Optimal Net Contribution and Investment Rules for a Pension Reserve Fund*

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1 Extended Abstract

Governments everywhere have amassed large pools of resources held by and managed for the public sector, usually called "sovereign wealth funds" (SWFs). Among the different types of the SWFs (Mitchell et al., 2008), Pension Reserve Funds (PRFs) have emerged has an attractive alternative for countries that wish to (either fully, or partially) fund the implicit liabilities contained in the non-contributory pillar of their pension systems. Nonetheless, there are several features surrounding the management of a PRF that make the task of setting up one anything but simple. Moreover, flaws in its design can have a relevant impact on the sustainability of the PRF, and thus on welfare. This paper studies two of these features, namely, the determination of its optimal injection/withdrawal and investment rules, from the point of view of a benevolent social planner.

We study the decision problem of a benevolent social planner, who wishes to maximize social welfare of the representative consumer. The central planner's problem is cast as a continuous-time consumption portfolio problem (Merton, 1969, 1971), and it considers a fairly general financial market (time-varying expected returns, volatilities, and correlations), with several assets classes (approximated through domestic and international bond and equity indices, commodities, currencies, etc.), as well as the (partially) unhedgable longevity risk, implicit in the PRF's liabilities.

Our methodology is based on an extension of Detemple and Rindisbacher (2008), who consider an Asset-Liability-Management (ALM) problem within a dynamically complete financial market.[1] We extend their analysis by introducing market incompleteness and portfolio constraint. A key ingredient of our methodology is the use of nearly-optimal rules, which are based upon the analytical representation of the optimal policies. We use the nearly-optimal policies to study, among others, the welfare loss caused by the presence of unhedgeable longevity risk, currency mismatches between assets and liabilities, as well as some prohibitions contained in the investment policy of the PRF to held specific asset classes (e.g., derivatives).

Our aim is to apply our methodology to assess the optimality of the current injection/withdrawal and investment rules of the Chilean's PRF, for which we have developed (reduced form) stochastic models that describe its future behavior for the next 50 years.

1.1 A simple model

We illustrate the main insights of our study by the following example.

[1] Our methodology could be easily adapted to fit other approaches to deal with ALM problems, such as, Hoevenaars et al. (2008), Martellini and Milhau (2012) and van Binsbergen and Brandt (2012).

1. *Uncertainty.* Consider an stochastic environment where all economic quantities depend on stochastic factors, summarized by the vector $Y \in \mathbb{R}^2$, whose dynamics is represented by a diffusion process,

$$\mathrm{d}Y_t = \mu_Y(t, Y_t)\mathrm{d}t + \sigma_Y(t, Y_t)\mathrm{d}W_t, \quad Y_0 \in \mathbb{R}^2 \text{ given},$$

where the drift vector ($\mu_Y \in \mathbb{R}^2$) and volatility matrix ($\sigma_Y \in \mathbb{R}^{2\times 2}$) satisfy regular conditions (in order for Y to be an integrable vector process), and $W \in \mathbb{R}^2$ is a standard 2-dimensional Brownian motion process.

2. *Financial market.* The financial market of comprised by three (non-redundant) assets: a (locally) riskless money market account, a stock index, and a financial contract exposed to the longevity risk of the liabilities. The prices of these assets evolve according to the following laws of motion:

$$\begin{aligned} \mathrm{d}B_t/B_t &= r(t, Y_t)\mathrm{d}t, \quad B_0 = 1, \\ \mathrm{d}S_{i,t}/S_{i,t} &= (\mu_i(t, Y_t) - \delta_i(t, Y_t))\mathrm{d}t + \sigma_i(t, Y_t)\mathrm{d}W_t, \quad S_{i,0} > 0 \text{ given}, \end{aligned}$$

for $i = 1, 2$. The coefficients (r, μ, δ, σ) stand for the (instantaneous) short-term interest rate, the ex-dividend expected return, the dividend rate, and the volatility vector of each risky asset, respectively. These coefficients satisfy regular conditions for S_i to be integrable processes. Without further restrictions, the volatility vectors of the different processes are statistically not identified. Therefore, we impose the volatility matrix $\sigma_t \equiv (\sigma_{1t}, \sigma_{2t})' \in \mathbb{R}^{2\times 2}$ to be lower triangular.

