eta_x = 1 \ \text{ and } \ \sum_{x,t} \gamma_{t-x} = 0 .$$

- After obtaining the mortality indices and cohort effects, we obtain their corresponding parameters by maximizing the log-likelihood function

$$\sum_t \log\left(f\left(\varepsilon_t\right)\right) \ \text{ and } \ \sum_{s=t-x} \log\left(f\left(z_s\right)\right)$$

Empirical Analysis

- Model Comparison
- In-Sample Goodness of Fit
- Mortality Projection

Model Comparison

- Model Criteria:
  - LLF: Log Likelihood Function
  - AIC: Akaike Information Criterion
  - BIC: Bayesian Information Criterion
- Goodness-of-fit Tests:
  - KS: Kolmogorov-Smirnov test
  - AD: Anderson-Darling test
  - CvM: Cramér-von-Mises test

In-Sample Goodness of Fit

- Goodness-of-fit Measures for the Number of Deaths
- Goodness-of-fit Tests for the Residuals of the RH Model
- Goodness-of-fit Measures for the First Difference in Mortality Indices
- Goodness-of-fit Measures for the Residuals of Cohort Effects

Goodness-of-fit Measures for the Number of Deaths

Panel A: the Finland Mortality Data

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-17533.8	17849.8	18798.8	4	4	3
JD	-17524.6	17843.6	18801.7	3	3	4
VG	-17442.8	17760.8	18715.8	1	1	1
NIG	-17491.4	17809.4	18764.4	2	2	2

Panel B: the France Mortality Data

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-31475.1	31791.1	32740.1	4	4	4
JD	-31426.1	31745.1	32703.1	2	2	2
VG	-31355.9	31673.9	32628.9	1	1	1
NIG	-31450.6	31768.6	32723.6	3	3	3

Goodness-of-fit Measures for the Number of Deaths

Panel C: the Italy Mortality Data

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-44206.6	44522.6	45471.6	4	4	4
JD	-43202.4	43521.4	44479.4	2	2	2
VG	-42629.8	42947.8	43902.8	1	1	1
NIG	-43974.4	44292.4	45247.4	3	3	3

Panel D: the Netherlands Mortality Data

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-18427.2	18743.2	19692.2	4	4	3
JD	-18419.8	18738.8	19696.8	3	3	4
VG	-18353.0	18671.0	19626.0	1	1	1
NIG	-18403.3	18721.3	19676.3	2	2	2

Goodness-of-fit Tests for the Residuals of the RH Model

Panel A: the Finland Mortality Data

Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.044**	0.025	0.030	15.095**	2.524	3.860	2.405**	0.469	0.731
JD	0.011	0.025	0.030	0.451	2.475	3.804	0.073	0.465	0.738
VG	0.014	0.025	0.030	0.886	2.475	3.804	0.091	0.465	0.738
NIG	0.009	0.025	0.030	0.274	2.475	3.804	0.031	0.465	0.738

Panel B: the France Mortality Data

Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.045**	0.025	0.030	13.377**	2.524	3.860	2.152**	0.469	0.731
JD	0.015	0.025	0.030	0.406	2.475	3.804	0.066	0.465	0.738
VG	0.014	0.025	0.030	0.585	2.475	3.804	0.077	0.465	0.738
NIG	0.014	0.025	0.030	0.381	2.475	3.804	0.060	0.465	0.738

Note: * and ** denote significance at the 5% and 1% level, respectively.

Goodness-of-fit Tests for the Residuals of the RH Model

Panel C: the Italy Mortality Data

Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.088**	0.025	0.030	55.510**	2.524	3.860	9.935**	0.469	0.731
JD	0.034**	0.025	0.030	4.493**	2.475	3.804	0.793**	0.465	0.738
VG	0.022	0.025	0.030	3.057*	2.479	3.803	0.374	0.464	0.737
NIG	0.013	0.025	0.030	0.391	2.480	3.825	0.051	0.464	0.737

Panel D: the Netherlands Mortality Data

Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.048**	0.025	0.030	16.518**	2.524	3.860	2.618**	0.469	0.731
JD	0.015	0.025	0.030	0.623	2.475	3.804	0.102	0.465	0.738
VG	0.015	0.025	0.030	0.981	2.475	3.804	0.107	0.465	0.738
NIG	0.014	0.025	0.030	0.487	2.475	3.804	0.068	0.465	0.738

Note: * and ** denote significance at the 5% and 1% level, respectively.

