The impact of joint mortality modelling on hedging effectiveness of mortality derivatives

Jackie Li
Michel Dacorogna
Chong It Tan

Jackie Li is an Assistant Professor at Insurance Risk and Finance Research Centre, Nanyang Business School, Nanyang Technological University, Singapore. Michel Dacorogna is the Scientific Advisor at SCOR. Chong It Tan is a postgraduate researcher at Insurance Risk and Finance Research Centre, Nanyang Business School, Nanyang Technological University, Singapore. The authors can be contacted via e-mail: jackieli@ntu.edu.sg, mdacorogna@scor.com, and citan1@e.ntu.edu.sg.

Abstract

Standardised mortality derivatives are an innovative and practical tool for annuity providers and pension plan sponsors to manage their longevity risk. There is, however, a major concern over the existence of basis risk, which arises from mismatching between the characteristics of the hedging instrument and the portfolio to be hedged. In this project, we compare a variety of joint mortality models and conduct a quantitative evaluation of their impact on assessing the hedging effectiveness of standardised mortality derivatives. We broadly categorise the models into five groups, and apply them to UK assured lives, pensioners, annuitants, and population data to measure the goodness-of-fit and conduct an out-of-sample analysis. We then use each model in turn to perform simulation of future outcomes for computing the level of risk reduction for an annuity portfolio in a longevity hedge composed of *q*-forwards. In the simulations, process error, parameter error, and different portfolio sizes are taken into account, and via the comparison of using different models, the extent of model error is also considered.

Joint Mortality Models

In this section, we provide a review of several joint mortality models that have been proposed in the demographic and actuarial literature for handling multiple populations. These models have different characteristics and generally stem from single-population mortality models. In the following, we group the models into five broad categories: (I) associated mortality indices, (II) common and specific factors, (III) ratio of death rates, (IV) historical simulation, and (V) continuous-time models. Note that the models in different categories may have similar elements; in fact, it can be seen that certain model features may be interchanged or combined. The term $m_{x,t,i}$ used below is the central death rate at age x in year t, and the subscript i refers to one of the populations. The term $q_{x,t,i}$ is the corresponding one-year probability of death. Assuming the force of mortality $\mu_{x,t,i}$ is constant over each age-year band and so $m_{x,t,i} = \mu_{x,t,i}$, the death probability can be computed as $q_{x,t,i} = 1 - \exp(-m_{x,t,i})$. All the error terms are supposed to be independent across time t.

Category I – Associated Mortality Indices

In this category, the same single-population model is first fitted to each population. The resulting two sets of associated parameter series are then modelled as a bivariate time series process. The first approach in Carter and Lee (1992) is to fit the Lee and Carter (1992) model to each population separately and then measure the dependence between the two mortality indices. Since the Lee-Carter mortality index is usually highly linear over time and treated as a random walk with drift, one may express this approach as:

$$\ln m_{x,t,i} = \alpha_{x,i} + \beta_{x,i}\,\kappa_{t,i}; \qquad \text{(Lee-Carter model)}$$

$$\mathrm{K}_t = \Theta + \mathrm{K}_{t-1} + \Delta_t, \qquad \text{(bivariate random walk with drift)} \qquad (1)$$

where $\alpha_{x,i}$ is the overall age schedule, $\kappa_{t,i}$ is the mortality index, $\beta_{x,i}$ is the sensitivity measure, $\mathrm{K}_t = (\kappa_{t,1}, \kappa_{t,2})'$, Θ is the vector drift term, and Δ_t is the vector error term which is normally distributed. On the other hand, the third approach in Carter and Lee (1992) is to model the two mortality indices as a co-integrated process (Li and Hardy, 2011) instead:

$$\ln m_{x,t,i} = \alpha_{x,i} + \beta_{x,i}\,\kappa_{t,i}; \qquad \text{(Lee-Carter model)}$$

$$\kappa_{t,1} = \theta + \kappa_{t-1,1} + \delta_t; \qquad \text{(random walk with drift)}$$

$$\kappa_{t,2} = a_0 + a_1\,\kappa_{t,1} + \omega_t, \qquad \text{(co-integrated process)} \qquad (2)$$

in which θ, a_0, and a_1 are the parameters of the co-integrated process, and δ_t and ω_t are independent normal error terms. This approach can be used when there exists a stationary linear combination of $\kappa_{t,1}$ and $\kappa_{t,2}$.