3. *Contributions and liabilities.* Both injections to, and withdrawals from the PRF over time, which we denote by $e \equiv (e_t)_{t\geq 0}$ and $l \equiv (l_t)_{t\geq 0}$, respectively, are represented by diffusion processes,

$$\mathrm{d}a_t/a_t = \mu_a(t, Y_t)\mathrm{d}t + \sigma_a(t, Y_t)\mathrm{d}W_t, \quad a_0 > 0 \text{ given},$$

for $a \in \{e, l\}$.

4. *PRF's dynamics.* The market value process of the assets is denoted by $X \equiv (X_t)_{t\geq 0}$. Its dynamics is dictated by the rules that govern the injections/withdrawals to/from the PRF, as well as for the investments made in the financial assets. We denote these policies by (c, π). For a given policy (c, π), the dynamics of the PRF is dictated by the system:

$$\left\{ \begin{array}{l} \mathrm{d}X_t = X_t(1 - \pi_t'\mathbf{1})r_t\mathrm{d}t + X_t\pi_t'\left[\mu_t\mathrm{d}t + \sigma_t\mathrm{d}W_t\right] + (e_t - l_t - c_t)\mathrm{d}t; \\ X_0 \geq 0; \quad c_t \in [-C_t, +\infty), \, \pi_t \in \mathcal{K}, \ X_t \geq 0, \ t \geq 0; \end{array} \right. \tag{1}$$

where π_t is the vector of proportions of total value of the PRF invested in the 2 risky assets at time t, the symbol ($'$) denotes the transposition of vectors and matrices, $X_t(1-\pi_t'\mathbf{1})$ is the dollar amount invested in the money market account, $\mathbf{1} \equiv (1,1)' \in \mathbb{R}^2$, c is the process of injections/withdrawals made to/taken from the PRF, over and above the predetermined policy, $e - l$, $C \equiv (C_t \geq 0)_{t\geq 0}$ is an exogenous consumption level (to be introduced below), $X_t \geq 0$ is a solvency constraint that ensures the absence of arbitrage opportunities (see, e.g., Dybvig and Huang, 1988), and

$$\mathcal{K} \equiv \left\{\pi_t \in \mathbb{R}^2 : \pi_{1,t} \in [0,1], \;\; \pi_{2,t} = 0\right\}$$

describes the portfolio restrictions to which the PRF is subject to. In particular, the investment decisions of the PRF are subject to a no-short sales of the stock index (first coordinate of π_t), a no-borrowing constraint on the money market account ($0 \leq 1 - \pi_t'\mathbf{1} \leq 1$), and the impossibility to held the second risky asset. This last restriction may be due to either market incompleteness, or simply because the second risky asset has been excluded from the investment policy of the PRF.

5. *Preferences.* We model the social planner's preferences by means of a time-additive von-Neumann Morgenstern (vN-M) index,

$$U(c) \equiv \mathbf{E}\left[\int_0^{+\infty} u(C_t + c_t, t)\mathrm{d}t\right], \tag{2}$$

where $\mathbf{E}[\cdot]$ is the mathematical expectation, conditional on the information available at time-0, C is an exogenous endowment of the consumption good, which accounts for the un-modeled parts of the economy, and $u(\cdot, t) : \mathbb{R}_+ \to \mathbb{R}$ is a strictly increasing and strictly concave utility function, that satisfy regular Inada end-point conditions. The second argument in $u(c,t)$ allows us to include a subjective time-discount factor. We set $u(C + c, t) = -\infty$, for $c < -C$, and model C as a diffusion process,

$$\mathrm{d}C_t/C_t = \mu_C(t, Y_t)\mathrm{d}t + \sigma_C(t, Y_t)\mathrm{d}W_t, \quad C_0 \geq 0 \text{ given},$$

where $\mu_C \in \mathbb{R}$ and $\sigma_C' \in \mathbb{R}^2$ satisfy regular conditions.