Goodness-of-fit Measures for the First Difference in Mortality Indices

Panel A: the Finland Mortality Index

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-182.46	184.46	187.05	4	1	1
JD	-181.97	186.97	193.46	1	4	4
VG	-182.10	186.10	191.29	2	2	2
NIG	-182.18	186.18	191.37	3	3	3

Panel B: the France Mortality Index

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-193.05	195.05	197.64	4	1	1
JD	-190.96	195.96	202.45	1	4	4
VG	-191.41	195.41	200.60	2	2	2
NIG	-191.66	195.66	200.85	3	3	3

Goodness-of-fit Measures for the First Difference in Mortality Indices

Panel C: the Italy Mortality Index

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-201.24	203.24	205.84	4	3	1
JD	-198.87	203.87	210.35	2	4	4
VG	-198.78	202.78	207.97	1	1	2
NIG	-198.93	202.93	208.12	3	2	3

Panel D: the Netherlands Mortality Index

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-211.01	213.01	215.61	4	4	4
JD	-187.16	192.16	198.65	3	3	3
VG	-186.02	190.02	195.21	1	1	1
NIG	-186.10	190.10	195.29	2	2	2

Goodness-of-fit Measures for the Residuals of Cohort Effects

Panel A: the Finland Cohort Effect

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-184.24	187.24	191.50	4	4	4
JD	-168.16	174.16	182.70	1	2	3
VG	-168.47	173.47	180.58	2	1	1
NIG	-169.67	174.67	181.78	3	3	2

Panel B: the France Cohort Effect

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-123.09	126.09	130.35	4	4	4
JD	-101.90	107.90	116.43	1	1	2
VG	-115.61	120.61	127.72	3	3	3
NIG	-103.40	108.40	115.51	2	2	1

Goodness-of-fit Measures for the Residuals of Cohort Effects

Panel C: the Italy Cohort Effect

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-146.70	149.70	153.97	4	4	4
JD	-114.46	120.46	128.99	1	1	1
VG	-126.47	131.47	138.58	3	3	3
NIG	-123.55	128.55	135.66	2	2	2

Panel D: the Netherlands Cohort Effect

Model	LLF	AIC	BIC	LLF Rank	AIC Rank	BIC Rank
Normal	-79.39	82.39	86.66	4	4	4
JD	-66.60	72.60	81.13	2	3	3
VG	-64.22	69.22	76.33	1	1	1
NIG	-67.40	72.40	79.51	3	2	2

Goodness-of-fit Tests for the First Difference in Mortality Indices

Panel A: the Finland Mortality Index

Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.052	0.130	0.158	0.229	2.477	3.933	0.039	0.461	0.737
JD	0.040	0.131	0.157	0.114	2.491	3.859	0.017	0.465	0.735
VG	0.040	0.130	0.159	0.131	2.540	3.929	0.021	0.465	0.780
NIG	0.045	0.131	0.156	0.152	2.497	3.893	0.025	0.457	0.733

Panel B: the France Mortality Index

Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.070	0.130	0.158	0.503	2.477	3.933	0.090	0.461	0.737
JD	0.052	0.131	0.157	0.198	2.491	3.859	0.031	0.465	0.735
VG	0.046	0.130	0.159	0.221	2.540	3.929	0.034	0.464	0.780
NIG	0.053	0.131	0.156	0.250	2.496	3.892	0.040	0.457	0.733

Note: * and ** denote significance at the 5% and 1% level, respectively.