Similarly, Yang and Wang (2013) assumed that the Lee-Carter mortality indices followed a vector error correction model:

$\ln m_{x,t,i} = \alpha_{x,i} + \beta_{x,i}\ \kappa_{t,i}$; (Lee-Carter model)

$\mathrm{K}_t - \mathrm{K}_{t-1} = \Theta + \Pi\,\mathrm{K}_{t-1} + \Gamma\left(\mathrm{K}_{t-1} - \mathrm{K}_{t-2}\right) + \Delta_t$, (vector error correction model) (3)

where $\mathrm{K}_t = \left(\kappa_{t,1}, \kappa_{t,2}\right)'$, Θ is the vector term, Π and Γ are the matrix terms, and Δ_t is the normal vector error. Yang and Wang (2013) further assumed that the first differences of the single-age error terms of the Lee-Carter structure are multivariate normal. Another idea proposed by Zhou, Li, and Tan (2013) is to adopt a common sensitivity measure for both populations and a stationary AR(1) process for the difference between the two mortality indices, so that the projected (central estimate) ratio of death rates between both populations at each age converges in the long run:

$\ln m_{x,t,i} = \alpha_{x,i} + \beta_x\ \kappa_{t,i}$; (adjusted Lee-Carter model)

$\kappa_{t,1} = \theta + \kappa_{t-1,1} + \delta_t$; (random walk with drift)

$\kappa_{t,1} - \kappa_{t,2} = b_0 + b_1\left(\kappa_{t-1,1} - \kappa_{t-1,2}\right) + \omega_t$, (AR(1) process) (4)

in which β_x is the common sensitivity measure, θ, b_0, and b_1 are the parameters of the time series processes, and δ_t and ω_t are bivariate normal error terms. Zhou, Li, and Tan (2013) also suggested incorporating the jump count and jump severity into the time series processes.

Cairns, Blake, Dowd, Coughlan, and Khalaf-Allah (2011) modelled the period and cohort parameter series of the age-period-cohort model (Bray, 2002; Currie, 2006) between a large population and a small subpopulation jointly:

$\ln m_{x,t,i} = \alpha_{x,i} + n_a^{-1}\ \kappa_{t,i} + n_a^{-1}\ \iota_{t-x,i}$; (age-period-cohort model)

$\kappa_{t,1} = \theta + \kappa_{t-1,1} + \delta_t$; (random walk with drift)

$\kappa_{t,1} - \kappa_{t,2} = b_0 + b_1\left(\kappa_{t-1,1} - \kappa_{t-1,2}\right) + \omega_t$; (AR(1) process)

$\tilde{\iota}_{h,1} = c_{0,1} + c_{1,1}\ \tilde{\iota}_{h-1,1} + c_{2,1}\ \tilde{\iota}_{h-2,1} + \varepsilon_{h,1}$; (AR(2) process)

$\iota_{h,1} - \iota_{h,2} = c_{0,2} + c_{1,2}\left(\iota_{h-1,1} - \iota_{h-1,2}\right) + c_{2,2}\left(\iota_{h-2,1} - \iota_{h-2,2}\right) + \varepsilon_{h,2}$, (AR(2) process) (5) where $i=1$ (2) refers to the large population (small subpopulation), $\alpha_{x,i}$ is the age effect, $\kappa_{t,i}$ is the period effect, $\iota_{h,i}$ is the cohort effect of year of birth h, n_a is the number of ages, and $\tilde{\iota}_{h,1}=\iota_{h,1}-\rho_0-\rho_1\left(h-\bar{h}\right)$ where ρ_0 and ρ_1 define a linear trend and $\bar{h}$ is the average year of birth plus one. The terms θ, b_0, b_1, $c_{0,1}$, $c_{1,1}$, $c_{2,1}$, $c_{0,2}$, $c_{1,2}$, and $c_{2,2}$ are the parameters of the time series processes, δ_t and ω_t are bivariate normal error terms, and $\varepsilon_{h,1}$ and $\varepsilon_{h,2}$ are also bivariate normal error terms (independent of δ_t and ω_t). The autoregressive processes are assumed to be stationary so that the projected ratio of the two populations' death rates at each age converges over time. In a similar vein, Dowd, Cairns, Blake, Coughlan, and Khalaf-Allah (2011) used the gravity model instead for the period and cohort parameter series above:

$$\ln m_{x,t,i}=\alpha_{x,i}+n_a^{-1}\,\kappa_{t,i}+n_a^{-1}\,\iota_{t-x,i}\,; \qquad \text{(age-period-cohort model)}$$

$$\kappa_{t,1}=\theta+\kappa_{t-1,1}+\delta_t\,; \qquad \text{(random walk with drift)}$$

$$\kappa_{t,2}-\kappa_{t-1,2}=b_0+b_1\left(\kappa_{t-1,1}-\kappa_{t-1,2}\right)+\omega_t\,; \qquad \text{(gravity model)}$$

$$\iota_{h,1}-\iota_{h-1,1}=c_{0,1}+c_{1,1}\left(\iota_{h-1,1}-\iota_{h-2,1}\right)+\varepsilon_{h,1}\,; \qquad \text{(AR(1) process)}$$

$$\iota_{h,2}-\iota_{h-1,2}=c_{0,2}+c_{1,2}\left(\iota_{h-1,2}-\iota_{h-2,2}\right)+c_{2,2}\left(\iota_{h-1,1}-\iota_{h-1,2}\right)+\varepsilon_{h,2}\,. \qquad \text{(gravity model)} \qquad (6)$$