1.2 The optimal policy

We follow the duality approach of Cvitanic and Karatzas (1992), and solved the utility maximization of (2), subject to the dynamic budget constraint (1), and the condition $|U(c)| < +\infty$, in a fictitious unconstrained economy with modified interest rate, $\tilde{r}_t = r_t + \nu_{1t}^-$, $x^- \equiv \max(-x, 0)$, and expected return $\tilde{\mu}_{1t} = \mu_{1t} + \nu_{1t}^+$ for the stock index, $x^+ \equiv \max(x, 0)$, and

$\tilde{\mu}_{2t} = \mu_{2t} + \nu_{1t}^{-} + \nu_{2t}$ for the contract exposed to longevity risk.[2] Intuitively, $\nu \equiv (\nu_{1t}, \nu_{2t})_{t\geq 0}$ is the shadow cost of the constraints, when the constraints bind they change the relative attractiveness of the risky assets, so that their indivisual demand is contained within the boundaries of $\mathcal{K}$.

In this fictitious economy, the *liquidity unconstrained* optimal rules,[3] as function of ν, are given by:[4]

$$c_t^{\nu,*} = \begin{cases} I(y\tilde{\xi}_t, t) - C_t & \text{if} \quad \tilde{\xi}_t \geq \xi_{Ct} \\ 0 & \text{if} \quad \tilde{\xi}_t < \xi_{Ct} \end{cases} \tag{3}$$

where $I(\cdot, t) \equiv \partial_1 u^{-1}(\cdot, t)$, with ∂_1 as the derivative of the first argument, $\tilde{\xi}_t$ is the stochastic discount factor (SDF), which solves $\mathrm{d}\tilde{\xi}_t = -\tilde{\xi}_t(\tilde{r}_t \mathrm{d}t + \tilde{\theta}_t \mathrm{d}W_t)$, $\tilde{\theta}_t = \sigma_t^{-1}(\mu_t - r_t \mathbf{1}) + \sigma_t^{-1}\nu_t$, $\xi_{Ct} = \partial_1 u(C_t, t)/y$, $y > 0$ solves

$$\mathcal{X}(y) = X_0 + \mathbf{E}\left[\int_0^{+\infty} \tilde{\xi}_t (e_t - l_t)\mathrm{d}t\right],$$

with $\mathcal{X}(y) \equiv \mathbf{E}\left[\int_0^{+\infty} \tilde{\xi}_t c_t^{\nu,*} \mathrm{d}t\right]$,

$$\begin{aligned} \pi_t^{\nu,*} \quad = \quad & (X_t^{\nu,*}\sigma_t\sigma_t')^{-1}(\tilde{\mu}_t - \tilde{r}_t\mathbf{1})\underbrace{\mathbf{E}_t\left[\int_t^{+\infty} \tilde{\xi}_{t,s}\{(C_s + c_s^{\nu,*})/R_s^u\}\mathrm{d}s\right]}_{V_{1t}^{c^{\nu,*}}} \\ & -(X_t^{\nu,*}\sigma_t')^{-1}\underbrace{\mathbf{E}_t\left[\int_t^{+\infty} \tilde{\xi}_{t,s}(C_s + c_s^{\nu,*})(1 - 1/R_s^u)H_{t,s}^{\tilde{\xi}}\mathrm{d}s\right]}_{V_{2t}^{c^{\nu,*}}} \\ & -(X_t^{\nu,*}\sigma_t')^{-1}\underbrace{\mathbf{E}_t\left[\int_t^{+\infty} \tilde{\xi}_{t,s}C_s\{\sigma_{Ct} + H_{t,s}^{C} + H_{t,s}^{\tilde{\xi}}\}\mathrm{d}s\right]}_{V_t^C} \\ & -(X_t^{\nu,*}\sigma_t')^{-1}\underbrace{\mathbf{E}_t\left[\int_t^{+\infty} \tilde{\xi}_{t,s}e_s\{\sigma_{et} + H_{t,s}^{e} + H_{t,s}^{\tilde{\xi}}\}\mathrm{d}s\right]}_{V_t^e} \\ & +(X_t^{\nu,*}\sigma_t')^{-1}\underbrace{\mathbf{E}_t\left[\int_t^{+\infty} \tilde{\xi}_{t,s}l_s\{\sigma_{lt} + H_{t,s}^{l} - H_{t,s}^{\tilde{\xi}}\}\mathrm{d}s\right]}_{V_t^l} \end{aligned} \tag{4}$$

[2]If the second risky asset is not available (i.e., it is a fictitious asset), we would set $\mu_{2t} = 0$.