Goodness-of-fit Tests for the First Difference in Mortality Indices

Panel C: the Italy Mortality Index

Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.072	0.130	0.158	0.475	2.477	3.933	0.076	0.461	0.737
JD	0.057	0.131	0.157	0.166	2.491	3.858	0.026	0.465	0.735
VG	0.042	0.130	0.159	0.130	2.540	3.929	0.017	0.465	0.780
NIG	0.049	0.131	0.156	0.142	2.496	3.892	0.021	0.457	0.733

Panel D: the Netherlands Mortality Index

Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.128*	0.130	0.158	3.702*	2.477	3.933	0.640*	0.461	0.737
JD	0.072	0.131	0.157	0.388	2.491	3.855	0.069	0.464	0.734
VG	0.063	0.130	0.159	0.406	2.540	3.923	0.045	0.464	0.779
NIG	0.060	0.131	0.156	0.230	2.499	3.901	0.028	0.456	0.733

Note: * and ** denote significance at the 5% and 1% level, respectively.

Goodness-of-fit Tests for the Residuals in Cohort Effects

Panel A: the Finland Cohort Effect

Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.108	0.115	0.137	2.541*	2.453	3.816	0.470*	0.450	0.712
JD	0.032	0.115	0.137	0.106	2.495	3.786	0.015	0.460	0.724
VG	0.055	0.116	0.143	0.166	2.518	3.936	0.030	0.463	0.752
NIG	0.034	0.116	0.139	0.171	2.522	3.836	0.023	0.463	0.729

Panel B: the France Cohort Effect

Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.120*	0.115	0.137	2.562*	2.453	3.816	0.391	0.450	0.712
JD	0.069	0.115	0.137	0.453	2.495	3.786	0.077	0.459	0.724
VG	0.087	0.116	0.143	1.513	2.520	3.936	0.278	0.463	0.753
NIG	0.066	0.116	0.139	0.553	2.521	3.827	0.090	0.463	0.729

Note: * and ** denote significance at the 5% and 1% level, respectively.

Goodness-of-fit Tests for the Residuals in Cohort Effects

Panel C: the Italy Cohort Effect

Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.129*	0.115	0.137	3.147*	2.453	3.816	0.419	0.450	0.712
JD	0.054	0.115	0.137	0.345	2.495	3.786	0.052	0.459	0.724
VG	0.068	0.116	0.143	0.744	2.520	3.936	0.078	0.464	0.753
NIG	0.061	0.116	0.139	0.597	2.521	3.827	0.066	0.463	0.730

Panel D: the Netherlands Cohort Effect

Model	KS			AD			CvM		
	Statistic	Critical Value		Statistic	Critical Value		Statistic	Critical Value	
		5%	1%		5%	1%		5%	1%
Normal	0.094	0.115	0.137	1.890	2.453	3.816	0.329	0.450	0.712
JD	0.044	0.115	0.137	0.258	2.495	3.786	0.035	0.459	0.724
VG	0.073	0.115	0.143	0.817	2.516	3.934	0.150	0.462	0.753
NIG	0.054	0.116	0.139	0.388	2.521	3.836	0.066	0.462	0.729

Note: * and ** denote significance at the 5% and 1% level, respectively.

Mortality Projection

- Fitting period: 1900-1999
- Forecasting period: 2000-2007
- Simulation Paths: 1,000,000
- Mean Absolute Percentage Error

$$MAPE = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{A_i - F_i}{A_i}\right|$$

where A_i is historical mortality rate and F_i is the forecast mortality rate

MAPE of Mortality Projection

Panel A: the Finland Mortality Data (Unit: %)

Model	Mean	Median	Mean Rank	Median Rank
Original RH	3.883	3.461	5	5
VG-Normal	3.865	3.440	4	4
VG-JD	3.858	3.415	1	1
VG-VG	3.858	3.419	2	2
VG-NIG	3.863	3.435	3	3

Panel B: the France Mortality Data (Unit: %)

Model	Mean	Median	Mean Rank	Median Rank
Original RH	3.247	2.782	3	4
VG-Normal	3.253	2.787	5	5
VG-JD	3.252	2.772	4	3
VG-VG	3.242	2.753	2	2
VG-NIG	3.233	2.744	1	1

Note: Original RH is the same as M2 of Cairns et al. (2009). X-Y model means that the best model for the number of death is X model and the error terms in Equations (2) and (3) are Y model.