The larger the two gravity parameters (b_1 and $c_{2,2}$ in (6)), the more the small subpopulation's death rates tend to move in line with those of the large population. However, the projected ratio of death rates between the two populations at each age does not necessarily converge.

Recently, Tan, Li, Li, and Balasooriya (2014) suggested a three-factor extension of the Cairns, Blake, and Dowd (2006) model and the use of a VARMA(r,s) model for the parameter series:

$$\ln\left(\tfrac{q_{x,t,i}}{1-q_{x,t,i}}\right)=\kappa_{t,i}^{(1)}+\kappa_{t,i}^{(2)}\left(x-\bar{x}\right)+\kappa_{t,i}^{(3)}\left(\left(x-\bar{x}\right)^2-\hat{\sigma}_x^2\right); \qquad \text{(3-factor CBD model)}$$

$$\mathrm{Z}_t=\Theta+\sum_{l=1}^{r}\Phi_l\,\mathrm{Z}_{t-l}+\sum_{u=1}^{s}\Lambda_u\,\Delta_{t-u}+\Delta_t\,, \qquad \text{(VARMA(}r,s\text{) model)} \qquad (7)$$

in which $\kappa_{t,i}^{(1)}$, $\kappa_{t,i}^{(2)}$, and $\kappa_{t,i}^{(3)}$ are the CBD mortality indices, $\bar{x}$ is the average age in the sample, $\hat{\sigma}_x^2$ is the average value of $\left(x-\bar{x}\right)^2$, $\mathrm{Z}_t=\left({}^d\kappa_{t,1}^{(1)},\,{}^d\kappa_{t,1}^{(2)},\,{}^d\kappa_{t,1}^{(3)},\,{}^d\kappa_{t,2}^{(1)},\,{}^d\kappa_{t,2}^{(2)},\,{}^d\kappa_{t,2}^{(3)}\right)'$, ${}^d\kappa_{t,i}^{(\cdot)}$ is the d^{th} difference of $\kappa_{t,i}^{(\cdot)}$, Θ is the intercept vector, Φ_l and Λ_u are the autoregressive and moving-average coefficient matrices, and Δ_t is the normal vector error. The order (r,s) can be identified via the Tiao and Box (1981) procedure.

Category II – Common and Specific Factors

There are another group of models which incorporate a common factor for the combined population as a whole, as well as additional factors for each population. The common factor describes the main long-term trend in mortality change while the additional factors depict the short-term discrepancy from the main trend for each population. Li and Lee (2005) proposed the augmented common factor model:

$$\ln m_{x,t,i} = \alpha_{x,i} + B_x\, K_t + \beta_{x,i}\, \kappa_{t,i}\,; \qquad \text{(augmented common factor model)}$$

$$K_t = \theta + K_{t-1} + \delta_t\,; \qquad \text{(random walk with drift)}$$

$$\kappa_{t,i} = c_{0,i} + c_{1,i}\, \kappa_{t-1,i} + \varepsilon_{t,i}\,, \qquad \text{(AR(1) process)} \qquad (8)$$

where $\alpha_{x,i}$ is the overall age schedule, K_t is the mortality index of the common factor with sensitivity B_x, $\kappa_{t,i}$ is the time component of the additional factor for population i with sensitivity $\beta_{x,i}$, θ, $c_{0,i}$, and $c_{1,i}$ are the parameters of the time series processes, and δ_t, $\varepsilon_{t,1}$, and $\varepsilon_{t,2}$ are independent normal error terms. The AR(1) processes are assumed to be stationary and so the projected ratio of the two populations' death rates tends to a constant at each age. Later, Li (2013) further generalised (8) as the Poisson common factor model and allowed for n additional factors:

$$\ln m_{x,t,i} = \alpha_{x,i} + B_x\, K_t + \sum_{j=1}^{n} \beta_{x,i,j}\, \kappa_{t,i,j}\,; \qquad \text{(generalised common factor model)}$$

$$K_t = \theta + K_{t-1} + \delta_t\,; \qquad \text{(random walk with drift)}$$

$$\kappa_{t,i,j} = c_{0,i,j} + c_{1,i,j}\, \kappa_{t-1,i,j} + \ldots + c_{r,i,j}\, \kappa_{t-r,i,j} + \varepsilon_{t,i,j}\,, \qquad \text{(AR(} r \text{) process)} \qquad (9)$$

in which the extra subscript j refers to the j^{th} additional factor. The optimal number of additional factors n is based on the Bayesian Information Criterion (BIC) values, the patterns of the residual plots, the resulting trends of the parameters, and the amount of data. The orders r of the autoregressive processes are determined by whether the projected values converge (and so the projected ratio of death rates converges) and whether the autocorrelations of the residuals are insignificant. Recently, Yang, Li, and Balasooriya (2014) incorporated six kinds of cohort expressions into (9) to cater for the cohort effect.