[3]In this case, the admissible policies satisfy: $X_t + \mathbf{E}_t\left[\int_t^{+\infty} \tilde{\xi}_{t,s}(e_s - l_s)\mathrm{d}s\right] \geq 0$. We will deal with the liquidity *constrained* version of the problem (e.g., El Karoui and Jeanblanc-Picque, 1998, Detemple and Serrat, 2003) in the final draft.

[4]Given the completeness of the fictitious financial market, the optimal policies can be determined by an application of the martingale approach to portfolio problems; see, for instance, Karatzas et al. (1987) and Cox and Huang (1989).

where $\tilde{\xi}_{t,s} \equiv \tilde{\xi}_s/\tilde{\xi}_t$,

$$X_t^{\nu,*} = \mathbf{E}_t\left[\int_t^{+\infty} \tilde{\xi}_{t,s}(c_s^{\nu,*} - (e_s - l_s))\mathrm{d}s\right] \tag{5}$$

is the optimal value at time-t of the PRF, $R_s^u \equiv (\partial_1^2 u(c,s)/\partial c^2)/(\partial_1 u(c,s)/\partial c)$ is the relative risk aversion (RRA) coefficient of the vN-M index,

$$\begin{aligned} H_{t,s}^{\tilde{\xi}} &= \int_t^s \mathscr{D}_t\left(\tilde{r}_u + \frac{1}{2}\tilde{\theta}_u'\tilde{\theta}_u\right)\mathrm{d}u + \int_t^s \mathscr{D}(\tilde{\theta}_u)\mathrm{d}W_u \in \mathbb{R}^2, \\ H_{t,s}^b &= \int_t^s \mathscr{D}_t\left(\tilde{\mu}_{bu} - \frac{1}{2}\sigma_{bu}'\sigma_{bu}\right)\mathrm{d}u + \int_t^s \mathscr{D}(\sigma_{bu})\mathrm{d}W_u \in \mathbb{R}^2, \end{aligned}$$

$b \in \{C, e, l\}$, and $\mathscr{D}_t(\cdot)$ is the Malliavin derivative operator.[5]

The optimal portfolio has the usual structure (e.g., Detemple et al., 2003, 2005a). The first term on the right hand side is the mean-variance demand, and it is followed by four hedging demand terms. The first term of the hedging demand is motivated for the optimal consumption policy, and it depends upon the (time-varying) RRA of the social planner. Under myopic behavior ($R_s^u = 1$) this term vanishes from the optimal policy. The second, third and fourth hedging terms are motivated by the predictable changes in the cost of the future stream of C, e, and l, respectively. In the case of the former, for instance, the hedging policy is motivated by three concerns: the local (stochastic) change in value (measured by $\sigma_{Ct} = \sigma_C(t, Y_t)$), and the predictable changes in its value over the interval $(t, s]$, which may be due to changes in either its future level (measured by $H_{t,s}^C$), or the future SDF used to price its market value (measured by $H_{t,s}^{\tilde{\xi}}$). The structure of the third and fourth hedging terms follows the same logic.

The optimal policy $(c^{\nu,*}, \pi^{\nu,*})$ is parametrized by the shadow cost process ν, which can be found from the Kuhn-Tucker conditions of optimality. In particular, from the conditions $\pi_{1t}^{\nu,*} \in [0,1]$ and $\pi_{2t}^{\nu,*} = 0$, we can obtain conditions for ν to enforce the constraints that bear over $\pi^{\nu,*}$. Yet, these conditions result in a system of two (backward-forward stochastic differential equations) BFSDEs for ν, whose solution is unknown.[6]

[5] The introduction of this operator in asset allocation problems is due to Ocone and Karatzas (1991). The Malliavin derivative operator is an extension of the classical notion, that extends the concept to functions of the trajectories of W. In the same way that the classical derivative measures the local change in the function, due to a local change in the underlying variable, the Malliavin derivative measures the change in the function implied by a small change in the trajectory of W. The interested reader is referred to Detemple et al. (2005a) for a brief introduction to this operator in the context of a portfolio choice problem, and to Nualart (2006) for a comprehensive treatment.