MAPE of Mortality Projection

Panel C: the Italy Mortality Data (Unit: %)

Model	Mean	Median	Mean Rank	Median Rank
Original RH	3.517	2.978	5	5
VG-Normal	3.509	2.971	4	4
VG-JD	3.501	2.970	3	3
VG-VG	3.494	2.951	1	1
VG-NIG	3.496	2.956	2	2

Panel D: the Netherlands Mortality Data (Unit: %)

Model	Mean	Median	Mean Rank	Median Rank
Original RH	3.923	3.379	4	4
VG-Normal	3.925	3.381	5	5
VG-JD	3.671	2.998	2	2
VG-VG	3.486	2.951	1	1
VG-NIG	3.691	3.045	3	3

Note: Original RH is the same as M2 of Cairns et al. (2009). X-Y model means that the best model for the number of death is X model and the error terms in Equations (2) and (3) are Y model.

Conclusions and Suggestions

- We attempt to provide an iterative fitting algorithm for estimating the Cox regression model under which death rates adhere to the RH model with three heavy-tailed distributions —JD, VG and NIG.

- Using mortality data from the four countries, Finland, France, Italy and the Netherlands, we employ the KS, AD and CvM tests and find consistent support for the non-Gaussian residuals of the RH model.

Conclusions and Suggestions

- When we calibrate the parameters of the RH model, the VG model is the best one for the four countries according to the BIC criterion.

- The residuals of the mortality indices and cohort effects come from non-Gaussian distributions.

Conclusions and Suggestions

- In the four countries, the non-Gaussian distributions provide good mortality projections.

- For applications of the RH model, the heavy-tailed distributions appear to be the most appropriate choices for modeling long-term mortality data.

Thanks for your attention.

The proof of theorem 1

$$LLF = \sum_{x,t} \int_{-\infty}^{\infty} \log f\left(D_{x,t} = d_{x,t} \mid e_{x,t} = y\right) f_{e_{x,t}}(y)\,dy$$

$$= \sum_{x.t} \Big(\int_{-\infty}^{\infty} d_{x,t} \left(\log E_{x,t} + \alpha_x + \beta_x k_t + \eta_x \gamma_{t-x} + y\right) f_{e_{x,t}}(y)\,dy$$

$$- \int_{-\infty}^{\infty} E_{x,t} \exp\left(\alpha_x + \beta_x k_t + \eta_x \gamma_{t-x}\right) \exp(y) f_{e_{x,t}}(y)\,dy - \log\left(d_{x,t}!\right) \Big).$$

$$E\left(e_{x,t}\right) = 0$$

$$= \sum_{x,t} \Big[d_{x,t} \left(\log E_{x,t} + \alpha_x + \beta_x k_t + \eta_x \gamma_{t-x}\right)$$

$$- E_{x,t} \exp\left(\alpha_x + \beta_x k_t + \eta_x \gamma_{t-x}\right) \int_{-\infty}^{\infty} \exp(y) f_{e_{x,t}}(y)\,dy - \log\left(d_{x,t}!\right) \Big]$$

$$= \sum_{x.t} \left[d_{x,t} \left(\alpha_x + \beta_x k_t + \eta_x \gamma_{t-x}\right) - \left(E_{x,t} \exp\left(\alpha_x + \beta_x k_t + \eta_x \gamma_{t-x}\right)\right) M_{e_{x,t}} \right]$$

$$+ \sum_{x.t} \left[d_{x,t} \log E_{x,t} - \log\left(d_{x,t}!\right) \right].$$