On the other hand, Li (2012) constructed the two-population logistic model:

$$\ln\left(\frac{m_{x,t,i}}{1-m_{x,t,i}}\right) = K_t^{(1)} + K_t^{(2)}\, x + \kappa_{t,i}^{(1)} + \kappa_{t,i}^{(2)}\, x\,; \qquad \text{(two-population logistic model)}$$

$$\begin{pmatrix} K_t^{(1)} \\ K_t^{(2)} \end{pmatrix} = \begin{pmatrix} \theta^{(1)} \\ \theta^{(2)} \end{pmatrix} + \begin{pmatrix} K_{t-1}^{(1)} \\ K_{t-1}^{(2)} \end{pmatrix} + \begin{pmatrix} \delta_t^{(1)} \\ \delta_t^{(2)} \end{pmatrix}; \qquad \text{(bivariate random walk with drift)}$$

$$\begin{pmatrix} \kappa_{t,i}^{(1)} \\ \kappa_{t,i}^{(2)} \end{pmatrix} = \begin{pmatrix} c_{0,i}^{(1)} \\ c_{0,i}^{(2)} \end{pmatrix} + \begin{pmatrix} c_{1,i}^{(1)}\, \kappa_{t-1,i}^{(1)} \\ c_{1,i}^{(2)}\, \kappa_{t-1,i}^{(2)} \end{pmatrix} + \begin{pmatrix} \varepsilon_{t,i}^{(1)} \\ \varepsilon_{t,i}^{(2)} \end{pmatrix}, \qquad \text{(bivariate AR(1) process)} \qquad (10)$$

where $K_t^{(1)}$ and $K_t^{(2)}$ are the common mortality indices, $\kappa_{t,i}^{(1)}$ and $\kappa_{t,i}^{(2)}$ are the time components for population i, $\theta^{(1)}$, $\theta^{(2)}$, $c_{0,i}^{(1)}$, $c_{0,i}^{(2)}$, $c_{1,i}^{(1)}$, and $c_{1,i}^{(2)}$ are the parameters of the time series processes, and $\left(\delta_t^{(1)}, \delta_t^{(2)}\right)$, $\left(\varepsilon_{t,1}^{(1)}, \varepsilon_{t,1}^{(2)}\right)$, and $\left(\varepsilon_{t,2}^{(1)}, \varepsilon_{t,2}^{(2)}\right)$ are independent normal vector errors. The discrepancy between the two populations' projected death rates would reduce if the AR(1) processes are stationary.

There is a note regarding the following models which are extended from the Lee-Carter model. The second approach in Carter and Lee (1992) involves the model structure $\ln m_{x,t,i} = \alpha_{x,i} + \beta_{x,i}\, K_t$. The initial common factor model in Li and Lee (2005) is $\ln m_{x,t,i} = \alpha_{x,i} + B_x\, K_t$. The model structure proposed in Delwarde, Denuit, Guillén, and Vidiella-i-Anguera (2006) is $\ln m_{x,t,i} = A + \alpha_i + A_x + \alpha_{x,i} + \left(B_x + \beta_{x,i}\right) K_t$, in which A_x and $\alpha_{x,i}$ are age-specific but A and α_i are not. Debón, Montes, and Martínez-Ruiz (2011) used $\ln m_{x,t,i} = \alpha_i + A_x + B_x\, K_t$. Russolillo, Giordano, and Haberman (2011) suggested a three-way structure $\ln m_{x,t,i} = \alpha_{x,i} + B_x\, K_t\, \varphi_i$, where φ_i is the parameter for population i. All these models only have a single common period effect K_t for all populations. As argued in Carter and Lee (1992), this simple arrangement may enforce greater consistency between the two populations' mortality levels and is a parsimonious way to model both populations jointly. However, it also implies that the death rates of the two populations are perfectly associated, which would lead to an underestimation of the basis risk. As such, these models are not suitable for our analysis here.