[6] Except for some stilyzed cases, closed-form solutions of dynamic consumption-portfolio choice problems are rare in the literature (e.g., Detemple and Rindisbacher, 2005, 2010).

1.3 An approximate optimal policy

Inspired in the recent work by Bick et al. (2013), we follow the path of using the analytical expressions in equations (3) and (4) to derive an approximate optimal policy. In particular, we derive what we called the "local" shadow cost, $\hat{\nu}$, which is the shadow cost of the constraints that enforce the conditions $\pi_{1t}^{\nu,*} \in [0,1]$ and $\pi_{2t}^{\nu,*} = 0$, for the specific case where the pair $(\tilde{\xi}_{t,s}, H_{t,s}^{\tilde{\xi}})$ is replaced by its unconstrained analog $(\xi_{t,s}, H_{t,s}^{\xi})$ in equations (3), (4), and (5). Using $\pi_{1t}^{u,*}$ to denote the unconstrained optimal investment for the risky asset 1, we obtain:

- if $\pi_{1t}^{u,*} \in [0,1]$, $\hat{\nu}_{1t}^{+} = \hat{\nu}_{1t}^{-} = 0$, and $\hat{\nu}_{2t}$ solves:

$$\begin{aligned} \begin{pmatrix} 0 \\ \hat{\nu}_{2t} \end{pmatrix} &= (V_{1t}^{c^{\nu,*}})^{-1}(X_t^{\nu,*}\sigma_t\sigma_t') \begin{pmatrix} \pi_{1t}^{u,*} \\ 0 \end{pmatrix} + (V_{1t}^{c^{\nu,*}})^{-1}\sigma_t V_{2t}^{c^{\nu,*}} \\ &\quad + (V_{1t}^{c^{\nu,*}})^{-1}\sigma_t V_t^C + (V_{1t}^{c^{\nu,*}})^{-1}\sigma_t V_t^e - (V_{1t}^{c^{\nu,*}})^{-1}\sigma_t V_t^l - (\mu_t - r_t\mathbf{1}), \end{aligned}$$

- if $\pi_{1t}^{u,*} > 1$, $\hat{\nu}_{1t}^{+} = 0$, and $(\hat{\nu}_{1t}^{-}, \hat{\nu}_{2t})$ solve:

$$\begin{aligned} \begin{pmatrix} -\hat{\nu}_{1t}^{-} \\ \hat{\nu}_{1t}^{-} + \hat{\nu}_{2t} \end{pmatrix} &= (V_{1t}^{c^{\nu,*}})^{-1}(X_t^{\nu,*}\sigma_t\sigma_t') \begin{pmatrix} (\pi_{1t}^{u,*} - 1)^{+} \\ 0 \end{pmatrix} + (V_{1t}^{c^{\nu,*}})^{-1}\sigma_t V_{2t}^{c^{\nu,*}} \\ &\quad + (V_{1t}^{c^{\nu,*}})^{-1}\sigma_t V_t^C + (V_{1t}^{c^{\nu,*}})^{-1}\sigma_t V_t^e - (V_{1t}^{c^{\nu,*}})^{-1}\sigma_t V_t^l - (\mu_t - r_t\mathbf{1}), \end{aligned}$$