Category III – Ratio of Death Rates

Under this category, a single-population model is fitted to one of the populations, and the ratio of the two populations' death rates is then modelled as a function of age and time. Plat (2009) adopted the CBD model for the reference population and a function of age and period factors for the ratio of death rates between the portfolio and the reference population:

$$\ln\left(\frac{q_{x,t,1}}{1-q_{x,t,1}}\right)=\kappa_t^{(1)}+\kappa_t^{(2)}\left(x-\bar{x}\right); \qquad \text{(CBD model)}$$

$$\frac{q_{x,t,2}}{q_{x,t,1}}=1+\psi_x(1)\gamma_t(1)+\psi_x(2)\gamma_t(2)+\ldots+\psi_x(n)\gamma_t(n); \qquad \text{(age and period factors)}$$

$$\begin{pmatrix}\kappa_t^{(1)}\\ \kappa_t^{(2)}\end{pmatrix}=\begin{pmatrix}\theta^{(1)}\\ \theta^{(2)}\end{pmatrix}+\begin{pmatrix}\kappa_{t-1}^{(1)}\\ \kappa_{t-1}^{(2)}\end{pmatrix}+\begin{pmatrix}\delta_t^{(1)}\\ \delta_t^{(2)}\end{pmatrix}; \qquad \text{(bivariate random walk with drift)}$$

$$\gamma_t(j)=c_0(j)+c_1(j)\gamma_{t-1}(j)+\varepsilon_t(j), \qquad \text{(AR(1) process)} \qquad (11)$$

in which $i=1$ (2) refers to the reference population (portfolio), $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$ are the CBD mortality indices, $\bar{x}$ is the average age, $\psi_x(j)$ and $\gamma_t(j)$ are the age and period parameters, $\theta^{(1)}$, $\theta^{(2)}$, $c_0(j)$, and $c_1(j)$ are the parameters of the time series processes, $\delta_t^{(1)}$ and $\delta_t^{(2)}$ are bivariate normal error terms, and $\varepsilon_t(j)$'s are independent normal error terms (independent of $\delta_t^{(1)}$ and $\delta_t^{(2)}$). The optimal number of factors n can be decided through examining the BIC values. The ratio of projected death rates at each age converges to a constant if the AR(1) processes are stationary.

Ngai and Sherris (2011) applied a logit model to the reference population and a simple linear model to the ratio of death rates between the portfolio and the reference population:

$$\ln\left(\frac{m_{x,t,1}}{1-m_{x,t,1}}\right)-\ln\left(\frac{m_{x-1,t-1,1}}{1-m_{x-1,t-1,1}}\right)=a_0+a_1\,x+\omega_{x,t}; \qquad \text{(logit model)}$$

$$\frac{q_{x,t,2}}{q_{x,t,1}}=b_0+b_1\,x+\varepsilon_{x,t}, \qquad \text{(linear model)} \qquad (12)$$

where $i=1$ (2) refers to the reference population (portfolio), a_0, a_1, b_0, and b_1 are the parameters of the two models, $\omega_{x,t}$'s are multivariate normal error terms, and $\varepsilon_{x,t}$'s are also multivariate normal error terms (independent of $\omega_{x,t}$'s). The ratio of projected death rates at each age is constant.

Hyndman, Booth, and Yasmeen (2013) proposed the product-ratio model, which exploits the fact that the sum of and the difference between two random variables are roughly uncorrelated if the variances of the two variables are approximately equal:

$$\ln\sqrt{m_{x,t,1}\, m_{x,t,2}} = \eta_x + \phi_x(1)\lambda_t(1) + \phi_x(2)\lambda_t(2) + \ldots + \phi_x(n_p)\lambda_t(n_p); \quad \text{(product model)}$$

$$\ln\sqrt{m_{x,t,1}/m_{x,t,2}} = \xi_x + \psi_x(1)\gamma_t(1) + \psi_x(2)\gamma_t(2) + \ldots + \psi_x(n_r)\gamma_t(n_r); \quad \text{(ratio model)}$$

$${}^{d}\lambda_t(j) = \tau_0(j) + \sum_{l=1}^{r}\tau_l(j)\,{}^{d}\lambda_{t-l}(j) + \sum_{u=1}^{s}\chi_u(j)\,\omega_{t-u}(j) + \omega_t(j); \quad \text{(ARMA}(r,s)\text{ model)}$$

$$\gamma_t(j) = c_0(j) + \sum_{l=1}^{r}c_l(j)\gamma_{t-l}(j) + \sum_{u=1}^{s}\varsigma_u(j)\varepsilon_{t-u}(j) + \varepsilon_t(j), \quad \text{(ARMA}(r,s)\text{ model)} \quad (13)$$

in which η_x, $\phi_x(j)$, and $\lambda_t(j)$ are the product model's parameters, ξ_x, $\psi_x(j)$, and $\gamma_t(j)$ are the ratio model's parameters, ${}^{d}\lambda_t(j)$ is the d^{th} difference of $\lambda_t(j)$, $\tau_0(j)$ and $c_0(j)$ are the intercepts, $\tau_l(j)$ and $c_l(j)$ are the autoregressive coefficients, $\chi_u(j)$ and $\varsigma_u(j)$ are the moving-average coefficients, and $\omega_t(j)$'s and $\varepsilon_t(j)$'s are all independent normal error terms. Hyndman, Booth, and Yasmeen (2013) set n_p = n_r = 6, as they discovered that using more than six factors makes almost no impact on the resulting forecasts but using too few reduces forecast accuracy. Using stationary ARMA processes for the ratio model ensures that the projected ratio of death rates converges at each age.

Villegas and Haberman (2014) fitted the cohort extension of the Lee-Carter model (Renshaw and Haberman, 2006) to the reference population and a function of age and period factors to the log ratio of death rates between the portfolio and the reference population:

$$\ln m_{x,t,1} = \alpha_x + \beta_x\,\kappa_t + \iota_{t-x}; \quad \text{(Lee-Carter model with cohort)}$$

$$\ln\tfrac{m_{x,t,2}}{m_{x,t,1}} = a_x + b_x\,k_t; \quad \text{(age and period factors)}$$

$$\kappa_t = \theta + \kappa_{t-1} + \delta_t; \quad \text{(random walk with drift)}$$

$$k_t = c + k_{t-1} + \varepsilon_t, \quad \text{(random walk with drift)} \quad (14)$$

where $i=1$ (2) refers to the reference population (portfolio), α_x is the overall age schedule, κ_t is the mortality index, β_x is the sensitivity measure, ι_h is the cohort effect of year of birth h, a_x, b_x, and k_t are the age and period parameters, θ and c are the drift terms, and δ_t and ε_t are independent normal error terms. Though Villegas and Haberman (2014) did not specify the way to project ι_h, one may exploit an autoregressive process as in the other models discussed earlier.

There are a few other models which have similar characteristics. Jarner and Kryger (2011) fitted a frailty model to the reference population and a function of age and period factors to the log ratio of death rates. Hatzopoulos and Haberman (2013) expressed the log death rate of the combined population, the log ratio of death rates between females and males, and also the residuals for each country as functions of age and period factors. These approaches were tailored to particular applications and datasets, and we do not consider them in our computations.

Category IV – Historical Simulation

Besides fitting a model to the mortality data and using the fitted model to generate random samples, one can also adopt a model-independent approach called historical simulation. This approach does not assume any model setting; instead it involves repeated sampling from the historical data. One useful technique is the block bootstrap method, in which the data are divided into overlapping or non-overlapping blocks of equal size, and then random samples of blocks are drawn (with replacement) and put end to end to construct pseudo data. By sampling blocks rather than individual data points, the underlying serial dependence can be preserved in the simulation. In the context of handling multiple populations, this approach requires gathering the different populations' death rates (or their rates of decline) at each age-time cell as an individual data point, so that the relationships between the populations can be maintained in the simulated samples. Some application examples of historical simulation in the actuarial literature include Coughlan, Khalaf-Allah, Ye, Kumar, Cairns, Blake, and Dowd (2011) and Li and Ng (2011).

Category V – Continuous-Time Models

All the mortality models we have covered so far are discrete-time. There are also a few continuous-time models for modelling multiple populations jointly. Cox, Lin, and Wang (2006) expressed each population's mortality as a combination of a Brownian motion and a compound Poisson process and assumed the Brownian motions are correlated between different populations. Barbarin (2008) applied the Heath-Jarrow-Morton framework to model longevity bond prices and take basis risk into account. Dahl, Melchior, and Møller (2008) and Dahl, Glar, and Møller (2011) adopted Cox-Ingersoll-Ross processes for modelling the mortality intensities of an insurance portfolio and a large population. In this project, we focus on discrete-time models as our data are recorded by year.

Goodness-of-fit

In this section, we measure the goodness-of-fit of the various joint mortality models stated earlier to both industry and population data. The industry data include the experience of male assured lives, pensioners of insured pension schemes, and annuitants collected from the Continuous Mortality Investigation (CMI). The pensioners and annuitants data are available only for years 1983-2006, and so we set this period as the data sampling period. We take the age range of 65-90, as the data outside this range are often sparse and volatile for modelling and projection. Correspondingly, we use English and Welsh male population data obtained from the Human Mortality Database (HMD) with the same age range and data period.