- if $\pi_{1t}^{u,*} < 0$, $\hat{\nu}_{1t}^{-} = 0$, and $(\hat{\nu}_{1t}^{+}, \hat{\nu}_{2t})$ solve:

$$\begin{aligned} \begin{pmatrix} \hat{\nu}_{1t}^{+} \\ \hat{\nu}_{2t} \end{pmatrix} &= (V_{1t}^{c^{\nu,*}})^{-1}(X_t^{\nu,*}\sigma_t\sigma_t') \begin{pmatrix} (\pi_{1t}^{u,*})^{-} \\ 0 \end{pmatrix} + (V_{1t}^{c^{\nu,*}})^{-1}\sigma_t V_{2t}^{c^{\nu,*}} \\ &\quad + (V_{1t}^{c^{\nu,*}})^{-1}\sigma_t V_t^C + (V_{1t}^{c^{\nu,*}})^{-1}\sigma_t V_t^e - (V_{1t}^{c^{\nu,*}})^{-1}\sigma_t V_t^l - (\mu_t - r_t\mathbf{1}). \end{aligned}$$

Using these local shadow cost, $\hat{\nu}$, the nearly-optimal policy, $(\hat{c}_t^*, \hat{\pi}_t^*)$, is given by:

$$\hat{c}_t^* = \begin{cases} I(\hat{y}\hat{\xi}_t, t) - C_t & \text{if } \ \hat{\xi}_t \geq \hat{\xi}_{Ct} \\ 0 & \text{if } \ \hat{\xi}_t < \hat{\xi}_{Ct} \end{cases}$$

where $\hat{\xi}_{Ct} = \partial_1 u(C_t, t)/\hat{y}$, $\hat{y}$ is adjusted accordingly, $\hat{\xi}_t$ solves $\mathrm{d}\hat{\xi}_t = -\hat{\xi}_t(\hat{r}_t \mathrm{d}t + \hat{\theta}_t \mathrm{d}W_t)$, $\hat{\theta}_t = \sigma_t^{-1}(\mu_t - r_t\mathbf{1}) + \sigma_t^{-1}\hat{\nu}_t$, while $\hat{\pi}_t^*$ corresponds to the solution in (4), for the case where $(\tilde{\xi}_{t,s}, H_{t,s}^{\tilde{\xi}})$ is replaced by its unconstrained analog $(\xi_{t,s}, H_{t,s}^{\xi})$.

1.4 Future steps

We plan to apply a generalized version of the model developed in the previous sections to assess the optimality of the injection/withdrawal and investment policy of the Chilean PRF.

1.4.1 The Chilean PRF

As of February 2014, the PRF has assets for nearly USD 7bn. In addition, current regulation entails a predetermined rule for the injection of funds, which can range between 0.2% and 0.5% of the GDP, and a discretionary rule for the disbursements, which can range from zero and up to a third of the expenditures generated by the 2008 Pension Reform, directed to increase the coverage of the first pillar. The PRF's investment policy entails a strategic asset allocation of 65% in sovereign debt, 20% in corporate bonds, and 15% in equity. In addition, all assets are held in foreign currency.

1.4.2 Assessment of the current policies

Based on the structural model developed by Castañeda et al. (2014), which provides a first-principles description of the injection and withdrawal (stochastic) processes of the Chilean PRF,[7] our assessment will entail:

1. to determine (by means of regression analysis) reduced-form approximation (i.e., polynomials on the identified factors) of the "structural" models of the predetermined injection and withdrawal processes;[8]

2. to derive and compute the approximate optimal policies ($\hat{c}^*$ and $\hat{\pi}^*$), provided parametric models for e, l, C, S, and r;

3. to measure the welfare loss due to (among others) the unhedged portion of longevity risk, the currency mismatch between assets and liabilities, and the exclusion of some asset classes from the current investment policy.[9]

[7] The first and second pillars in Chile are integrated. This means that the maximum disbursement of the PRF (defined at the individual level as a top up benefit) depends upon the savings of each individual in the country. This makes the task of projecting its future evolution more much more involved relative to first pillar systems based almost excluseively on demographics.

[8] The introduction of longevity risk in the structural model of Castañeda et al. (2014) will be done within the confines of the Brownian uncertainty, along the lines of, for instance, Luciano et al. (2012).

[9] The assessment of the welfare cost from excluded asset classes in the investment policy paralels the point raised by Caballero and Panageas (2008), regarding the absence of (protective put) option contracts in the portfolio held by the Chilean Internation Reserves.

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