Table 1 lists the computed values of Bayesian Information Criterion (BIC), which is defined as $-2l + p \ln n_d$, where l is the log-likelihood value based on the assumption of the number of deaths $D_{x,t,i} \sim \text{Poisson}(e_{x,t,i}\, m_{x,t,i})$, $e_{x,t,i}$ is the matching exposure, p is the number of parameters, and n_d is the number of data points. In general, a lower value of BIC, which penalises both poor data fitting and excessive use of parameters, is preferred. The rankings of the models are also stated in the brackets.

A number of interesting results are observed. First, it can be seen that those models with cohort parameters ((5), (6), and (14)) produce the best fit. This result agrees with the significant cohort effects found in the residuals of most of the other models for English and Welsh population data. Earlier work such as Renshaw and Haberman (2006) also showed prominent cohort effects in modelling data of England and Wales. The performance of the other models may be improved by incorporating cohort parameters. However, no particular cohort patterns are discovered for assured lives, pensioners, and annuitants data. Second, it is important to note that the BIC value should not be taken as the sole indicator for the suitability of a model. For example, model (12) produces a satisfactory fit compared to many of the other models in terms of BIC, and since it handles the death rates in the cohort direction, the residuals are randomly scattered along cohort year. Nevertheless, the residuals demonstrate obvious patterns along age and calendar year, indicating the insufficiency of this model in capturing the main features. It is thus necessary to examine carefully the residuals along age, calendar year, and cohort year, and even in heat maps, besides inspecting the BIC values. Other statistical criteria can also be applied and the conclusions may be different.

Table 1

Computed BIC Values of Joint Mortality Models

Models	Assured Lives with England & Wales	Pensioners with England & Wales	Annuitants with England & Wales
(1) / (2) / (3)	16,113 [9]	16,644 [8]	14,283 [9]
(4)	16,020 [7]	16,485 [6]	14,172 [7]
(5) / (6)	14,443 [2]	14,860 [2]	12,633 [2]
(7)	15,688 [4]	16,191 [4]	13,851 [4]
(8) / (9)	15,906 [6]	16,520 [7]	14,065 [6]
(10)	16,107 [8]	16,755 [9]	14,228 [8]
(11)	15,722 [5]	16,152 [3]	13,877 [5]
(12)	14,835 [3]	16,341 [5]	13,127 [3]
(13)	19,381 [10]	18,676 [10]	19,510 [10]
(14)	14,025 [1]	14,534 [1]	12,312 [1]

Note: The rankings (1 to 10) of the models are given in the brackets.

Note: Models (1) to (3) have the same Lee-Carter structure. Models (5) and (6) have the age-period-cohort structure. Model (9) involves only one additional factor as determined by the lowest BIC value.

A satisfactory model fit does not necessarily mean that the model is also suitable for projection purposes. In the next steps of our project, we will divide the data into two periods and conduct an out-of-sample analysis. We will also use each model in turn to simulate future scenarios for calculating the level of risk reduction for an annuity portfolio in a longevity hedge constructed by standardised *q*-forwards.

References

Barbarin, J., 2008, Heath-Jarrow-Morton Modelling of Longevity Bonds and the Risk Minimization of Life Insurance Portfolios, *Insurance: Mathematics and Economics*, 43: 41-55.

Bray, I., 2002, Application of Markov Chain Monte Carlo Methods to Projecting Cancer Incidence and Mortality, *Applied Statistics*, 51: 151-164.

Cairns, A. J. G., D. Blake, and K. Dowd, 2006, A Two-Factor Model for Stochastic Mortality with Parameter Uncertainty: Theory and Calibration, *Journal of Risk and Insurance*, 73: 687-718.

Cairns, A. J. G., D. Blake, K. Dowd, G. D. Coughlan, and M. Khalaf-Allah, 2011, Bayesian Stochastic Mortality Modelling for Two Populations, *ASTIN Bulletin*, 41: 29-59.

Carter, L. R., and R. D. Lee, 1992, Modeling and Forecasting US Sex Differentials in Mortality, *International Journal of Forecasting*, 8: 393-411.

Coughlan, G. D., M. Khalaf-Allah, Y. Ye, S. Kumar, A. J. G. Cairns, D. Blake, and K. Dowd, 2011, Longevity Hedging 101: A Framework for Longevity Basis Risk Analysis and Hedge Effectiveness, *North American Actuarial Journal*, 15: 150-176.

Cox, S. H., Y. Lin, and S. Wang, 2006, Multivariate Exponential Tilting and Pricing Implications for Mortality Securitization, *Journal of Risk and Insurance*, 73: 719-736.

Currie, I., 2006, Smoothing and Forecasting Mortality Rates with P-Splines, Heriot Watt University.

Dahl, M., S. Glar, and T. Møller, 2011, Mixed Dynamic and Static Risk-Minimization with An Application to Survivor Swaps, *European Actuarial Journal*, 1: 233-260.

Dahl, M., M. Melchior, and T. Møller, 2008, On Systematic Mortality Risk and Risk-Minimization with Survivor Swaps, *Scandinavian Actuarial Journal*, 2-3: 114-146.

Debón, A., F. Montes, and F. Martínez-Ruiz, 2011, Statistical Methods to Compare Mortality for A Group with Non-Divergent Populations: An Application to Spanish Regions, *European Actuarial Journal*, 1: 291-308.

Delwarde, A., M. Denuit, M. Guillén, and A. Vidiella-i-Anguera, 2006, Application of the Poisson Log-Bilinear Projection Model to the G5 Mortality Experience, *Belgian Actuarial Bulletin*, 6: 54-68.

Dowd, K., A. J. G. Cairns, D. Blake, G. D. Coughlan, and M. Khalaf-Allah, 2011, A Gravity Model of Mortality Rates for Two Related Populations, *North American Actuarial Journal*, 15: 334-356.

Hatzopoulos, P., and S. Haberman, 2013, Common Mortality Modeling and Coherent Forecasts. An Empirical Analysis of Worldwide Mortality Data, *Insurance: Mathematics and Economics*, 52: 320-337.

Hyndman, R. J., H. Booth, and F. Yasmeen, 2013, Coherent Mortality Forecasting: The Product-Ratio Method with Functional Time Series Models, *Demography*, 50: 261-283.

Jarner, S. F., and E. M. Kryger, 2011, Modelling Adult Mortality in Small Populations: The Saint Model, *ASTIN Bulletin*, 41: 377-418.

Lee, R. D., and L. R. Carter, 1992, Modeling and Forecasting U.S. Mortality, *Journal of the American Statistical Association*, 87: 659-671.

Li, J., 2012, Mortality Projection Methods: Extensions and Modifications, Macquarie University, Australia.

Li, J., 2013, A Poisson Common Factor Model for Projecting Mortality and Life Expectancy Jointly for Females and Males, *Population Studies*, 67: 111-126.

Li, J. S. H., and M. R. Hardy, 2011, Measuring Basis Risk in Longevity Hedges, *North American Actuarial Journal*, 15: 177-200.

Li, N., and R. Lee, 2005, Coherent Mortality Forecasts for A Group of Populations: An Extension of the Lee-Carter Method, *Demography*, 42: 575-594.

Li, J. S. H., and A. C. Y. Ng, 2011, Canonical Valuation of Mortality-Linked Securities, *Journal of Risk and Insurance*, 78: 853-884.

Ngai, A., and M. Sherris, 2011, Longevity Risk Management for Life and Variable Annuities: The Effectiveness of Static Hedging Using Longevity Bonds and Derivatives, *Insurance: Mathematics and Economics*, 49: 100-114.

Plat, R., 2009, Stochastic Portfolio Specific Mortality and the Quantification of Mortality Basis Risk, *Insurance: Mathematics and Economics*, 45: 123-132.

Renshaw, A. E., and S. Haberman, 2006, A Cohort-Based Extension to the Lee-Carter Model for Mortality Reduction Factors, *Insurance: Mathematics and Economics*, 38: 556-570.

Russolillo, M., G. Giordano, and S. Haberman, 2011, Extending the Lee-Carter Model: A Three-Way Decomposition, *Scandinavian Actuarial Journal*, 2: 96-117.

Tan, C. I., J. Li, J. S. H. Li, and U. Balasooriya, 2014, Parametric Mortality Indexes: From Index Construction to Hedging Strategies, Insurance Risk and Finance Research Centre, Nanyang Business School, Singapore.

Tiao, G. C., and G. E. P. Box, 1981, Modelling Multiple Time Series with Applications, *Journal of the American Statistical Association*, 76: 802-816.

Villegas, A. M., and S. Haberman, 2014, On the Modeling and Forecasting of Socioeconomic Mortality Differentials: An Application to Deprivation and Mortality in England, *North American Actuarial Journal,* 18: 168-193.

Yang, B., J. Li, and U. Balasooriya, 2014, Cohort Extensions of the Poisson Common Factor Model for Modelling Both Genders Jointly, *Scandinavian Actuarial Journal* (in press).

Yang, S. S., and C. W. Wang, 2013, Pricing and Securitization of Multi-Country Longevity Risk with Mortality Dependence, *Insurance: Mathematics and Economics*, 52: 157-169.

Zhou, R., J. S. H. Li, and K. S. Tan, 2013, Pricing Standardized Mortality Securitizations: A Two-Population Model with Transitory Jump Effects, *Journal of Risk and Insurance*, 80: 733-774.